基于平衡截断法的降阶模型

① 输入输出映射

Causality time-invariant

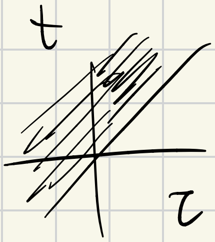
$$h(t,\tau)=0 \quad \forall \tau>t$$

$$h(t-\tau)=0 \quad \forall \tau>t \qquad h(-\tau)=0 \quad \forall\ \tau>0$$

$$y(t)=\int_{-\infty}^{t} h(t-\tau)\, w(\tau)\, d\tau$$

$$h = h_0\delta(t) + h_1 + h_2\frac{t}{1} + h_3\frac{t^2}{2!} + h_4\frac{t^3}{3!} \cdots\cdots$$

$$\mathcal{L}y(\xi) = \int_0^{\infty} y(t)\, e^{-\xi t}\, dt$$



$$= \int_{-\infty}^{t} w(\tau) \int_{\tau}^{\infty} h(t-\tau)\, e^{-\xi t}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} w(\tau)\, e^{-\xi\tau}\, d\tau \int_{\tau}^{\infty} h(t-\tau)\, e^{-\xi(t-\tau)}\, dt$$

$$= \int_{-\infty}^{\infty} w(\tau)\, e^{-\xi\tau}\, d\tau \int_0^{\infty} h(t)\, e^{-\xi t}\, dt$$

$$\mathcal{L}h(\xi) = h_0 + \sum_{i=1} h_i \int_0^{\infty} \frac{t^{i-1}}{(i-1)!}\, e^{-t\xi}\, dt$$

$$= h_0 + \sum_{i=1} h_i\, \xi^{-i}$$

② 线性动力系统

$$h(t) = \begin{cases} C e^{At} B + \delta(t) D & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$H(\xi) = \int_0^\infty h(t)\, e^{-t\xi}\, dt$$

$$= \int_0^\infty \left[C e^{At} B + \delta(t) D \right] e^{-t\xi}\, dt$$

$$= D + \int_0^\infty C\, e^{(A-\xi I)t} B\, dt$$

$$= D + \underline{C \int_0^\infty e^{-(\xi I - A)t}\, dt\, B} \quad \text{Formal}$$

$$= D + C(\xi I - A)^{-1} B$$

$$= D + C\left(I - \frac{A}{\xi}\right)^{-1} B \frac{1}{\xi}$$

$$= D + C\left(I + \frac{A}{\xi} + \frac{A^2}{\xi^2} \cdots\right) B \frac{1}{\xi}$$

$$= D + C\left(\frac{I}{\xi} + \frac{A}{\xi^2} + \frac{A^2}{\xi^3} \cdots\right) B$$

坐标变化 $\tilde{u} = Tu$

$$\frac{d}{dt}\tilde{u} = TAu + TBw = TAT^{-1}\tilde{u} + TBw$$

$$y = Cu + Dw = CT^{-1}\tilde{u} + Dw$$

$$\tilde{u}(\infty) = 0$$

$$\begin{pmatrix}\tilde{A} & \tilde{B}\\ \tilde{C} & \tilde{D}\end{pmatrix} \Rightarrow \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D\end{pmatrix}$$

$\tilde{D} = D$ $\quad \tilde{C}\tilde{B} = CB$ $\quad \tilde{C}\tilde{A}^k\tilde{B} = CT^{-1}(TAT^{-1})^k TB$

$\qquad = CA^kB$

③ 可达状态

$$u(t, w; 0, 0) = \int_0^t e^{A(t-\tau)} B w(\tau) d\tau$$

若 $u(k+1) = A u(k) + Bw(k)$

$= [B \ AB \cdots A^kB]\begin{bmatrix} w(k)\\ \vdots \\ w(0)\end{bmatrix}$

$(u(0) = 0)$

i) $\text{Range } P(t) = \text{Range } R(A, B)$

$q \perp P(t) \iff q \perp R(A, B)$

$P(t)$ 对称且半正定

$q \perp P(t) \iff q^* P(t) q = 0$

$\iff \int_0^t \| B^* e^{A^*\tau} q \|^2 d\tau = 0$

$\iff B^* e^{A^*\tau} q = 0 \quad \forall\ \tau < t$

因为指数函数在0附近解析

$\iff B^* e^{A^*\tau} q$ 和它的导数在0附近都为0

$\iff B^* (A^*)^{i-1} q = 0 \quad i > 0$

$$\Longleftrightarrow q \perp R(A,B)$$

ii) X^{reach} 是线性空间

$$u_1 = u(\bar{T}_1, w_1, 0, 0)$$

$$u_2 = u(\bar{T}_2, w_2, 0, 0) \qquad \bar{T}_1 \geq \bar{T}_2$$

$$\hat{w}_2 = \begin{cases} 0 & t \in [0, \bar{T}_1 - \bar{T}_2] \\ w_2(t - \bar{T}_1 + \bar{T}_2) & t \in [\bar{T}_1 - \bar{T}_2, \bar{T}_1] \end{cases}$$

