卡尔曼滤波

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本堂课大纲

- 课程内容简介

  - 卡尔曼滤波(Kalman filter)
    - 扩展卡尔曼滤波(Extended Kalman filter)
    - 无迹卡尔曼滤波(Unscented Kalman filter)
    - 集合卡尔曼滤波(Ensemble Kalman filter)

  - 高维数据同化问题
    - 平方根类卡尔曼滤波(Square root Kalman filter)
    - 卡尔曼滤波的问题与改进

数据同化问题

- 数据同化问题

演化方程: $x_{n+1} = \mathcal{F}(x_n) + \omega_{n+1}$

观测方程: $y_{n+1} = \mathcal{H}(x_{n+1}) + \eta_{n+1}$

- 假设

高斯演化噪声: $\rho_\omega = \mathcal{N}(x; 0, \Sigma_\omega)$

高斯观测噪声: $\rho_\eta = \mathcal{N}(x; 0, \Sigma_\eta)$

高斯先验分布: $\rho_{\text{prior}}(x) = \mathcal{N}(x; r_0, \Sigma_0)$

贝叶斯反问题与数据同化问题

- 参数化方法

高斯近似：

$$\rho^*(x) \approx \mathcal{N}(x; m, C)$$

混合高斯近似：

$$\rho^*(x) \approx \sum_{j=1}^{J} w_j \mathcal{N}(x; m_j, C_j) \qquad \sum_{j=1}^{J} w_j = 1$$

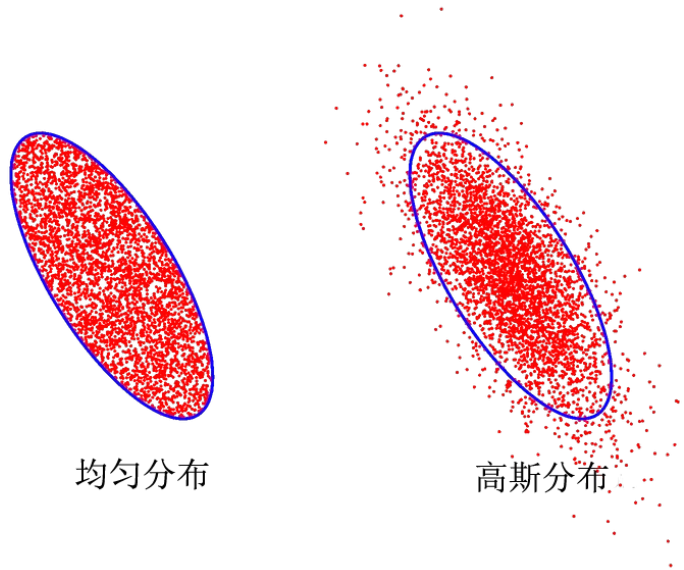
……

- 非参数化方法

粒子近似 $J \gg 1$：

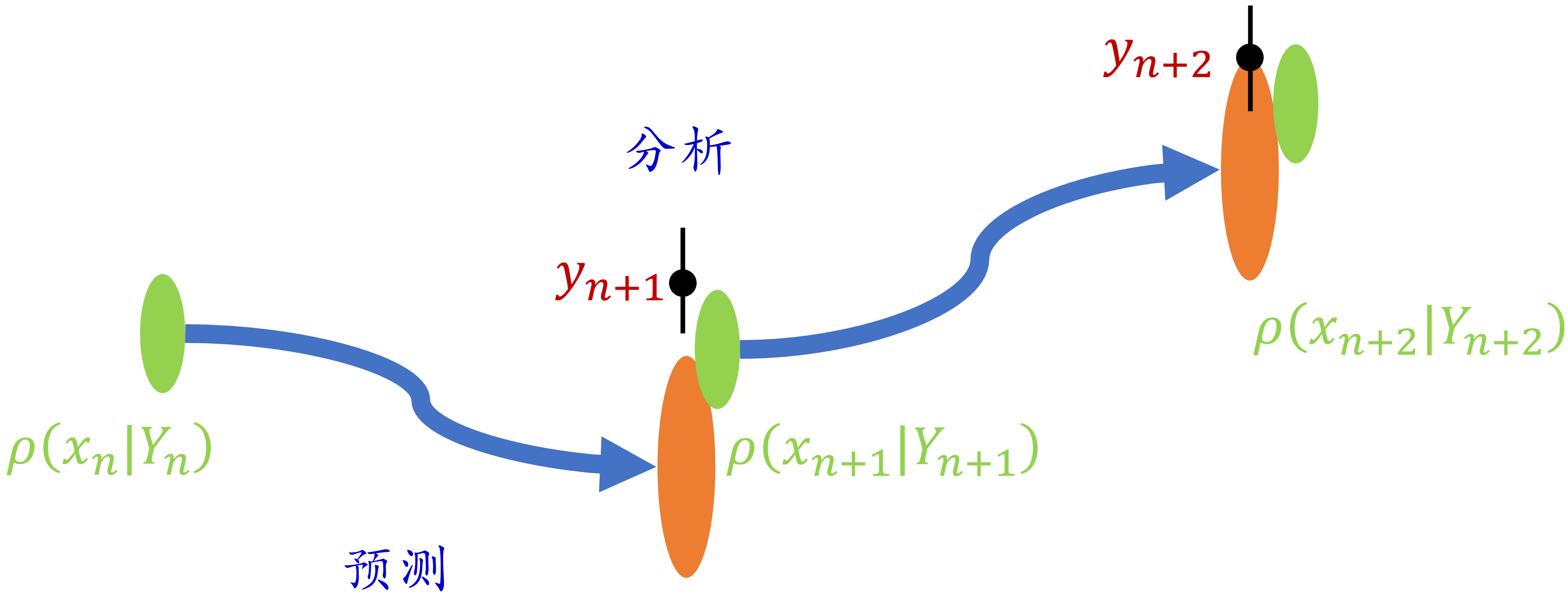
$$\rho^*(x) \approx \{x_j\}_{j=1}^{J}$$

$$\rho^*(x) \approx \frac{1}{J}\sum_{j=1}^{J} \delta(x - x_j)$$



数据同化问题

➤ 卡尔曼方法



预测(Prediction)

- 演化方程

演化方程: $x_{n+1} = \mathcal{F}(x_n) + \omega_{n+1}$

高斯演化噪声: $\omega_{n+1} \sim \rho_\omega = \mathcal{N}(x; 0, \Sigma_\omega)$

- 当前状态

高斯近似: $x_n | Y_n \sim \mathcal{N}(x; m_n, C_n)$

粒子近似: $x_n | Y_n \sim \frac{1}{J} \sum_{j=1}^{J} \delta(x - x_n^j)$

- 预测

高斯近似: $x_{n+1} | Y_n \sim \mathcal{N}(x; \widehat{m}_n, \hat{C}_n)$

粒子近似: $x_{n+1} | Y_n \sim \frac{1}{J} \sum_{j=1}^{J} \delta(x - \hat{x}_n^j)$

分析(Analysis)

- 观测方程

观测方程: $y_{n+1} = \mathcal{H}(x_{n+1}) + \eta_{n+1}$

高斯观测噪声: $\eta_{n+1} \sim \rho_\eta = \mathcal{N}(x; 0, \Sigma_\eta)$

- 预测状态

高斯近似: $x_{n+1}|Y_n \sim \mathcal{N}(x; \widehat{m}_{n+1}, \hat{C}_{n+1})$

粒子近似: $x_{n+1}|Y_n \sim \frac{1}{J}\sum_{j=1}^{J} \delta(x - \hat{x}_{n+1}^{j})$

- 分析

高斯近似: $x_{n+1}|Y_{n+1} \sim \mathcal{N}(x; m_{n+1}, C_{n+1})$

粒子近似: $x_{n+1}|Y_{n+1} \sim \frac{1}{J}\sum_{j=1}^{J} \delta(x - x_{n+1}^{j})$

扩展(Extended)卡尔曼滤波

- 泰勒展开线性化

$$x_n|Y_n \sim \mathcal{N}(x; m_n, C_n)$$

$$\mathcal{F}(x_n) \approx \mathcal{F}(m_n) + \nabla\mathcal{F}(m_n)(x_n - m_n)$$

$$x_{n+1}|Y_n \sim \mathcal{N}\left(x; \widehat{m}_{n+1}, \hat{C}_{n+1}\right) \qquad \omega_{n+1} \sim \mathcal{N}(0, \Sigma_\omega)$$

$$\widehat{m}_{n+1} = \mathbb{E}[\mathcal{F}(x_n) + \omega_{n+1}] \approx \mathcal{F}(m_n)$$

$$\hat{C}_{n+1} = \mathrm{Cov}[\mathcal{F}(x_n) + \omega_{n+1}] \approx \nabla\mathcal{F}(m_n)^T C_n \nabla\mathcal{F}(m_n) + \Sigma_\omega$$

卡尔曼方法

- 先验分布

$$x_{n+1}|Y_n \sim \mathcal{N}\left(\hat{m}_{n+1}, \hat{C}_{n+1}\right)$$

- x_{n+1} 和 η 的联合分布

$$\rho\left(x_{n+1}, \mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n\right) \approx \mathcal{N}\left(\begin{bmatrix}\hat{m}_{n+1} \\ \hat{y}_{n+1}\end{bmatrix}, \begin{bmatrix}\hat{C}_{n+1} & \hat{C}_{n+1}^{xy} \\ \hat{C}_{n+1}^{xy^T} & \hat{C}_{n+1}^{yy}\end{bmatrix}\right)$$

- 后验分布（条件分布）

$$\rho\left(x_{n+1}|\mathcal{H}(x_{n+1}) + \eta_{n+1} = y_{n+1}, Y_n\right) = \mathcal{N}\left(m_{n+1}, C_{n+1}\right)$$

$$m_{n+1} = \hat{m}_{n+1} + \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}(y_{n+1} - \hat{y}_{n+1})$$

$$C_{n+1} = \hat{C}_{n+1} - \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}\hat{C}_{n+1}^{xy^T}$$

