基于投影与斜投影

① 投影矩阵 $\Pi^2=\Pi$

$\text{range}\,\Pi = \{\Pi y\}$

$\ker \Pi = \{x : \Pi x = 0\}$

$x = \Pi x + (x - \Pi x)$

$x \in \text{Range}\,\Pi \cap \ker \Pi$

$\Pi x = 0 \qquad x = \Pi y = \Pi^2 y = \Pi x = 0$

$\text{Range}(\Pi) \oplus \ker(\Pi) = R^N$

$\Downarrow$ 特征值为 1 的特征向量

$\Pi(\Pi y) = \Pi(y)$

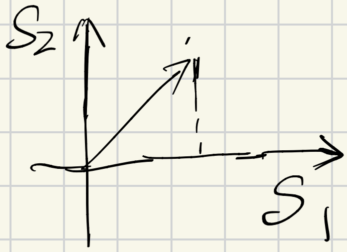
$\Downarrow$ 特征值为 0 的特征向量

$$\Pi = P \begin{bmatrix} I_m & \\ & O_{N-m} \end{bmatrix} P^{-1}$$

$$\Pi P = P \begin{bmatrix} I_m & \\ & O_{N-m} \end{bmatrix}$$

$\Pi P_1 = P_1 \Rightarrow S_1$

$\Pi P_2 = 0 \Rightarrow S_2$



② 三维螺旋

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Pi_{VV^T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

斜投影

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Pi_{V,W} = V(W^T V)^{-1} W^T$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Pi_{V,W} \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} = \begin{bmatrix} \cos t + t \\ \sin t + t \\ 0 \end{bmatrix}$$

③ 不变性

$W, V \in R^{N\times k}$

变化基底

$\tilde{W} = WQ_1 \quad \tilde{V} = VQ_2 \qquad Q_i \in R^{k\times k}$ 可逆矩阵

$$(\tilde{W}^T\tilde{V})^{-1}\tilde{W}^T = (Q_1^T W^T V Q_2)^{-1} Q_1^T W^T$$

$$= Q_2^{-1}(W^T V)^{-1} W^T$$

$$\tilde{q}(0) = Q_2^{-1}(W^T V)^{-1} W^T u_0 = Q_2^{-1} q(0)$$

$$\frac{d\tilde{q}(t)}{dt} = (\tilde{W}^T\tilde{V})^{-1}\tilde{W}^T f(\tilde{V}\tilde{q}(t), t, x)$$

$$= Q_2^{-1}(W^T V)^{-1} W^T f(VQ_2\tilde{q}(t), t, x)$$

$$\frac{d\,Q\tilde{q}(t)}{dt} = (W^T V)^{-1} W^T f(VQ_2\tilde{q}(t), t, x)$$

$$\Rightarrow \tilde{q}(t) = Q_2^{-1} q(t)$$

$$\Rightarrow \tilde{u}(t) = \tilde{V}\tilde{q}(t) = VQ_2\tilde{q}(t) = Vq(t) = u(t)$$

④ 误差分析

HDM:

$$\frac{du(t)}{dt} = A\,u(t) \qquad u(t) \in R^N$$

PROM:

$$u(t) = V\,q(t) \qquad V \in R^{N\times k} \qquad V^TV = I$$

$$\frac{dq(t)}{dt} = V^TAVq(t) \qquad q(0) = V^T u(0)$$

误差:

$$\varepsilon_V = VV^T u(t) - V q(t) \qquad \varepsilon_{V\perp} = u - VV^T u$$

$$[V\ V_c] \Rightarrow I = [V\ V_c][V\ V_c]^T = VV^T + V_cV_c^T$$

$$VV^T\varepsilon_V = \varepsilon_V \Rightarrow \|\varepsilon_V\|_2 = \|V^T\varepsilon_V\|_2$$

$$V_cV_c^T\varepsilon_{V\perp} = V_cV_c^T u = \varepsilon_{V\perp} \qquad \|\varepsilon_{V\perp}\|_2 = \|V_c^T\varepsilon_{V\perp}\|_2$$

$$\frac{d\varepsilon_V}{dt} = VV^TAu - VV^TAVq(t)$$

$$= VV^TA(u - Vq(t))$$

$$= VV^T(A\varepsilon_V) + \underline{VV^T(A\varepsilon_{V\perp})} \quad \text{外力}$$

ODE:

$$\dot{x}(t) = Ax + u \qquad x(0) = 0$$

$$\Rightarrow x(t) = \int_0^t e^{A(t-\tau)} u(\tau)\, d\tau$$

$$\|x\|_2 \le \|F(T; A)\| \, \|u\|_2$$

$$\frac{d V^T \varepsilon_v}{dt} = (V^T A V) V^T \varepsilon_v + (V^T A V_c) V_c^T \varepsilon_{v\perp}$$

$$\|\varepsilon_v\|_2 \le \|F(T, V^T A V)\|_2 \, \|V^T A V_c\|_2 \, \|\varepsilon_{v\perp}\|_2$$

⑤稳定性

$$\frac{du}{dt} = Au \qquad u = e^{At} u_0$$

$$A = \begin{bmatrix} 1 & -3.5 \\ 0.6 & -2 \end{bmatrix} \qquad \lambda(A) = \{-0.1127 \quad -0.8873\}$$

$$V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V^T A V = [1]$$

对于 A 是对称矩阵的情形

A 可对角化，特征值为实数，满足 $\lambda(A) \le 0$

$\lambda(A) \le 0$

$\Rightarrow \quad v^T A v \le 0 \quad \forall v$

因为 $A = Q^T \Sigma Q$

$v^T Q^T \Sigma Q v = \sum \lambda_i u_i^2 \le 0$

其中 $u = Qv$

$\Rightarrow u^T V^T A V u \le 0 \quad \forall u$

$\Rightarrow \quad V^T A V u = \lambda u$ 那么 $\lambda \le 0$

因为 $u^T V^T A V u = \lambda u^T u \le 0$