$$u(\bar{T}_1, \alpha_1 w_1 + \alpha_2 \hat{w}_2, 0, 0)$$
$$= \alpha_1 \int_0^{\bar{T}_1} e^{A(t-\tau)} B w_1(\tau) d\tau + \alpha_2 \int_0^{\bar{T}_1} e^{A(t-\tau)} B \hat{w}_2(\tau) d\tau$$
$$= \alpha_1 u_1 + \alpha_2 \int_{\bar{T}_1 - \bar{T}_2}^{\bar{T}_1} e^{A(t-\tau)} B w_2(\tau - \bar{T}_1 + \bar{T}_2) d\tau$$
$$= \alpha_1 u_1 + \alpha_2 u_2$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} \cdots\cdots$$

iii) $X^{reach} \subseteq \text{Range } R(A,B)$

$$u(t, w; 0, 0) = \int_0^t e^{A(t-\tau)} B w(\tau) d\tau$$
$$= \sum_{i>0} A^{i-1} B \int_0^t \frac{(t-\tau)^{i-1}}{(i-1)!} w(\tau) d\tau$$
$$\subseteq \text{Range } R(A,B)$$

iv) $\text{Range}\, R(A,B) \subseteq X^{reach}$ (可达，任意时间可达)

$$\text{Range}\, R(A,B) = \text{Range}\, P(\bar{T}) \quad \forall \bar{T}$$

$$\forall \bar{u} \in \text{Range}\, R(A,B) \Longleftrightarrow \bar{u} \in \text{Range}\, P(\bar{T})$$

$$\bar{u} = P(\bar{T})\bar{\xi} = \int_0^{\bar{T}} e^{A\tau} B B^* e^{A^*\tau} \bar{\xi}\, d\tau$$

$$= \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B\, w(\tau)\, d\tau$$

其中 $w(t) = B^* e^{A^*(\bar{T}-t)} \bar{\xi}$

最小能量:

$$\bar{u} = u(\bar{T}) = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B\, w(\tau)\, d\tau$$

可以选取 $w(t) = B^* e^{A^*(\bar{T}-t)} \bar{\xi}$

其中 $\bar{u} = P(\bar{T})\bar{\xi}$

对于其它 $\hat{w}(t)$ 也能抵达 $\bar{u}$，我们有

$$0 = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B\, (w-\hat{w})\tau\, d\tau$$

$$\Longrightarrow \int_0^{\bar{T}} w(\tau)^* (w-\hat{w})(\tau)\, d\tau = 0$$

$$\Rightarrow \|\hat{w}\|_2^2 = \|\hat{w} - w + w\|_2^2$$
$$= \|\hat{w} - w\|_2^2 + \|w\|_2^2$$
$$\geq \|w\|_2^2 = \bar{\xi}^* P(\bar{T})\, \bar{\xi}$$
$$= \bar{u}^* P(T)^{-1} \bar{u}$$

④ 可控状态

$\bar{u} \in X^{contr}$

$$0 = u(\bar{T}, w, 0, \bar{u})$$
$$\Leftrightarrow 0 = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B w(\tau)\, d\tau + e^{A\bar{T}} \bar{u}$$
$$\Leftrightarrow \bar{u} = -e^{-A\bar{T}} \underbrace{\int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B w(\tau)\, d\tau}_{\in X^{reach}}$$

因为 $X^{reach} = \mathrm{Range}\, R(A, B) = \mathrm{Range}\{A^{i-1} B\}$

$e^{-A\bar{T}} X^{reach} \subset X^{reach} \Rightarrow \bar{u} \in X^{reach}$

若 $\bar{u} \in X^{reach}$, $-e^{A\bar{T}} \bar{u} \in X^{reach}$

$$-e^{A\bar{T}} \bar{u} = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B\, w(\tau)\, d\tau$$ 选w到达