扩展卡尔曼滤波

➤ 泰勒展开线性化

$$\mathcal{H}(x_{n+1}) \approx \mathcal{H}(\widehat{m}_{n+1}) + \nabla\mathcal{H}(\widehat{m}_{n+1})(x_{n+1} - \widehat{m}_{n+1})$$

$$x_{n+1}|Y_n \sim \mathcal{N}\left(x; \widehat{m}_{n+1}, \hat{C}_{n+1}\right) \qquad \eta_{n+1} \sim \mathcal{N}(0, \Sigma_\eta)$$

$$\hat{y}_{n+1} = \mathbb{E}[\mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n] \approx \mathcal{H}(\widehat{m}_{n+1})$$

$$\begin{aligned} \hat{C}_{n+1}^{xy} &= \mathrm{Cov}[x_{n+1}, \mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n] \\ &\approx \hat{C}_{n+1}\nabla\mathcal{H}(\widehat{m}_{n+1})^T \end{aligned}$$

$$\begin{aligned} \hat{C}_{n+1}^{xx} &= \mathrm{Cov}[\mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n] \\ &\approx \nabla\mathcal{H}(\widehat{m}_{n+1})\hat{C}_{n+1}\nabla\mathcal{H}(\widehat{m}_{n+1})^T + \Sigma_\eta \end{aligned}$$

无迹(Unscented)卡尔曼方法

无迹变换(Unscented transform)

对于高斯分布 $\theta \sim \mathcal{N}(m, C) \in R^{N_\theta}$, 我们选取$2N_\theta + 1$ 个σ点， $\theta^0 = m$， 对 $j = 1,2\cdots, N_\theta$

$$\theta^j = m + c_j[\sqrt{C}]_j \quad \theta^{j+N_\theta} = m - c_j[\sqrt{C}]_j$$

其中 $[\sqrt{C}]_j$ 是C的Cholesky分解的第 j个列向量，那么

$$\mathbb{E}[\mathcal{G}(\theta)] \approx \sum_{i=0}^{2N_\theta} W_i^m \mathcal{G}(\theta^i)$$

$$\mathrm{Cov}[\mathcal{G}_1(\theta), \mathcal{G}_2(\theta)] \approx$$

$$\sum_{i=1}^{2N_\theta} W_i^c (\mathcal{G}_1(\theta^i) - \mathbb{E}[\mathcal{G}_1(\theta)])(\mathcal{G}_2(\theta^i) - \mathbb{E}[\mathcal{G}_2(\theta)])^T$$

参数： c_i， W_i^m, W_i^c

无迹卡尔曼方法

无迹变换(Unscented transform)

当 $1 \le j \le N_\theta$

$$W_j^m = W_{j+N_\theta}^m \qquad W_j^c = W_{j+N_\theta}^c = \frac{1}{2c_j^2} \qquad \sum_{i=0}^{2N_\theta} W_i^m = 1$$

$$\sum_{i=0}^{2N_\theta} W_i^m \mathcal{G}(\theta^i) = \mathbb{E}[\mathcal{G}(\theta)] + \mathcal{O}(\|C\|)$$

$$\sum_{i=0}^{2N_\theta} W_i^c (\mathcal{G}_1(\theta^i) - \mathbb{E}[\mathcal{G}_1(\theta)])(\mathcal{G}_2(\theta^i) - \mathbb{E}[\mathcal{G}_2(\theta)])^T = \\ = \mathrm{Cov}[\mathcal{G}_1(\theta), \mathcal{G}_2(\theta)] + \mathcal{O}(\|C\|^2)$$

我们选取 $W_0^m = 1$， $W_0^c = 0$，对于$1 \le j \le N_\theta$

$c_j = a\sqrt{N_\theta}$， $W_j^m = 0$， $W_j^c = \frac{1}{2a^2 N_\theta}$

无迹卡尔曼滤波

- 预测

$$x_n|Y_n \sim \mathcal{N}(x; m_n, C_n) \qquad \omega_{n+1} \sim \mathcal{N}(0, \Sigma_\omega)$$

$$\widehat{m}_{n+1} = \mathbb{E}[\mathcal{F}(x_n)|Y_n]$$

$$\hat{C}_{n+1} = \mathrm{Cov}[\mathcal{F}(x_n)|Y_n] + \Sigma_\omega$$

- 分析

$$x_{n+1}|Y_n \sim \mathcal{N}\left(x; \widehat{m}_{n+1}, \hat{C}_{n+1}\right) \qquad \eta_{n+1} \sim \mathcal{N}(0, \Sigma_\eta)$$

$$\hat{y}_{n+1} = \mathbb{E}[\mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n] = \mathbb{E}[\mathcal{H}(x_{n+1})|Y_n]$$

$$\hat{C}^{xy}_{n+1} = \mathrm{Cov}[x_{n+1}, \mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n]$$

$$= \mathrm{Cov}[x_{n+1}, \mathcal{H}(x_{n+1})|Y_n]$$

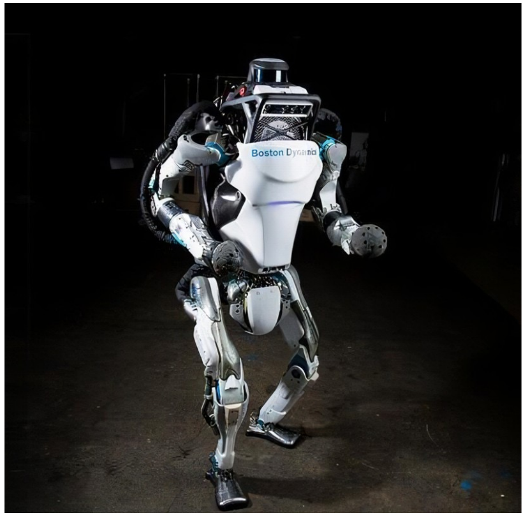
$$\hat{C}^{yy}_{n+1} = \mathrm{Cov}[\mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n]$$

$$= \mathrm{Cov}[\mathcal{H}(x_{n+1})|Y_n] + \Sigma_\eta$$

卡尔曼滤波

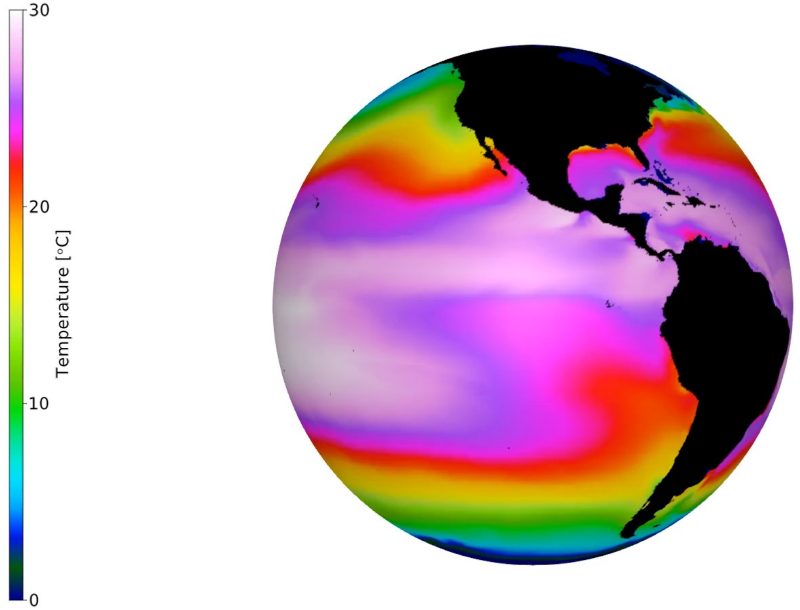
- 扩展卡尔曼滤波
  - 需要计算$\mathcal{F}$,$\mathcal{H}$的导数
  - 需要存放方差矩阵

- 无迹卡尔曼滤波
  - 方差近似更精确
  - 需要多次计算$\mathcal{F}$,$\mathcal{H}$
  - 需要存放方差矩阵



高维数据同化问题

- 气象、气候模型

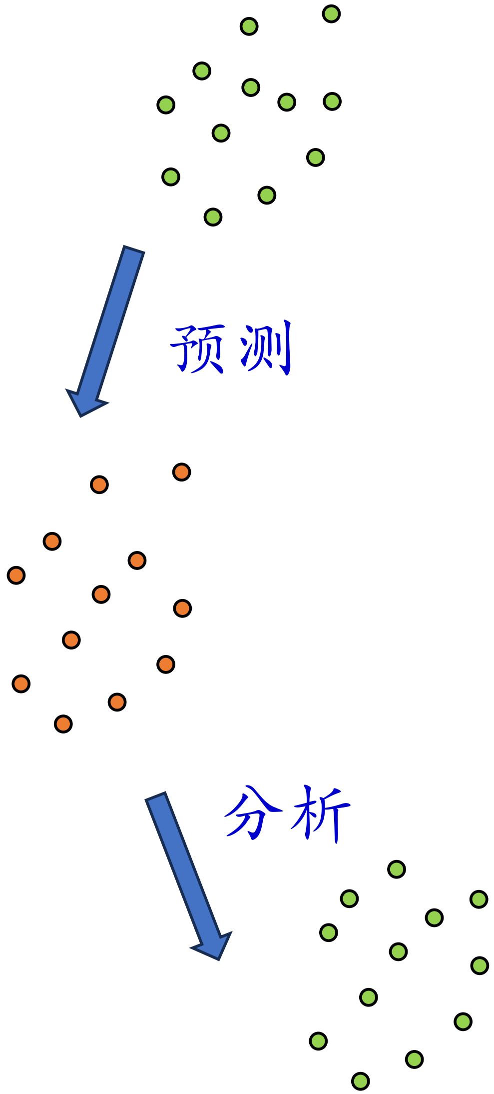


$$N_x = \mathcal{O}(10^7)$$

如何避免存放协方差矩阵？

集合(Ensemble)卡尔曼滤波

- 集合卡尔曼滤波



$$x_n|Y_n \sim \frac{1}{J}\sum_{j=1}^{J}\delta(x - x_n^j)$$

$$x_{n+1}|Y_n \sim \frac{1}{J}\sum_{j=1}^{J}\delta(x - \hat{x}_{n+1}^j)$$

$$x_{n+1}|Y_{n+1} \sim \frac{1}{J}\sum_{j=1}^{J}\delta(x - x_{n+1}^j)$$

集合卡尔曼滤波

- 预测

$$x_n|Y_n \sim \frac{1}{J}\sum_{j=1}^{J}\delta(x - x_n^j)$$

对于 $j = 1,2,\cdots J$

$$\hat{x}_{n+1}^j = \mathcal{F}\left(x_n^j\right) + \omega_{n+1}^j \qquad \omega_{n+1}^j \sim \mathcal{N}(x; 0, \Sigma_\omega)$$

$$x_{n+1}|Y_n \sim \frac{1}{J}\sum_{j=1}^{J}\delta(x - \hat{x}_{n+1}^j)$$

卡尔曼方法

- 先验分布

$$x_{n+1}|Y_n \sim \mathcal{N}\left(\widehat{m}_{n+1}, \hat{C}_{n+1}\right)$$

- x_{n+1} 和 η 的联合分布

$$\rho\left(x_{n+1}, \mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n\right) \approx \mathcal{N}\left(\begin{bmatrix}\widehat{m}_{n+1} \\ \hat{y}_{n+1}\end{bmatrix}, \begin{bmatrix}\hat{C}_{n+1} & \hat{C}_{n+1}^{xy} \\ \hat{C}_{n+1}^{xy^T} & \hat{C}_{n+1}^{yy}\end{bmatrix}\right)$$

