矩匹配方法

Petrov-Galerkin

$$\begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} = \begin{pmatrix} (W^TV)^{-1}W^TAV & (W^TV)^{-1}W^TB \\ CV & D \end{pmatrix}$$

① 不变性：

$W \Rightarrow WT_1 \quad V \Rightarrow VT_2 \quad , \quad T_1 , T_2 \in R^{k\times k}$

均为可逆矩阵。

$$\begin{pmatrix} \tilde{A}_r & \tilde{B}_r \\ \tilde{C}_r & \tilde{D}_r \end{pmatrix} = \begin{pmatrix} T_2^{-1}(W^TV)^{-1}W^TAVT_2 & T_2^{-1}(W^TV)W^TB \\ CVT_2 & D \end{pmatrix}$$

$$= \begin{pmatrix} T_2^{-1}A_rT_2 & T_2^{-1}B \\ C_rT_2 & D_r \end{pmatrix}$$

$$\begin{aligned} \tilde{H}(\xi) &= \tilde{C_r}(\xi I - \tilde{A_r})^{-1}\tilde{B_r} + \tilde{D_r} \\ &= C_rT_2(\xi I - T_2^{-1}A_rT_2)^{-1}T_2^{-1}B_r + D_r \\ &= C_r(\xi I - A_r)^{-1}B_r + C_r \\ &= H(\xi) \end{aligned}$$

考虑 $\begin{pmatrix} A & b \\ c & d \end{pmatrix} =$

② 矩匹配定理

$$\begin{pmatrix} A_r & b_r \\ C_r & d_r \end{pmatrix} = \begin{pmatrix} (W^TV)^{-1}W^TAV & (W^TV)^{-1}W^Tb \\ cV & D \end{pmatrix}$$

$$= \begin{pmatrix} W^TAV & W^Tb \\ cV & D \end{pmatrix}$$

i) $cA^{m-1}b = C_rA_r^{m-1}b_r$

$C_rA_r^{m-1}b_r = cV(W^TAV)^{m-1}W^Tb$

$V = [b \; Ab \cdots A^{k-1}b]$

$AV = [Ab \; A^2b \cdots A^kb]$

因为 $W^T V = I$，定义 $\alpha_k = W^T A^k b$

$$W^T A V = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & & & \alpha_k \\ \vdots & & \ddots & & \\ & & & 0 & \\ & & & 1 & 0 \end{bmatrix} \in R^{k\times k} \qquad W^T b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(W^T A V)^{m-1} W^T b = e_m$$ 使用了 $(m \leq k-1)$

$$cV e_m = c A^{m-1} b$$

(ii) Arnoldi 方法

$$V_k \Leftrightarrow \text{span}\{b, Ab, A^2 b \cdots A^{k-1} b\}$$

$$v_{k+1} = A v_k - V_k (V_k^T A v_k)$$

$$V_{k+1} = \left[V_k ; \frac{v_{k+1}}{\|v_{k+1}\|}\right]$$

$$AV_j = V_j H_j + f_j e_j^T \qquad e_j \in R^j$$

$$AV_1 = V_1 H_1 + f_1 e_1^T \qquad H_1 = [\alpha_1]$$

$$f_1 = Av_1 - V_1(V_1^T A v_1)$$

$$V_{j+1} = [V_j \quad v_{j+1}] \qquad v_{j+1} = \frac{f_j}{\|f_j\|}$$

$$AV_{j+1} = [AV_j \quad A v_{j+1}]$$

$$= [V_j H_j + f_j e_j^T \quad A v_{j+1}] \qquad \boxed{0 \cdots 1}$$

$$= [V_j \quad v_{j+1}] \left[\begin{array}{c|c} H_j & h \\ \hline \|f_j\| e_j^T & \end{array}\right] + f_{j+1} e_{j+1}^T$$

$$h = (V_{j+1}^T A v_{j+1})$$

$$f_{j+1} = A v_{j+1} - V_{j+1}(V_{j+1}^T A v_{j+1})$$

$$V_{j+1} h = A v_{j+1}, \quad f_j e_j^T = v_{j+1} e_j^T \|f_j\|$$

矩匹配定理：

$$O_k^T = [C^T \ A^TC^T \ A^{T^2}C^T \ \cdots \ A^{T^{k-1}}C^T]$$

$$R_k = [b \ Ab \ A^2b \ \cdots \ A^{k-1}b]$$

$$H_k = O_k R_k = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{k-1} \\ h_1 & h_2 & h_3 & \cdots & h_k \\ \vdots & & & & \\ h_{k-1} & h_k & \cdots & & h_{2k-2} \end{pmatrix} = \begin{pmatrix} cb & cAb & \cdots & \\ \vdots & & & \\ & & & cA^{2k-2}b \end{pmatrix}$$

目标

$$O_k A R_k = \begin{pmatrix} h_1 & h_2 & \cdots & h_k \\ h_2 & h_3 & \cdots & h_{k+1} \\ \vdots & & & \\ h_k & h_{k+1} & \cdots & h_{2k-1} \end{pmatrix}$$