$\bar{u} \in X^{contr}$

⑤ 可观测状态

若 $u(k+1) = A u(k)$

$y(k+1) = C u(k+1)$

$y = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix} u_0$

i) $\bar{u}_1, \bar{u}_2 \in X^{unobs}$

$$0 = C e^{At} \bar{u}_1 \qquad 0 = C e^{At} \bar{u}_2$$

$$\Rightarrow 0 = C e^{At} (\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2)$$

$$\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 \in X^{unobs}$$

ii) $\bar{u} \in X^{unobs} \Leftrightarrow Ce^{At} \bar{u} = 0 \quad \forall t$

由于 e^{At} 在 $t=0$ 是解析的

$$\Leftrightarrow \bar{u} \in \ker O(A, C)$$

iii) $\bar{u} \in \ker Q(t)$

$$\Leftrightarrow \bar{u}^* \int_0^t e^{A^* \tau} C^* C e^{A\tau} d\tau \, \bar{u} = 0$$

$$\Leftrightarrow \int_0^t \| C e^{A\tau} \bar{u} \|^2 d\tau = 0$$

$$\Leftrightarrow \bar{u} \in \ker O(A, C)$$

iv) $y(t) = Ce^{At} \bar{u}$

$$\|y\|^2 = \int_0^T \| C e^{At} \bar{u} \|^2 dt = \bar{u}^* Q(T) \bar{u}$$

⑥ 可达性可观测性规范分解

$$\tilde{u} = Tu \quad \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{pmatrix}$$

$\boxed{\text{空间是不变的}}$

可达 Range $R(A\ B) = \text{Range}(B,\ AB,\ A^2B \cdots)$

不可观测 $\ker O(A,C) = \ker(C, CA\ CA^2 \cdots)$

i) 可达性：

$$X = X^{reach} + X' \qquad \dim X^{reach} = m,\ \dim X' = N-m$$

$x_1 x_2 \cdots x_m$ 是 X^{reach} 的基底，$x_{m+1}\ x_{m+2} \cdots x_N$ 是 X' 的基底。

$$\tilde{A} X^{reach} \subset X^{reach} \Rightarrow \tilde{A} = \begin{pmatrix} \underset{\text{可达的}}{A_r} & A_{r\bar{r}} \\ 0 & \underset{\text{不可达的}}{A_{\bar{r}}} \end{pmatrix}$$

$$\text{im}\ \tilde{B} \subset X^{reach} \Rightarrow \tilde{B} = \begin{pmatrix} B_r \\ 0 \end{pmatrix}$$

$$\text{Rank } R(A_r, B_r) = \text{Rank } R(\tilde{A}\ \tilde{B}) = \text{Rank } R(A, B)$$

$$= \text{Rank } X^{reach} = m$$

ii) 可观测性：

$$X = X^{unobs} + X' \qquad \dim X^{unobs} = m \qquad \dim X' = N-m$$

$x_1, x_2 \cdots x_m$ 是 X^{unobs} 的基底，$x_{m+1}\ x_{m+2} \cdots x_N$ 是 X' 的基底。

$$\tilde{A} X^{unobs} \subset X^{unobs} \Rightarrow \tilde{A} = \begin{pmatrix} \underset{\text{不可观测}}{A_{\bar{o}}} & A_{\bar{o}o} \\ 0 & \underset{\text{可观测}}{A_o} \end{pmatrix}$$

$$\ker \tilde{C} \subset X^{unobs} \Rightarrow \tilde{C} = \begin{pmatrix} 0 & C_o \end{pmatrix}$$

$$m = \operatorname{Rank} \ker \begin{pmatrix} \tilde{A} \\ \tilde{C} \end{pmatrix} = \operatorname{Rank} \ker \begin{pmatrix} A_o \\ C_o \end{pmatrix} + m$$

$$\Rightarrow \ker \begin{pmatrix} A_o \\ C_o \end{pmatrix} = 0$$

iii) 可达性可观测性规范分解

$$X^{reach} = X^{reach} \cap X^{unobs} + X^{reach'}$$

$$X^{unobs} = X^{reach} \cap X^{unobs} + X^{unobs'}$$

$$X = X^{reach} \cap X^{unobs} + X^{reach'} + X^{unobs'} + X'$$

$$\Rightarrow \left(\begin{array}{cccc|c} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} & B_{r\bar{o}} \\ 0 & A_{ro} & 0 & A_{24} & B_{ro} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} & 0 \\ 0 & 0 & 0 & A_{\bar{r}o} & 0 \\ \hline 0 & C_{ro} & 0 & C_{\bar{r}o} & \end{array}\right) = \left(\begin{array}{cc|c} A_r & A_{r\bar{r}} & B_r \\ 0 & A_{\bar{r}} & 0 \\ \hline & & \end{array}\right)$$