- 后验分布（条件分布）

$$\rho\left(x_{n+1}|\mathcal{H}(x_{n+1}) + \eta_{n+1} = y_{n+1}, Y_n\right) = \mathcal{N}\left(m_{n+1}, C_{n+1}\right)$$

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}(y_{n+1} - \hat{y}_{n+1})$$

$$C_{n+1} = \hat{C}_{n+1} - \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}\hat{C}_{n+1}^{xy^T}$$

集合卡尔曼滤波

- 分析

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1})$$

$$C_{n+1} = \hat{C}_{n+1} - \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{xy^T}_{n+1}$$

根据 $x_{n+1}|Y_n \sim \frac{1}{J}\sum_{j=1}^{J}\delta(x - \hat{x}^j_{n+1})$，计算：

$$\hat{y}_{n+1} = \mathbb{E}[\mathcal{H}(x_{n+1})|Y_n] \approx \frac{1}{J}\sum_{j=1}^{J}\mathcal{H}(\hat{x}^j_{n+1})$$

$$\widehat{m}_{n+1} \approx \frac{1}{J}\sum_{j=1}^{J}\hat{x}^j_{n+1}$$

$$\hat{C}^{xy}_{n+1} = \mathrm{Cov}[x_{n+1}, \mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n]$$

$$\approx \frac{1}{J-1}\sum_{j=1}^{J}(\hat{x}^j_{n+1} - \widehat{m}_{n+1})\left(\mathcal{H}(\hat{x}^j_{n+1}) - \hat{y}_{n+1}\right)^T$$

$$\hat{C}^{yy}_{n+1} = \mathrm{Cov}[\mathcal{H}(x_{n+1}) + \eta_{n+1}|Y_n]$$

$$\approx \frac{1}{J-1}\sum_{j=1}^{J}\left(\mathcal{H}(\hat{x}^j_{n+1}) - \hat{y}_{n+1}\right)\left(\mathcal{H}(\hat{x}^j_{n+1}) - \hat{y}_{n+1}\right)^T + \Sigma_\eta$$

集合卡尔曼滤波

➢ 分析

$$m_{n+1} = \hat{m}_{n+1} + \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}(y_{n+1} - \hat{y}_{n+1})$$

$$C_{n+1} = \hat{C}_{n+1} - \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}\hat{C}_{n+1}^{xy^T}$$

$$x_{n+1}^{j} = \hat{x}_{n+1}^{j} + \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}\left(y_{n+1} - \mathcal{H}\left(\hat{x}_{n+1}^{j}\right)\right)$$

方差不能匹配

$$x_{n+1}^{j} = \hat{x}_{n+1}^{j} + \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}\left(y_{n+1} - \mathcal{H}\left(\hat{x}_{n+1}^{j}\right) - \eta_{n+1}^{j}\right)$$

其中 $\eta_{n+1}^{j} \sim \mathcal{N}(0, \Sigma_\eta)$

集合卡尔曼滤波

集合卡尔曼滤波误差估计

定义:

$$\widehat{m}_{n+1}=\frac{1}{J}\sum_{j=1}^{J}\hat{x}_{n+1}^{j}$$

$$\hat{C}_{n+1}=\frac{1}{J-1}\sum_{j=1}^{J}\left(\hat{x}_{n+1}^{j}-\widehat{m}_{n+1}\right)\left(\hat{x}_{n+1}^{j}-\widehat{m}_{n+1}\right)^{T}$$

$$m_{n+1}=\frac{1}{J}\sum_{j=1}^{J}x_{n+1}^{j}$$

$$C_{n+1}=\frac{1}{J-1}\sum_{j=1}^{J}\left(x_{n+1}^{j}-m_{n+1}\right)\left(x_{n+1}^{j}-m_{n+1}\right)^{T}$$

对集合卡尔曼滤波的分析步，我们有

$$m_{n+1}=\widehat{m}_{n+1}+\hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}(y_{n+1}-\hat{y}_{n+1})+\mathcal{O}(\frac{1}{\sqrt{J}})$$

$$C_{n+1}=\hat{C}_{n+1}-\hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}\hat{C}_{n+1}^{xy^{T}}+\mathcal{O}(\frac{1}{\sqrt{J}})$$

集合卡尔曼滤波

- 从优化角度的理解

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1})$$

$$C_{n+1} = \hat{C}_{n+1} - \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{xy^T}_{n+1}$$

$$x^j_{n+1} = \hat{x}^j_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \mathcal{H}(\hat{x}^j_{n+1}) - \eta^j_{n+1})$$

当 $\mathcal{H}(x) = Hx$

$$x^j_{n+1} = \operatorname{argmin}_{x_{n+1}} \frac{1}{2} \| \hat{C}^{-\frac{1}{2}}_{n+1}(x_{n+1} - \hat{x}^j_{n+1}) \|^2 + \frac{1}{2} \| \Sigma_\eta^{-\frac{1}{2}} (y_{n+1} - Hx_{n+1} - \eta^j_{n+1}) \|^2$$

Lorenz63系统

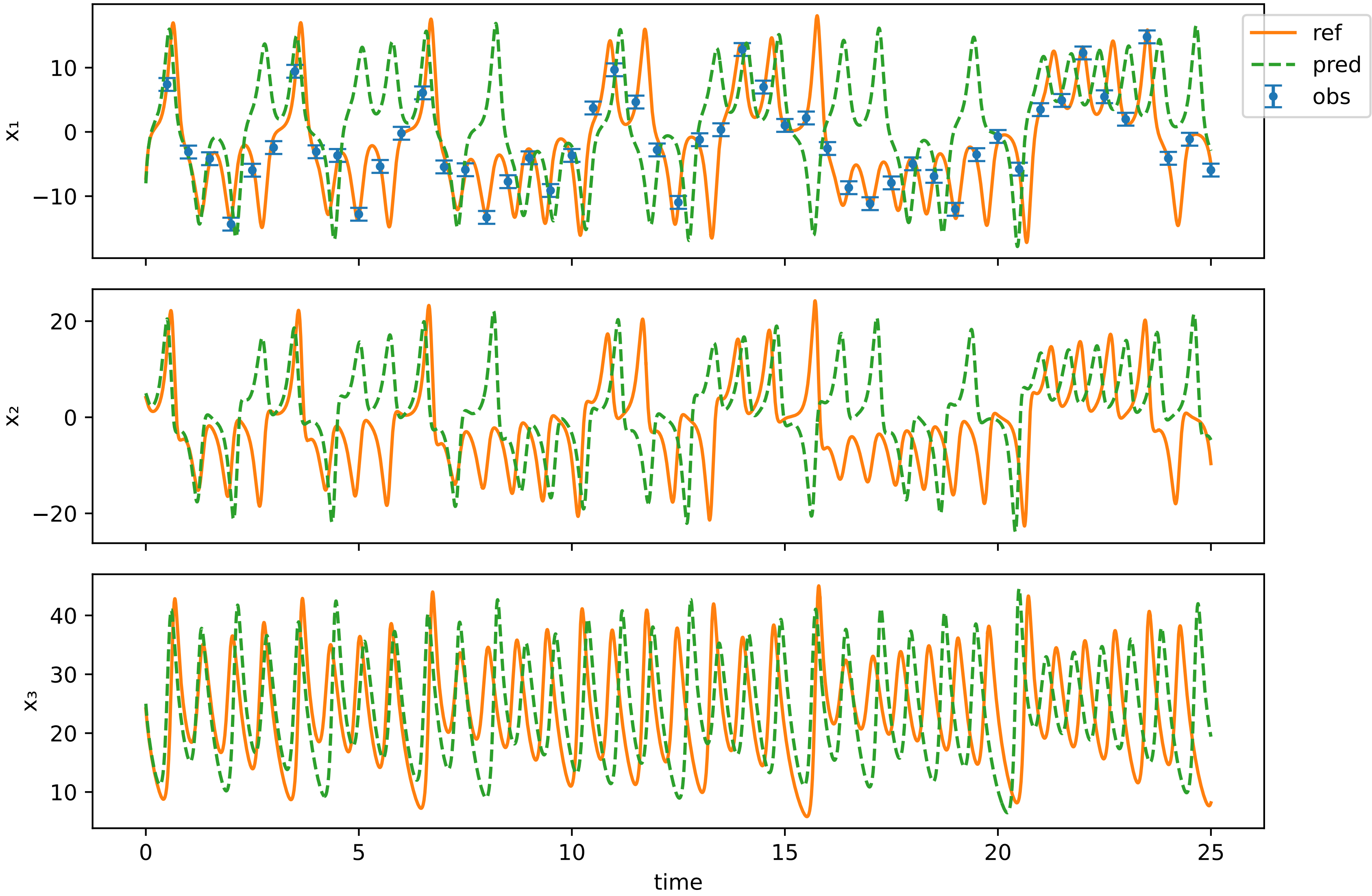
➢ 练习

$$\frac{dx_1}{dt} = \sigma(x_2 - x_1)$$
$$\frac{dx_2}{dt} = x_1(r - x_3) - x_2$$
$$\frac{dx_3}{dt} = x_1x_2 - \beta x_3$$