$H_k = LU$ （LU 分解）

$W^T = L^{-1}O_k \qquad V = R_k U^{-1}$

$W^T V = I \qquad V W^T = R_k U^{-1} L^{-1} O_k = R_k H_k^{-1} O_k$

$$\Rightarrow O_k V W^T = O_k$$

$$V W^T R_k = R_k$$

$A_r = W^T A V \qquad b_r = W^T b \qquad c_r = cV$

$$O_{rk}^T = [\,C_r^T \quad A_r^T C_r^T \quad A_r^{T^2} C_r^T \quad \cdots \quad A_r^{T^{k-1}} C_r\,]$$

$$R_{rk} = [\,b_r \quad A_r b_r \quad A_r^2 b_r \quad \cdots \quad A_r^{k-1} b_r\,]$$

$$O_{rk} = \begin{bmatrix} cV \\ cVW^TAV \\ cV(W^TAV)^2 \\ \vdots \\ cV(W^TAV)^{k-1} \end{bmatrix} = \begin{bmatrix} cV \\ cAV \\ \vdots \\ cA^{k-1}V \end{bmatrix} = O_k \cdot V$$

$$cA^{m-1}V(W^TAV) \quad (1 \le m \le k)$$

$$= cA^{m-1}(VW^T)AV = \underbrace{cA^{m-1}\,R_k}\,H_k^{-1}O_k\,AV$$

$$= cA^{m-1}AV$$

H_k的第 m 行 $\Rightarrow$ $e_m^T O_k AV$

$\Rightarrow$ $cA^{m-1}AV$

$$= cA^m V$$

$$R_{rk} = [\,b_r \quad A_r b_r \cdots A_r^{k-1} b_r\,]$$

$$= [\,W^Tb \quad W^TAVW^Tb \quad \cdots \quad (W^TAV)^{k-1}W^Tb\,]$$

$$= W^T R_k$$

$$(W^TAV)W^TA^{m-1}b = W^TA^m b \quad (1 \le m \le k)$$

目标

$$O_{rk} R_{rk} = O_k V W^T R_k = O_k R_k$$

$$O_{rk} A_r R_{rk} = O_k V W^T A V W^T R_k = O_k A R_k$$

$$\Rightarrow \quad h_{rm} = h_m \qquad 0 \le m \le 2k-1$$

双侧Lanczos方法

$V_m \Leftrightarrow \mathrm{span}\{b, Ab, \cdots A^{m-1}b\}$

$W_m \Leftrightarrow \mathrm{span}\{C^T, A^TC^T, \cdots A^{T^{m-1}}C^T\}$

$v_{m+1} = Av_m - \alpha_m v_m - \gamma_m v_{m-1}$

$w_{m+1} = A^T w_m - \alpha_m w_m - \beta_m w_{m-1}$

$v_{m+1} \perp w_m$, $w_{m+1} \perp v_m$ $\Rightarrow$ $\alpha_m = w_m^T A v_m$

$v_{m+1} \perp w_{m-1}$ $\Rightarrow$ $w_{m-1}^T A v_m = \gamma_m$

$w_{m+1} \perp v_{m-1}$ $\Rightarrow$ $v_{m-1}^T A^T w_m = \beta_m$

$v_{m+1} \perp w_j$ $\quad w_j^T A \in W_{j+1} \perp v_m \quad w_j \perp v_m, v_{m-1} \ (j \leq m-2)$

$w_{m+1}^T A v_{m+1} = \alpha_{m+1}$

$$w_{m+1}^T A v_m = \frac{g_{m+1}^T}{\gamma_{m+1}} A v_m$$

$$= \frac{g_{m+1}^T}{\gamma_{m+1}} (f_{m+1} + \alpha_m v_m + \gamma_m v_{m-1})$$

$$= \frac{g_{m+1}^T f_{m+1}}{\gamma_{m+1}} = \beta_{m+1}$$

$$\forall j \leq m-1 \quad w_{m+1}^T A v_j = \frac{g_{m+1}^T}{\gamma_{m+1}} (f_m + \alpha_{m-1} v_{m-1} + \gamma_{m-1} v_{m-2}) = 0$$

同理可证

$$w_m^T A v_{m+1} = \left(g_{m+1}^T + \alpha_m w_m^T + \beta_m w_{m-1}^T\right) v_{m+1}$$

$$= \frac{g_{m+1}^T f_{m+1}}{\beta_{m+1}} = \gamma_{m+1}$$

$$w_k^T b = \frac{c\, b}{\gamma_1} = \beta_1$$

$$c\, v_k = \frac{c\, b}{\beta_1} = \gamma_1$$

iii) $c\left(\xi_0 I - A\right)^{-(m+1)} b = c_r \left(\xi_0 I - A_r\right)^{-(m+1)} b_r$

$$c_r \left(\xi_0 I - A_r\right)^{-(m+1)} b_r = cV\left(W^T(\xi_0 I - A)V\right)^{-(m+1)} W^T b$$

$$V = \left[(\xi_0 I - A)^{-1} b,\ (\xi_0 I - A)^{-2} b,\ \cdots\ (\xi I - A)^{-k} b\right]$$

$$W^T\left(\xi_0 I - A_0\right) V = W^T\left[\ b,\ (\xi_0 I - A)^{-1} b\ \cdots\ (\xi I - A)^{-k+1} b\right]$$

$$= \begin{bmatrix} & 1 & 0 & & 0 \\ & 0 & 1 & & \vdots \\ W^T b & 0 & 0 & & \vdots \\ & \vdots & \vdots & & 0 \\ & \vdots & \vdots & & 1 \\ & 0 & 0 & & 0 \end{bmatrix}$$

$$\left(W^T(\xi_0 I - A)V\right)^{-(m+1)} W^T b = e_{m+1}$$

$$cV e_{m+1} = c\left(\xi_0 I - A\right)^{-(m+1)} b$$