$$\left(\begin{array}{cccc|c} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} & B_{r\bar{o}} \\ 0 & A_{ro} & 0 & A_{24} & B_{ro} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} & 0 \\ 0 & 0 & 0 & A_{\bar{r}o} & 0 \\ \hline 0 & C_{ro} & 0 & C_{\bar{r}o} & D \end{array}\right) = \left(\begin{array}{cc|c} A_{\bar{o}} & A_{\bar{o}o} & \\ 0 & A_o & \\ \hline 0 & C_o & \end{array}\right)$$

⑦ 平衡截断法

$$AP + PA^*$$
$$= \int_0^\infty A e^{A\tau} BB^* e^{A^*\tau} + e^{A\tau} BB^* e^{A^*\tau} A^* \, d\tau$$
$$= \int_0^\infty \frac{d}{d\tau} \left(e^{A\tau} BB^* e^{A^*\tau} \right)$$
$$= -BB^*$$

$$P = LL^* \qquad L^* Q L = K \Sigma^2 K^*$$

$$T_{bal} = \Sigma^{1/2} K^* L^{-1} \qquad T_{bal}^{-1} = L K \Sigma^{-1/2}$$

$$\tilde{P} = T_{bal} P T_{bal}^* = \Sigma^{1/2} K^* L^{-1} L L^T L^{-*} K \Sigma^{1/2 *}$$
$$= \Sigma$$

$$\tilde{Q} = T_{bal}^{-*} Q T_{bal}^{-1} = \Sigma^{-1/2} K^* L^* Q L K \Sigma^{-1/2}$$
$$= \Sigma^{-1/2} \Sigma^2 \Sigma^{-1/2}$$
$$= \Sigma$$

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} T_{bal} A T_{bal}^{-1} & T_{bal} B \\ C T_{bal}^{-1} & D \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{D} \end{pmatrix}$$

$\tilde{u} = T_{bal} u$

$$\frac{d\tilde{u}}{dt} = \tilde{A} \tilde{u} + \tilde{B} w \qquad \tilde{u}_0 = T_{bal} u_0$$
$$y = \tilde{C} \tilde{u} + D w$$

$\tilde{P} = \tilde{Q} = \Sigma = \mathrm{diag}\{\sigma_1\ \sigma_2 \cdots\ \sigma_N\}$

最小能量可达 : $\dfrac{\tilde{u}^T \tilde{P} \tilde{u}}{\tilde{u}^T \tilde{u}}$ 最大 $e_1, e_2 \cdots\ e_N$

最大观察能量 : $\dfrac{\tilde{u}^T \tilde{Q} \tilde{u}}{\tilde{u}^T \tilde{u}}$ 最大 $e_1\ e_2 \cdots\ e_N$

我们考虑子空间 $\{e_1\ e_2 \cdots e_k\}$ $V = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$ $\tilde{u} = V u_r$

$$\frac{du_r}{dt} = \tilde{A}_{11} u_r + \tilde{B}_1 w$$

$$y = \tilde{C}_1 u_r + \tilde{D} w$$

$u = T_{bal}^{-1} \tilde{u} = T_{bal}^{-1} V u_r$

$= T_{bal}^{-1} [:, 1:k]\ u_r$

Petrov-Galerkin $V = T_{bal}^{-1} [:, 1:k]$

$\tilde{A}_{11} = T_{bal} [1:k, :]\ A\ T_{bal}^{-1} [:, 1:k]$

petrov-Galerkin

$$= \begin{pmatrix} (W^T V)^{-1} W^T A V & (W^T V)^{-1} W^T B \\ CV & D \end{pmatrix}$$