其中$\sigma = 10,\ r = 28,\ \beta = \frac{8}{3}$，我们知道初始值满足$x(0) \sim \mathcal{N}([-8.0;\ 5.0;\ 25], I)$，我们能在$t = 0.5, 1, 1.5 \cdots$时刻观测到$x_1$，观测误差是$\mathcal{N}(0,1)$。

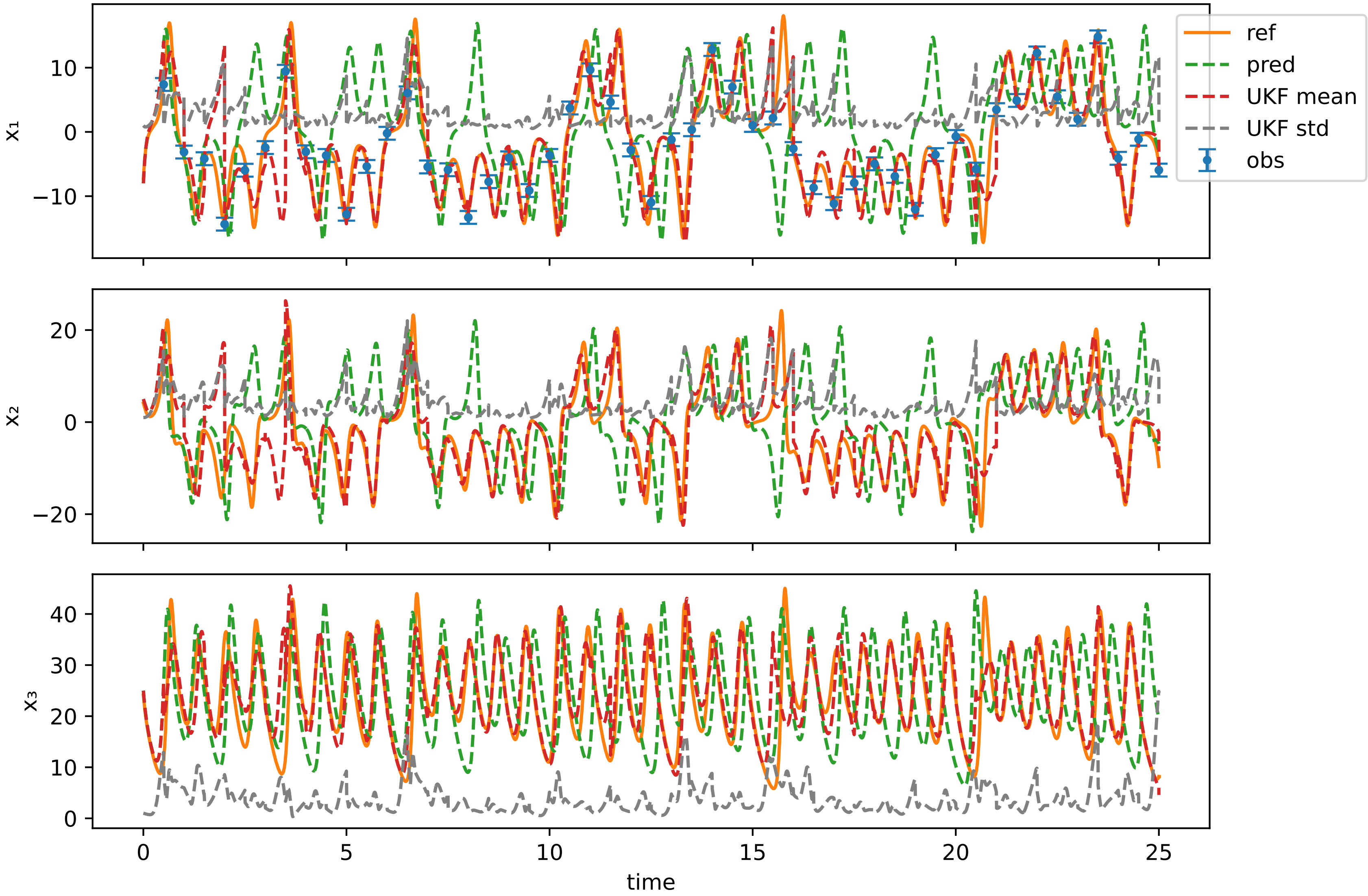
Lorenz63系统

➢ 直接预测（向前欧拉 $\Delta t = 10^{-3}$，$T = 25$）



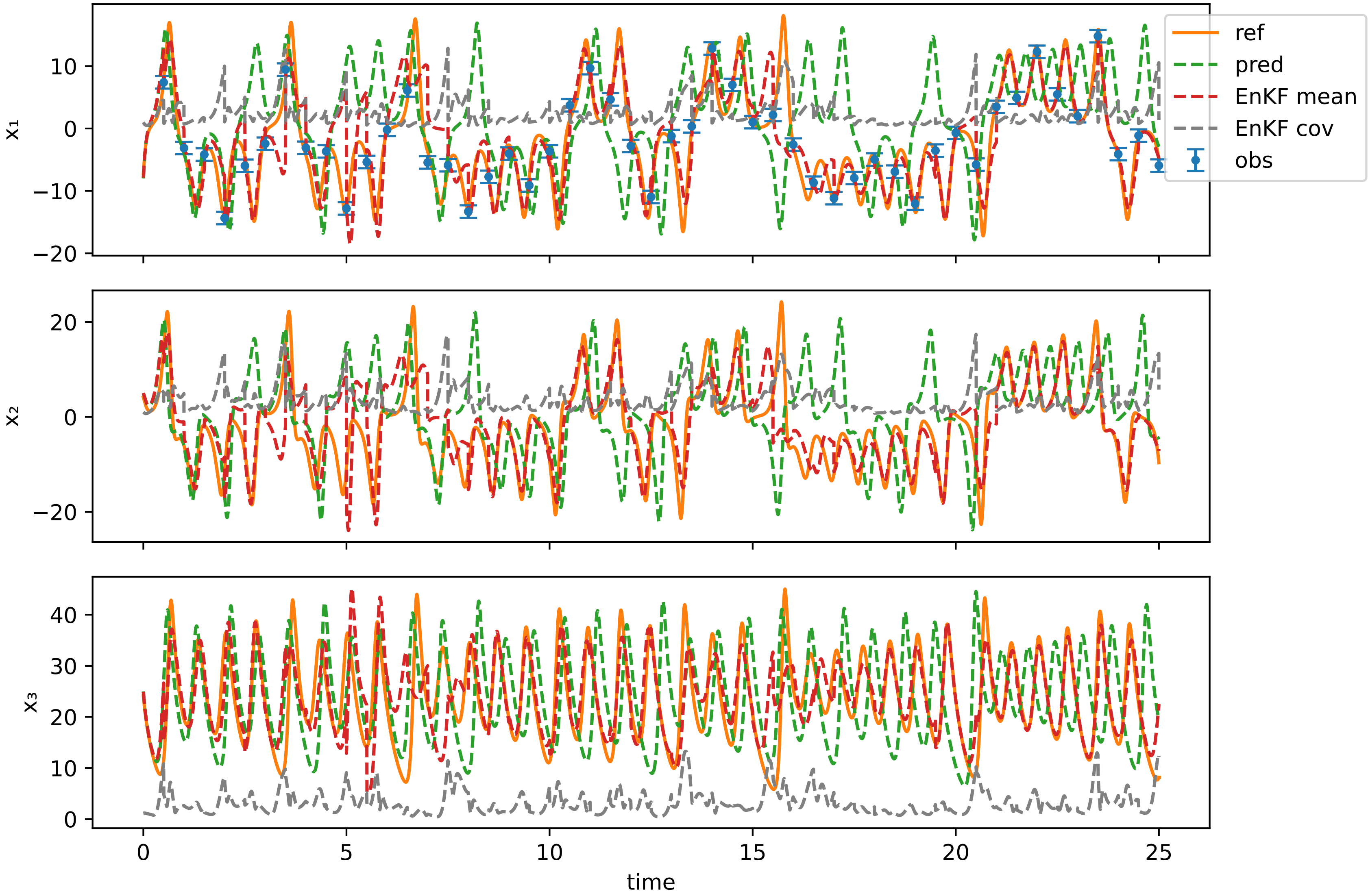
Lorenz63系统

➢ 无迹卡尔曼滤波($J = 7$)



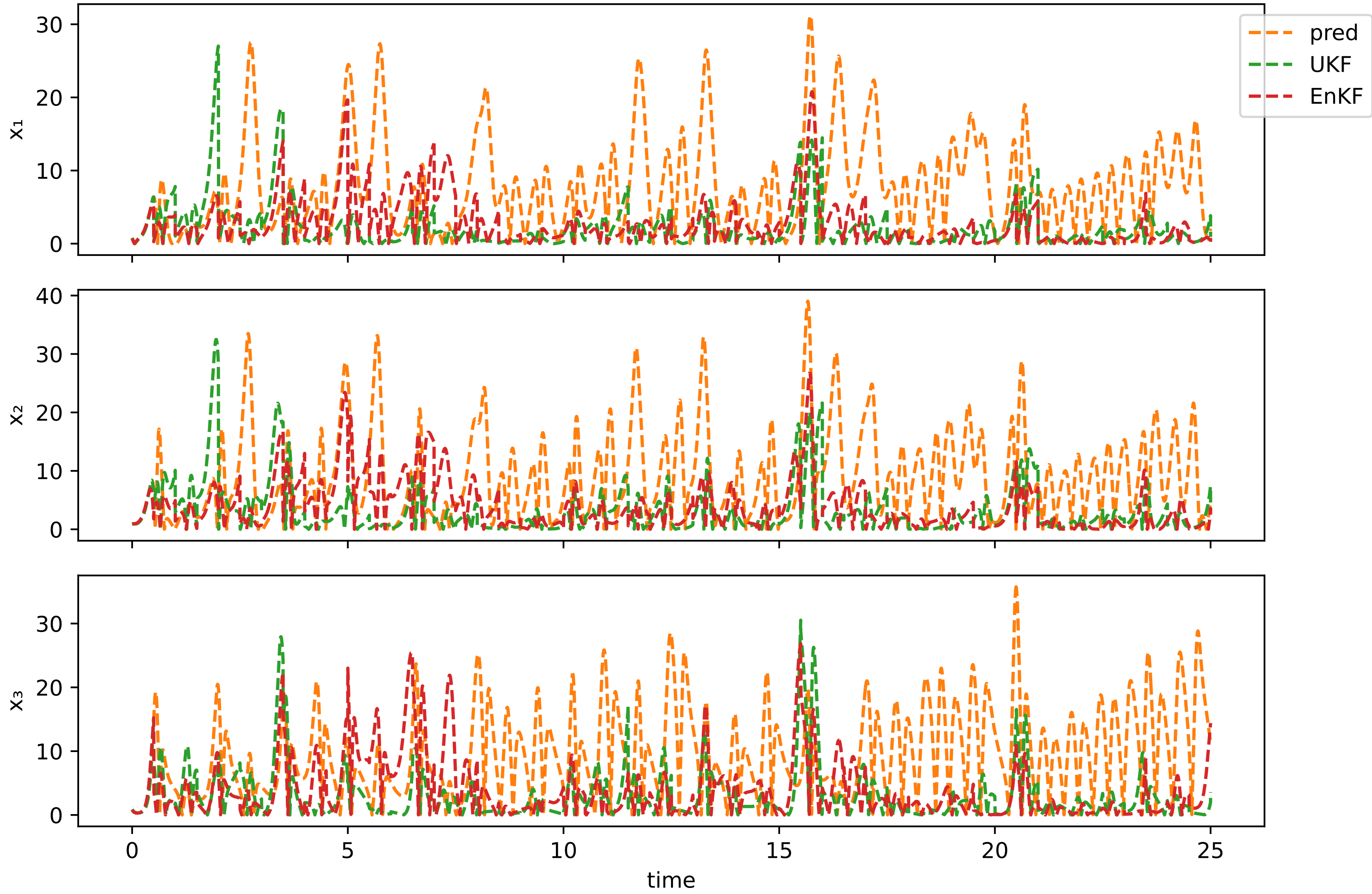
Lorenz63系统

➤ 集合卡尔曼滤波(J = 10)



Lorenz63系统

➤ 卡尔曼滤波绝对误差



7.84
2.21
2.71

9.18
3.31
3.93

9.68
3.10
3.83

卡尔曼滤波

- 扩展卡尔曼滤波
  - 需要计算$\mathcal{F},\mathcal{H}$的导数
  - 需要存放协方差矩阵

- 无迹卡尔曼滤波
  - 需要多次计算$\mathcal{F},\mathcal{H}$
  - 需要存放协方差矩阵

- 集合卡尔曼滤波
  - 需要多次计算$\mathcal{F},\mathcal{H}$
  - 不需要存放协方差矩阵
  - 引入随机误差

集合卡尔曼滤波

➢ 分析

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1})$$

$$C_{n+1} = \hat{C}_{n+1} - \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{xy^T}_{n+1}$$

加入随机噪音匹配期望和方差

$$x^j_{n+1} = \hat{x}^j_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \mathcal{H}(\hat{x}^j_{n+1}) - \eta^j_{n+1})$$

其中 $\eta^j_{n+1} \sim \mathcal{N}(0, \Sigma_\eta)$

集合卡尔曼滤波

- 分析

$$x_{n+1}^j = \hat{x}_{n+1}^j + \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}(y_{n+1} - \mathcal{H}(\hat{x}_{n+1}^j) - \eta_{n+1}^j)$$

定义:

$$\hat{Z}_{n+1} = \frac{1}{\sqrt{J-1}}[\hat{x}_{n+1}^1 - \widehat{m}_{n+1}; \ldots\ldots; \hat{x}_{n+1}^J - \widehat{m}_{n+1}]$$

$$\hat{Y}_{n+1} = \frac{1}{\sqrt{J-1}}[\mathcal{H}(\hat{x}_{n+1}^1) - \hat{y}_{n+1}; \ldots\ldots; \mathcal{H}(\hat{x}_{n+1}^J) - \hat{y}_{n+1}]$$

$$\hat{E}_{n+1} = \frac{1}{\sqrt{J-1}}[\eta_{n+1}^1 - \hat{\eta}_{n+1}; \ldots\ldots; \eta_{n+1}^J - \hat{\eta}_{n+1}]$$

$$Z_{n+1} = \frac{1}{\sqrt{J-1}}[x_{n+1}^1 - m_{n+1}; \ldots\ldots; x_{n+1}^J - m_{n+1}]$$

我们有:

低秩近似

$$\hat{C}_{n+1}^{xy} = \hat{Z}_{n+1}\hat{Y}_{n+1}^T \qquad \hat{C}_{n+1}^{yy} = \hat{Y}_{n+1}\hat{Y}_{n+1}^T + \Sigma_\eta$$

$$\hat{C}_{n+1} = \hat{Z}_{n+1}\hat{Z}_{n+1}^T \qquad C_{n+1} = Z_{n+1}Z_{n+1}^T$$

$$Z_{n+1} = \hat{Z}_{n+1}\left(I + \hat{Y}_{n+1}^T\Sigma_\eta^{-1}\hat{Y}_{n+1}\right)^{-1}(I - \hat{Y}_{n+1}^T\Sigma_\eta^{-1}\hat{E}_{n+1})$$

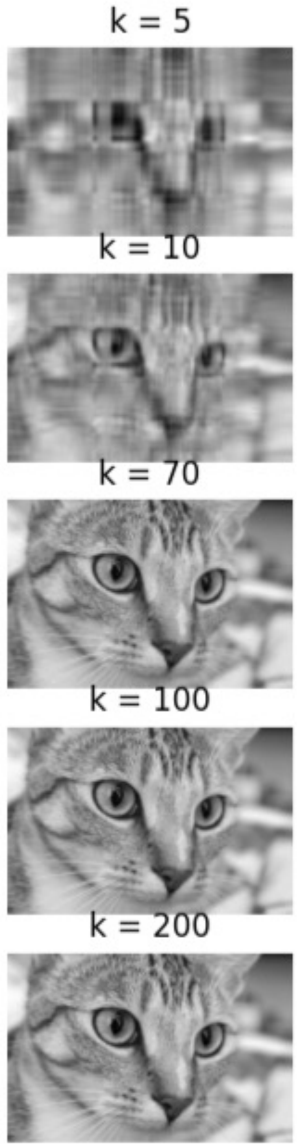
高维数据同化问题

- 奇异值分解

$$M_r = \text{argmin}_{\text{rank}(X)=r} \|M - X\|_F$$

$$M = U\Sigma V^T = \sum_i \sigma_i^2 u_i v_i^T$$

$$M_r = U_r \Sigma_r V_r^T = \sum_{i=1}^{r} \sigma_i^2 u_i v_i^T$$



集合卡尔曼滤波变体

- 卡尔曼滤波

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1})$$
$$C_{n+1} = \hat{C}_{n+1} - \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{xy^T}_{n+1}$$

- 集合变换卡尔曼滤波(Ensemble Transform Kalman Filter)

构造矩阵T:

$$Z_{n+1} = \hat{Z}_{n+1}T$$

$$\hat{Z}_{n+1}TT^T\hat{Z}^T_{n+1}$$
$$= \hat{Z}_{n+1}\hat{Z}^T{}_{n+1} - \hat{Z}_{n+1}\hat{Y}^T_{n+1}\left(\hat{Y}_{n+1}\hat{Y}^T_{n+1} + \Sigma_\eta\right)^{-1}\hat{Y}_{n+1}\hat{Z}^T_{n+1}$$

集合卡尔曼滤波变体

- 集合变换卡尔曼滤波(Ensemble Transform Kalman Filter)

$$TT^T = I - \hat{Y}_{n+1}^T \left(\hat{Y}_{n+1} \hat{Y}_{n+1}^T + \Sigma_\eta \right)^{-1} \hat{Y}_{n+1}$$

$$= \left(I + \hat{Y}_{n+1}^T \Sigma_\eta^{-1} \hat{Y}_{n+1} \right)^{-1}$$

奇异值分解：$\hat{Y}_{n+1}^T \Sigma_\eta^{-1} \hat{Y}_{n+1} = P \Gamma P^T$

$$T = P(\Gamma + \mathrm{I})^{-\frac{1}{2}} P^T$$

集合卡尔曼滤波变体

- 卡尔曼滤波

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}(y_{n+1} - \hat{y}_{n+1})$$
$$C_{n+1} = \hat{C}_{n+1} - \hat{C}_{n+1}^{xy}(\hat{C}_{n+1}^{yy})^{-1}\hat{C}_{n+1}^{xy^T}$$

- 集合调整卡尔曼滤波(Ensemble Adjustment Kalman Filter)

构造矩阵A:

$$x_{n+1}^{j} - m_{n+1} = A(\hat{x}_{n+1}^{j} - \widehat{m}_{n+1})$$

$$Z_{n+1} = A\hat{Z}_{n+1}$$

集合卡尔曼滤波变体

➤ 集合调整卡尔曼滤波(Ensemble Adjustment Kalman Filter)

$$A\hat{Z}_{n+1}\hat{Z}_{n+1}^T A^T$$

$$= \hat{Z}_{n+1}\hat{Z}_{n+1}^T - \hat{Z}_{n+1}\hat{Y}_{n+1}^T\left(\hat{Y}_{n+1}\hat{Y}_{n+1}^T + \Sigma_\eta\right)^{-1}\hat{Y}_{n+1}\hat{Z}_{n+1}^T$$

奇异值分解： $\hat{Z}_{n+1} = P\widehat{D}^{1/2}V^T$

奇异值分解： $V^T\left(I + \hat{Y}_{n+1}^T\Sigma_\eta^{-1}\hat{Y}_{n+1}\right)^{-1}V = UDU^T$

$$A = P\widehat{D}^{1/2}UD^{1/2}\widehat{D}^{-1/2}P^T$$
$$= P\widehat{D}^{\frac{1}{2}}V^T(VUD^{\frac{1}{2}}\widehat{D}^{-\frac{1}{2}}P^T)$$

集合卡尔曼滤波变体

- 卡尔曼滤波

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1})$$
$$C_{n+1} = \hat{C}_{n+1} - \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{xy^T}_{n+1}$$