$V = T_{bal}^{-1} [:, 1:k]$

$(W^T V)^{-1} W^T = T_{bal} [1:k, :]$

$W = T_{bal}^{*} [:, 1:k]$ $W^T V = I$

⑧ 平衡截断法的误差分析

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix} \qquad \begin{matrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{matrix}$$

$$A\Sigma + \Sigma A^* + BB^* = 0$$

$$A^*\Sigma + \Sigma A + C^*C = 0$$

$$\Rightarrow \begin{pmatrix} A_{11}\Sigma_1 & A_{12}\Sigma_2 \\ A_{21}\Sigma_1 & A_{22}\Sigma_2 \end{pmatrix} + \begin{pmatrix} \Sigma_1 A_{11}^* & \Sigma_1 A_{21}^* \\ \Sigma_2 A_{12}^* & \Sigma_2 A_{22}^* \end{pmatrix} + \begin{pmatrix} B_1 B_1^* & B_1 B_2^* \\ B_2 B_1^* & B_2 B_2^* \end{pmatrix} = 0$$

$$\Rightarrow \begin{matrix} A_{11}\Sigma_1 + \Sigma_1 A_{11}^* + B_1 B_1^* = 0 & A_{22}\Sigma_2 + \Sigma_2 A_{22}^* + B_2 B_2^* = 0 \\ A_{11}^*\Sigma_1 + \Sigma_1 A_{11} + C_1^* C_1 = 0 & A_{22}^*\Sigma_2 + \Sigma_2 A_{22} + C_2^* C_2 = 0 \end{matrix}$$

对输入：w，原动力系统的解：$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$，输出：$y$

降阶模型的解：$\overline{u}_1$　输出　$\overline{y}$

为了计算 $y-\bar{y}$，考虑动力系统

$$\Sigma_e = \left[\begin{array}{ccc|c} A_{11} & A_{12} & 0 & B_1 \\ A_{21} & A_{22} & 0 & B_2 \\ 0 & 0 & A_{11} & B_1 \\ \hline C_1 & C_2 & -C_1 & 0 \end{array}\right] \quad \text{解为} \begin{pmatrix} u_1 \\ u_2 \\ \bar{u}_1 \end{pmatrix}$$

假设 $\Sigma_2 = \gamma I$，

$$\frac{\int_0^\infty \|\Delta y\|^2}{\int_0^\infty \|w(t)\|^2} \le (2\gamma)^2$$

$$\iff \frac{\int_0^\infty \|\Delta y\|^2}{\int_0^\infty \|2\gamma w(t)\|^2} \le 1$$

$\Leftarrow$ 证明 $\left\|\frac{1}{2\gamma}\Sigma_e\right\|_{H_\infty} \le 1$

$$\frac{1}{2\gamma}\Sigma_e = \left(\begin{array}{ccc|c} A_{11} & A_{12} & 0 & B_1/2\gamma \\ A_{21} & A_{22} & 0 & B_2/2\gamma \\ 0 & 0 & A_{11} & B_1/2\gamma \\ \hline C_1 & C_2 & -C_1 & 0 \end{array}\right)$$

Lemma

$\Leftarrow$ 存在半正定矩阵 X

$$A^*X + XA + C^*C + XBB^*X = 0$$

$$\frac{d}{dt} u^*Xu = u^*A^*Xu + w^*B^*Xu + u^*XAu + u^*XBw \qquad (\dot{u} = Au + Bw)$$

$$= -u^*C^*Cu - u^*XBB^*Xu + w^*B^*Xu + w^*XBw$$

$$= w^*w - y^*y - \|w - B^*Xu\|^2$$

积分：$0 \le u^*(T) X u(T) \le \int_0^T w^*w - y^*y$

最后：

$$X = T^{-*} \underbrace{\begin{pmatrix} \Sigma_1 & & \\ & 2I\sigma & \\ & & \Sigma_1^{-1}\sigma^2 \end{pmatrix}}_{\tilde{X}} T^{-1}, \quad T = \begin{pmatrix} I & 0 & -I \\ 0 & I & 0 \\ I & 0 & I \end{pmatrix}$$

$$T^{-1} = \frac{1}{2}\begin{pmatrix} I & 0 & I \\ 0 & 2I & 0 \\ -I & 0 & I \end{pmatrix}$$

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ \frac{1}{2}A_{21} & A_{22} & \frac{1}{2}A_{21} \\ 0 & A_{12} & A_{11} \end{pmatrix}$$

$$\tilde{B} = TB = \begin{pmatrix} 0 \\ B_2/2\sigma \\ B_1/\sigma \end{pmatrix}$$

$$\tilde{C} = CT^{-1} = [C_1 \ \ C_2 \ \ 0]$$

$$\tilde{A}^* \tilde{X}^* + \tilde{X}^* \tilde{A} + \tilde{C}^* C + \tilde{X} \tilde{B} \tilde{B}^* \tilde{X} = 0$$