- 平方根类卡尔曼滤波(Square Root Kalman Filter)

$$m_{n+1} = \widehat{m}_{n+1} + \hat{Z}_{n+1}\hat{Y}^T_{n+1}(\hat{Y}_{n+1}\hat{Y}^T_{n+1} + \Sigma_\eta)^{-1}(y_{n+1} - \hat{y}_{n+1})$$
$$Z_{n+1} = \hat{Z}_{n+1}X$$

$$C_{n+1} \approx Z_{n+1}\, Z^T_{n+1} \qquad \hat{C}_{n+1} \approx \hat{Z}_{n+1}\hat{Z}^T_{n+1}$$

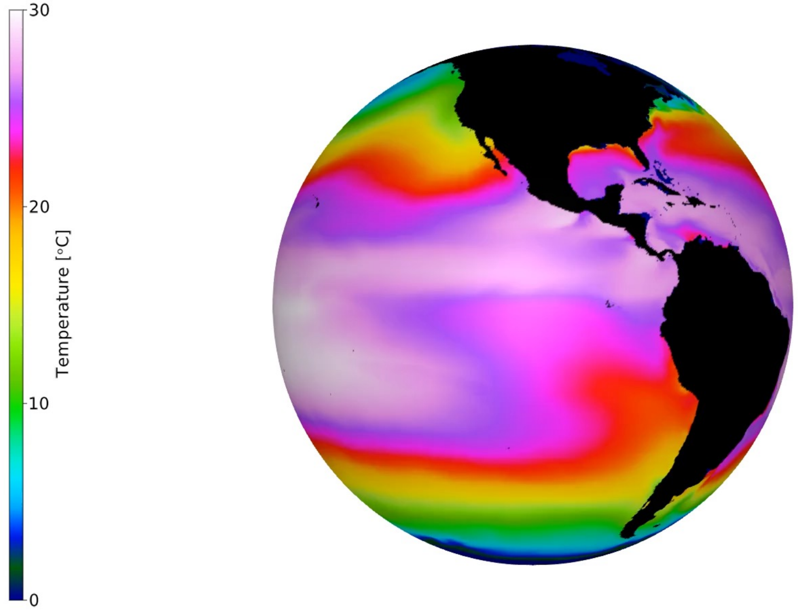
卡尔曼滤波

- 集合卡尔曼滤波及其变体
  - 集合变换卡尔曼滤波
  - 集合调整卡尔曼滤波

高维协方差矩阵的低秩近似

高维数据同化问题

- 气象、气候模型



$$N_x = \mathcal{O}(10^7)$$

平方根类卡尔曼滤波或类似的低维近似能解决高维数据同化问题吗？

对流问题例子

- 周期边界的对流方程

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=0 \quad x\in[0,L] \qquad\qquad L=1000$$

真实的初始值：

$$u_0(x)=\sum_{i=1}^{50} a_i \sin\frac{2\pi i x}{L}+b_i\cos\frac{2\pi i x}{L}$$

$$a_i，\; b_i \sim \mathcal{N}(0,1)$$

解：

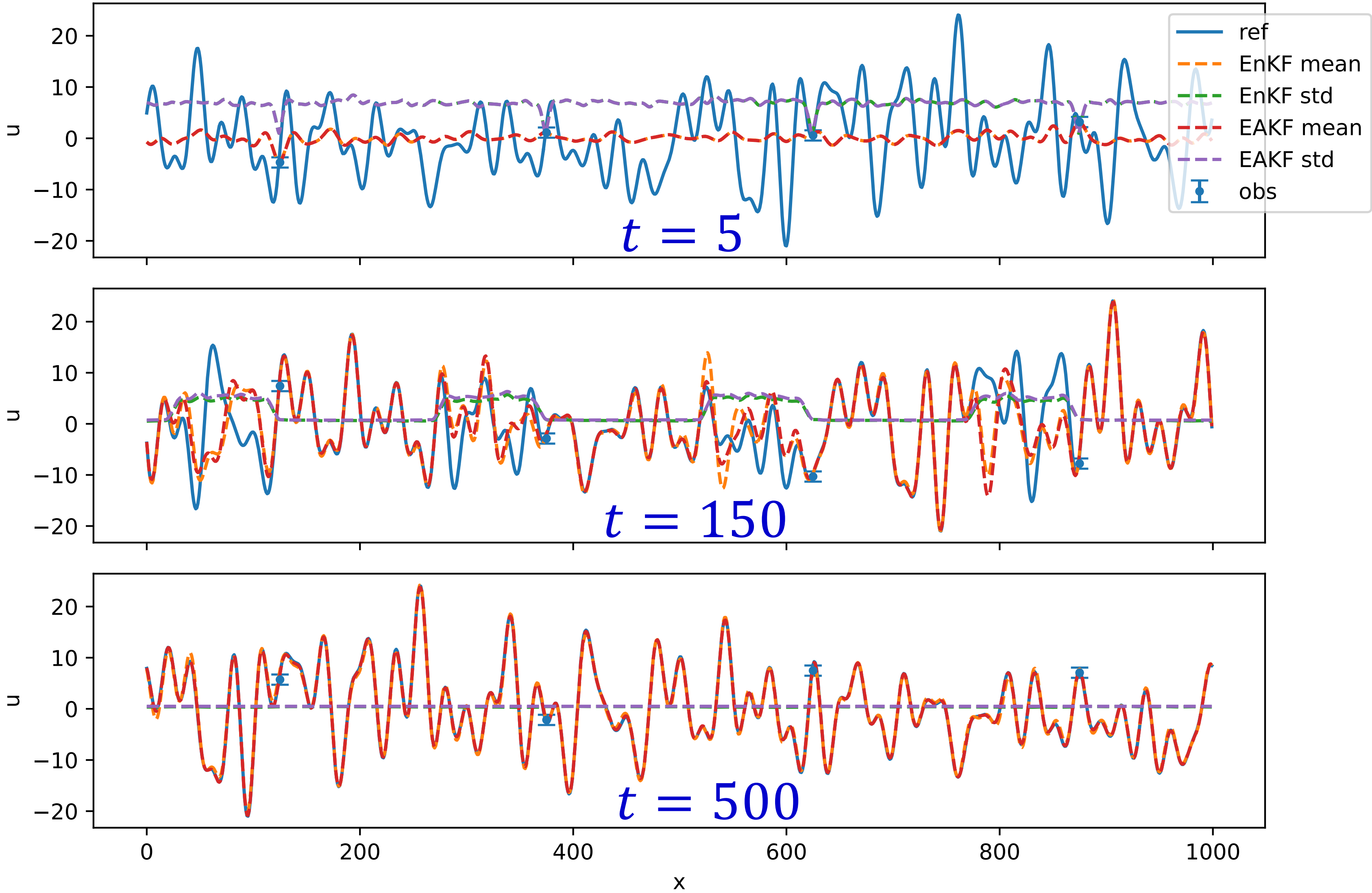
$$u(t,x)=u_0(x-t)，\quad 离散\ \Delta x=1$$

观测是 $u(t,125)$，$u(t,375)$，$u(t,625)$，$u(t,875)$

$t=5,10,15\ldots\ldots$ 观测误差是$\mathcal{N}(0,1)$

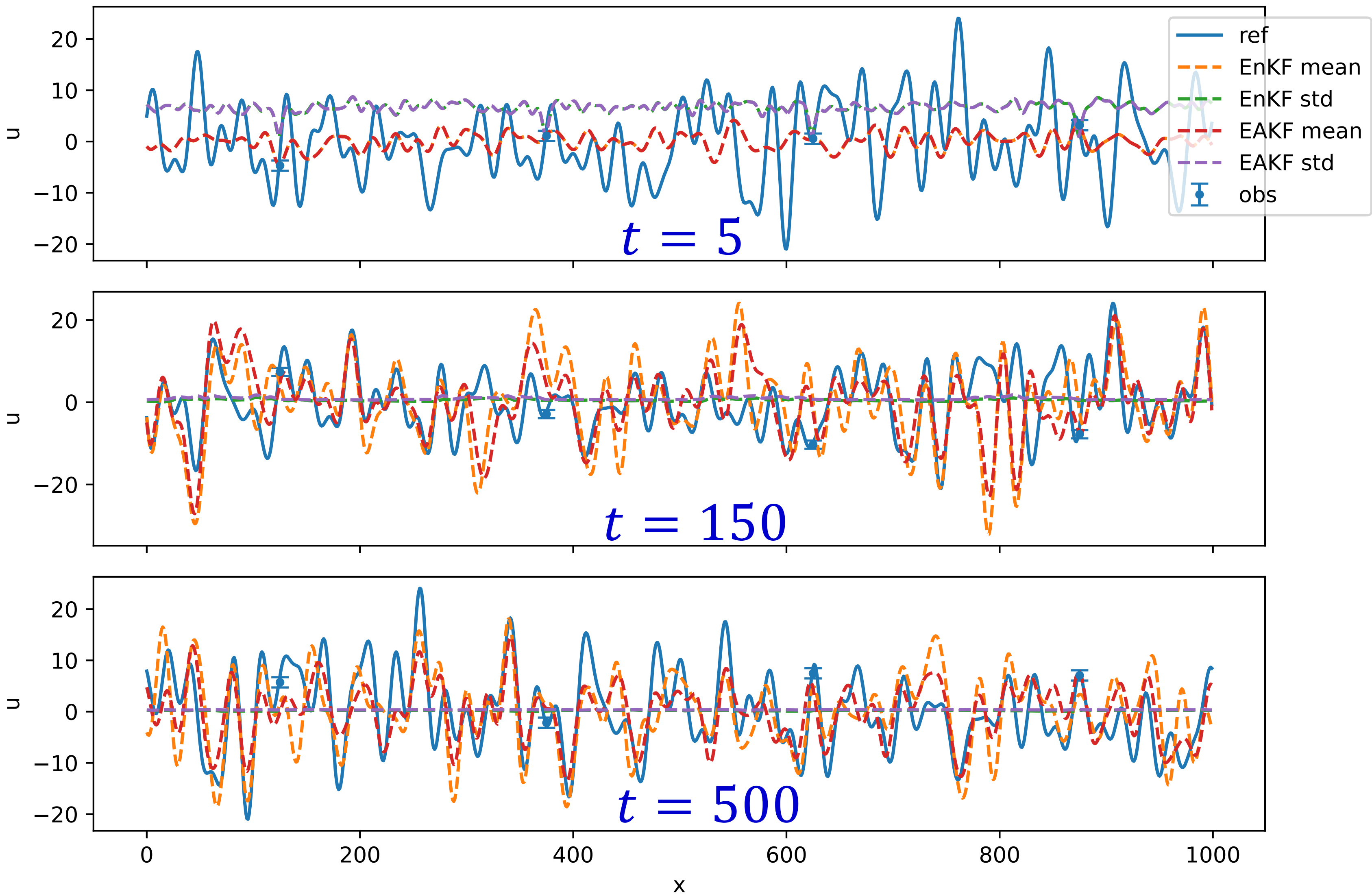
对流问题例子

$$J = 200: \ u(0,x) = \sum_{i=1}^{50} a_i \sin\frac{2\pi i x}{L} + b_i \cos\frac{2\pi i x}{L} \quad a_i, \ b_i \sim \mathcal{N}(0,1)$$



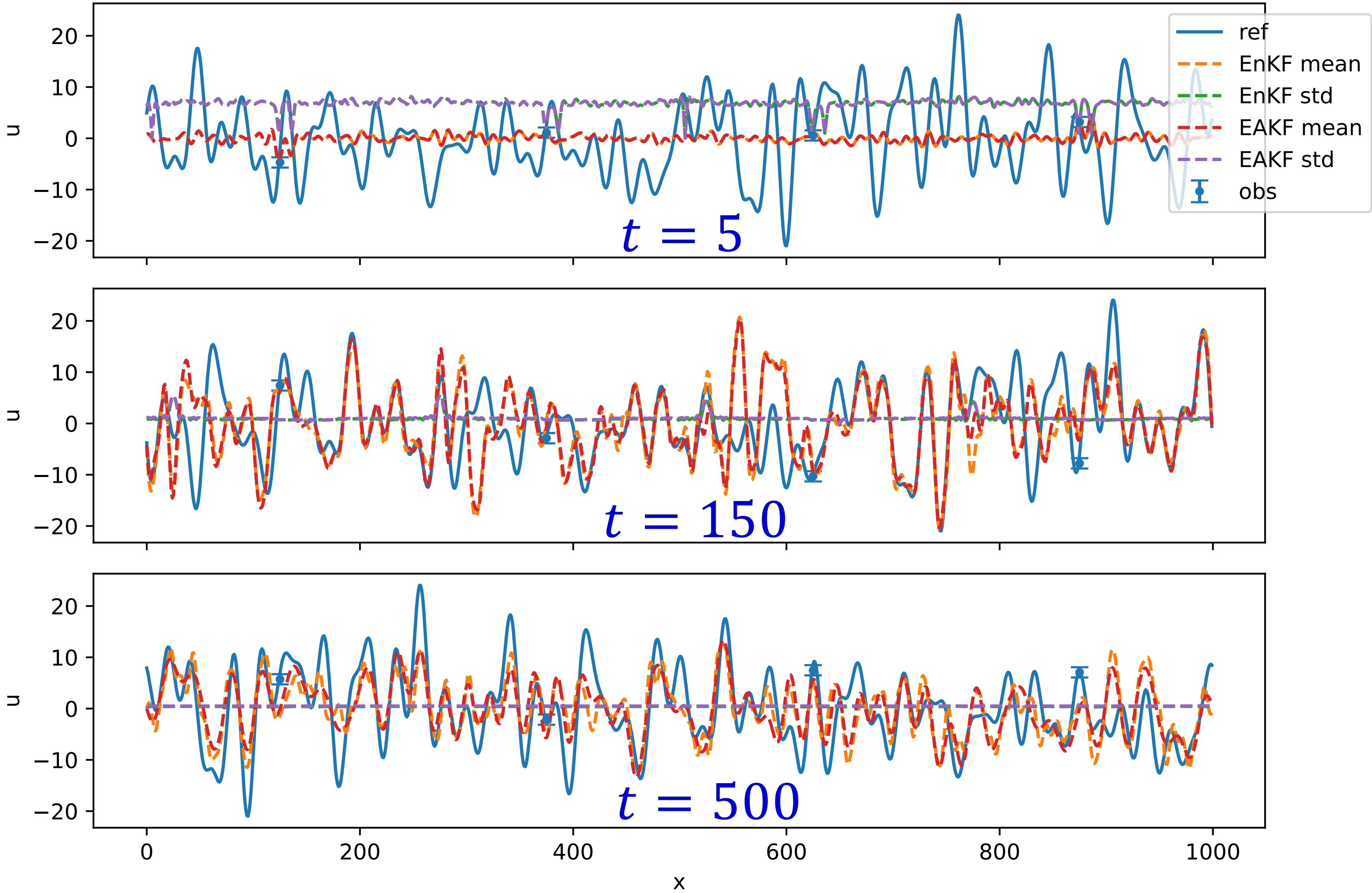
对流问题例子

$$J = 50: \ u(0,x) = \sum_{i=1}^{50} a_i \sin\frac{2\pi i x}{L} + b_i \cos\frac{2\pi i x}{L} \quad a_i, \ b_i \sim \mathcal{N}(0,1)$$



对流问题例子

$J = 200:\ u(0,x) = \sum_{i=1}^{100} a_i \sin\frac{2\pi i x}{L} \qquad a_i \sim \mathcal{N}(0,1)$



平方根类卡尔曼滤波

- 卡尔曼滤波

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1})$$
$$C_{n+1} = \hat{C}_{n+1} - \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{xy^T}_{n+1}$$

- 平方根类卡尔曼滤波(Square Root Kalman Filter)

$$m_{n+1} = \widehat{m}_{n+1} + \ \hat{Z}_{n+1}\hat{Y}^T_{n+1}(\hat{Y}_{n+1}\hat{Y}^T_{n+1} + \Sigma_\eta)^{-1}(y_{n+1} - \hat{y}_{n+1})$$
$$Z_{n+1} = \hat{Z}_{n+1}X$$

子空间限制：

$(\widehat{m}_{n+1}\ \hat{Z}_{n+1})$和$(m_{n+1}, Z_{n+1})$有相同的列向量空间。

平方根类卡尔曼滤波

- 伪相关(Spurious correlation) 和方差减小(Variance reduction)

演化方程： $\begin{bmatrix} x_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ b_n \end{bmatrix} \qquad b_0 \sim \mathcal{N}(0, I)$

观测方程： $y_{n+1} = \mathcal{H}(x_{n+1}) + \eta_{n+1}$

分析

$$\begin{bmatrix} Z_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} \hat{Z}_{n+1} \\ \hat{B}_{n+1} \end{bmatrix} X$$

$$C_{n+1} = \hat{C}_{n+1} - \begin{bmatrix} \hat{Z}_{n+1} \\ \hat{B}_{n+1} \end{bmatrix} \hat{Y}_{n+1}^T (\hat{Y}_{n+1} \hat{Y}_{n+1}^T + \Sigma_\eta)^{-1} \hat{Y}_{n+1} \begin{bmatrix} \hat{Z}_{n+1} \\ \hat{B}_{n+1} \end{bmatrix}^\mathrm{T}$$

会怎么变化？

平方根类卡尔曼滤波

- 协方差膨胀(Covariance inflation)

$$\hat{x}_{n+1}^{j} \to \rho\left(\hat{x}_{n+1}^{j} - \hat{m}_{n+1}\right) + \hat{m}_{n+1}$$

- 自适应的方差增长(Adaptive variance inflation)

$$\begin{bmatrix} Z_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} \hat{Z}_{n+1} \\ \hat{B}_{n+1} \end{bmatrix} X$$

根据 B_{n+1} 来选择ρ

平方根类卡尔曼滤波

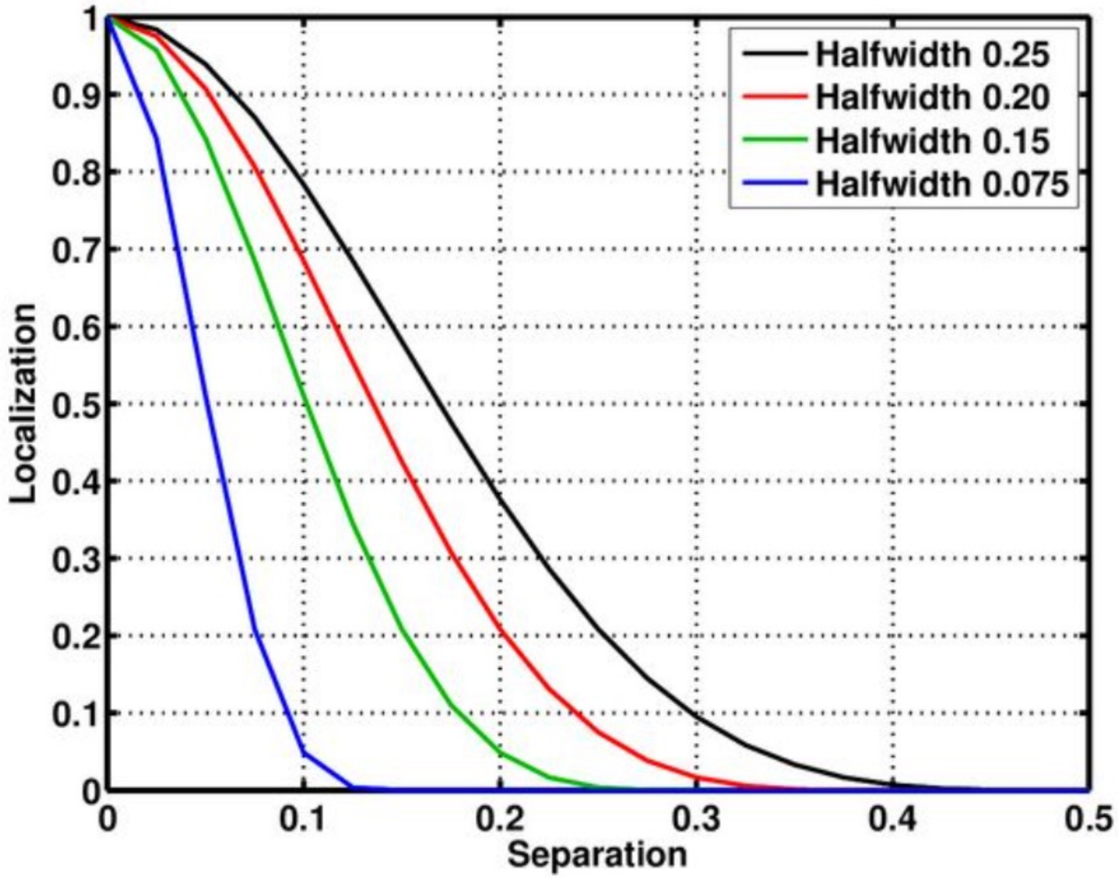
- 协方差局地化(Covariance localization)

$$\hat{C}_{n+1} \rightarrow \rho \odot \hat{C}_{n+1}$$

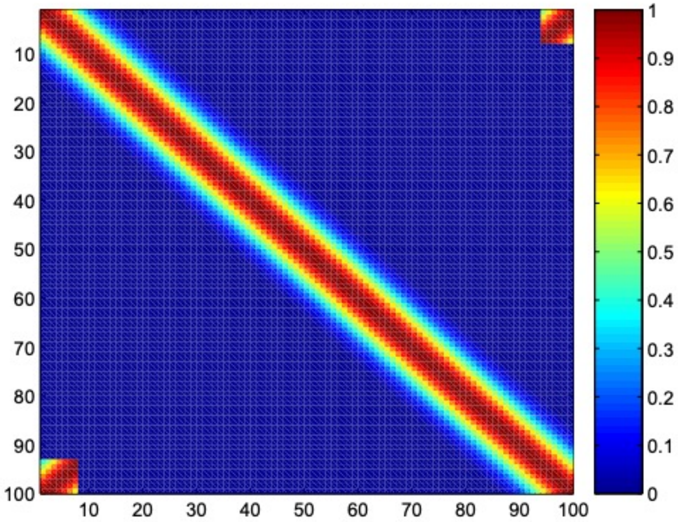
$$\rho[i,j] = K(\| x_i - x_j \|)$$

$$\hat{C}_{n+1}[i,j] \rightarrow \rho[i,j]\, \hat{C}_{n+1}\,[i,j]$$

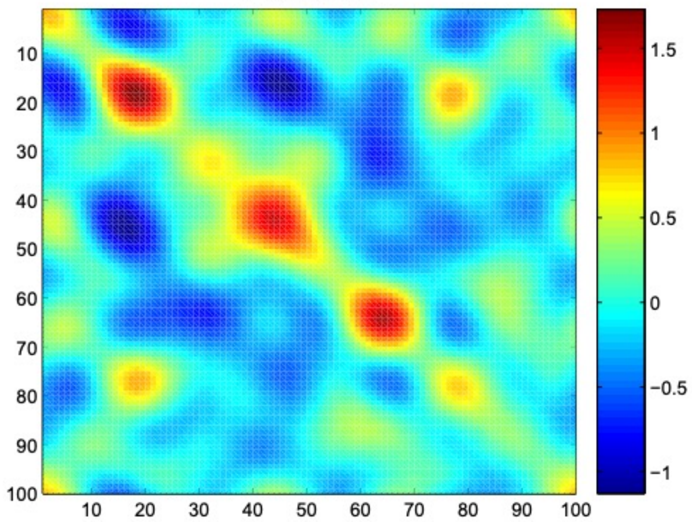
gaspari-cohn局地化函数



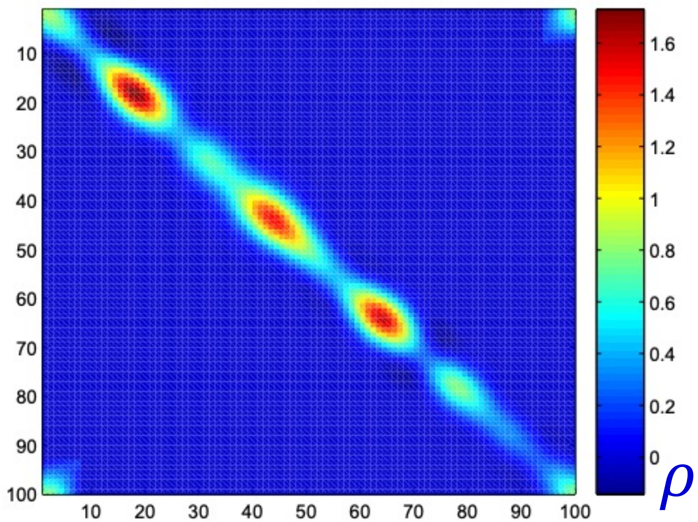
- 协方差局地化(Covariance localization)



ρ



$\hat{C}_{n+1}$

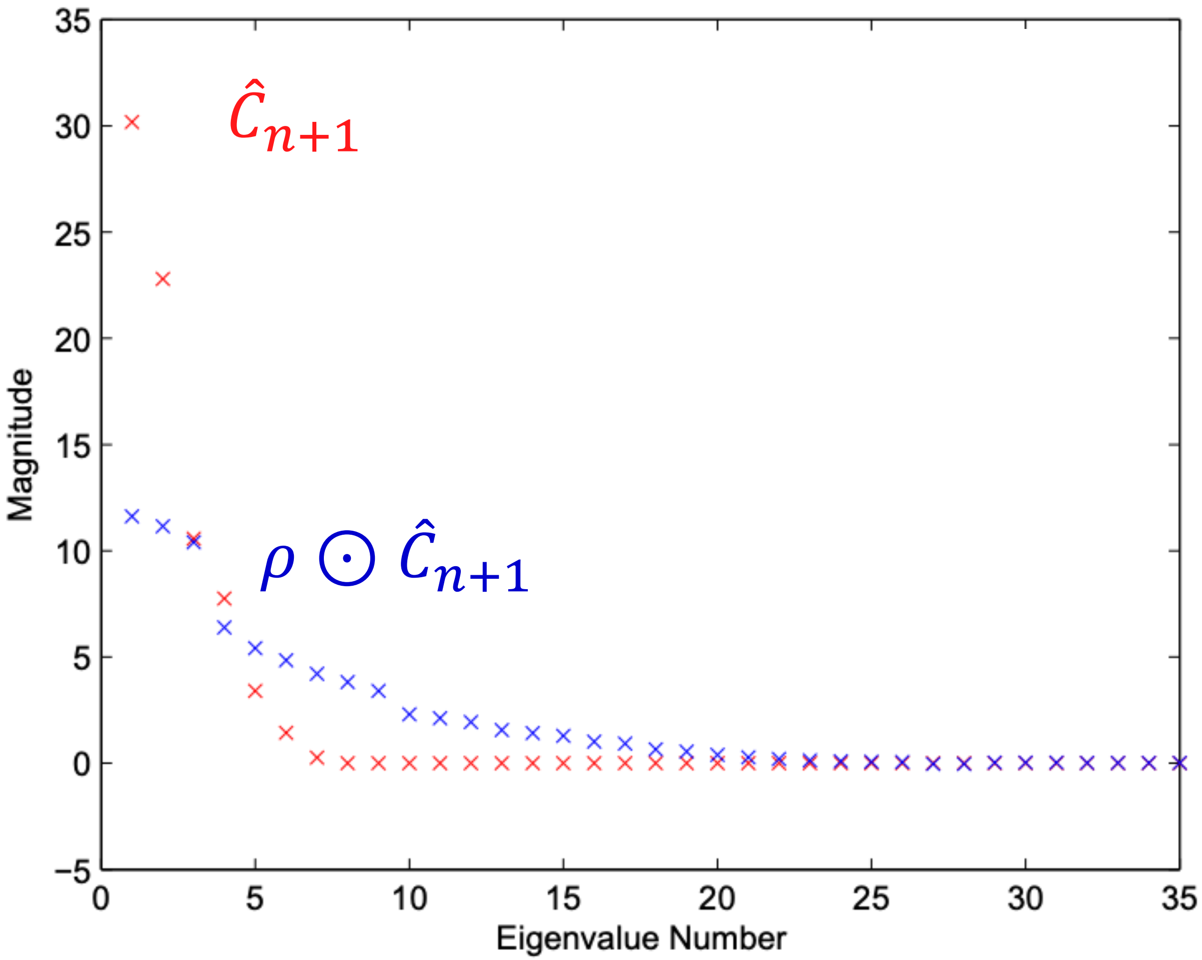


$\rho \odot \hat{C}_{n+1}$

平方根类卡尔曼滤波

➢ 协方差局地化(Covariance localization)

最大特征值下降，最小特征值上升，谱被压缩。



平方根类卡尔曼滤波

➤协方差局地化(Covariance localization)

$$m_{n+1} = \widehat{m}_{n+1} + \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1})$$
$$C_{n+1} = \hat{C}_{n+1} - \hat{C}^{xy}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{xy^T}_{n+1}$$

当 $\mathcal{H}(x) = Hx$

$$\hat{C}^{xy}_{n+1} \to \rho \odot \hat{C}_{n+1}H^T$$
$$\hat{C}^{yy}_{n+1} \to H(\rho \odot \hat{C}^{yy}_{n+1})H^T + \Sigma_\eta$$

采用一些特殊的近似

$$\rho = \check{\rho}\check{\rho}$$
$$\hat{Z}_{n+1} \to \check{\rho} \odot \hat{Z}_{n+1}$$

减小伪相关
打破子空间限制

卡尔曼滤波

- 集合卡尔曼滤波及其变体
  - 集合变换卡尔曼滤波
  - 集合调整卡尔曼滤波

高维协方差矩阵的低秩近似

- 集合卡尔曼滤波的问题（低秩近似）与改进
  - 子空间限制
  - 伪相关和方差减小
  - 方差膨胀
  - 协方差局地化

扩展阅读

- 卡尔曼滤波

EnKF书籍: Evensen, Geir. "The ensemble Kalman filter: Theoretical formulation and practical implementation." *Ocean dynamics* 53 (2003): 343-367.

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扩展阅读

- 卡尔曼滤波的改进

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