数据驱动的降阶模型

① 谱分解：

$$L v_k = \lambda_k v_k$$

$$\frac{d}{dt}\left(\sum a_k(t)\, v_k(x)\right) = \sum a_k(t)\lambda_k \quad v_k(x)$$

$$a_k(t) = a_k(0)\, e^{\lambda_k t}$$

$$u(t,x) = \sum a_k(0)\, e^{\lambda_k t}\, v_k(x)$$

② Koopman 特征函数

$$\frac{d K_t \varphi}{dt} = K\varphi = \lim \frac{K_t \psi - \psi}{t} = \lim \frac{(e^{\lambda t}-1)\psi}{t}$$

$$= \lambda \psi$$

$$K(\varphi_1 \cdot \varphi_2)(u) = \lim_{t\to 0} \frac{K_t\, \varphi(u) - \varphi(u)}{t}$$

$$= \frac{\varphi(F_t u) - \varphi(u)}{t}$$

$$= \frac{\varphi_1(F_t(u))\varphi_2(F_t(u)) - \varphi(u)}{t}$$

$$= \frac{(\varphi_1(F_t(u)) - \varphi_1(u))\,\varphi_2(F_t(u))}{t}$$

$$+ \frac{\varphi_1(u)(\varphi_2(F_t(u)) - \varphi_2(u))}{t}$$

$$= \lambda_1 (\varphi_1 \varphi_2)(u) + \lambda_2 (\varphi_1 \varphi_2)(u)$$

i) $$K\varphi(u) = \lim_{t \to 0} \frac{\varphi\, u(\to \Delta t) - \varphi(u)}{\Delta t}$$

$$= \nabla \varphi \cdot \frac{du}{dt} = \nabla \varphi \cdot (-u - u^3)$$

$$K = \frac{du}{dt} \cdot \nabla$$

i) $$\nabla \varphi \cdot (-u - u^3) = \lambda \varphi$$

$$\nabla \log \varphi = -\frac{\lambda}{u + u^3} = -\lambda \left(\frac{1}{u} - \frac{u}{1 + u^3} \right)$$

$$\log \varphi \propto -\lambda \left(\log u - \log(1 + u^2) \right)$$

$$\varphi = \frac{u}{1 + u^2}, \quad \lambda = -1$$

$$\varphi = \left(\frac{u}{1 + u^2} \right)^k \quad \lambda = -k$$

离散谱

③ Koopman 模式

例一.

$$\mu u_1 \frac{2\lambda}{2\mu-\lambda} u_1 + \lambda(u_2 - u_1^2)\cdot 1$$

$$\frac{2\lambda\mu}{2\mu-\lambda} - \lambda = \frac{\lambda^2}{2\mu-\lambda} \Rightarrow \lambda\left(u_2 + \frac{\lambda}{2\mu-\lambda}u_1^2\right)$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \varphi_1 \underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \varphi_2 \underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} + \varphi_3 \underline{\begin{pmatrix} 0 \\ -\frac{\lambda}{2\mu-\lambda} \end{pmatrix}}$$

Koopman 模式

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = e^{\mu t}\varphi_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{\lambda t}\varphi_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{2\mu}\varphi_3\begin{pmatrix} 0 \\ -\frac{\lambda}{2\mu-\lambda} \end{pmatrix}$$

$$\underline{u_1(0)} \qquad \underline{u_2(0) + \frac{\lambda}{2\mu-\lambda}u_1(0)^2} \qquad \underline{u_1(0)^2}$$

是观测

例二. $W^T V = I \Rightarrow V W^T = I$

koopman mode

$$id(u(0)) = \sum \underline{v_j}\, \underline{w_j^T u(0)}$$

koopman 算子特征函数 是一个观测

$$u(n) = K^n \sum v_j w_j^T u(0)$$

$$= \sum v_j K^n (w_j^T u(0))$$

$$= \sum v_j \lambda_j^n w_j^T u(0)$$

④ Koopman 算子的近似

$$\mathcal{G}_N \subset \mathcal{G} \qquad \mathcal{G}_N = \mathrm{span}\{\psi_1, \psi_2 \cdots \psi_N\}$$

$$\Pi : \mathcal{G} \to \mathcal{G}_N \qquad \Pi = \Psi \mathcal{I}$$

$$\mathcal{I} : \mathcal{G} \to \mathbb{C}^N$$

$$\mathcal{I}\psi_i = e_i \qquad \mathcal{I}\Psi = I$$

$$\mathcal{K}_N : \mathcal{G}_N \to \mathcal{G}_N$$

$$\mathcal{K}_N g = \Pi \mathcal{K} \Pi g = \Psi \underbrace{\boxed{\mathcal{I} \mathcal{K} \Psi}}_{K} \mathcal{I} g \qquad K : \mathbb{C}^N \to \mathbb{C}^N$$

动力学模态分解

$$\psi_i(u) = u(x_i)$$

$$Y = KX \qquad X = U \Sigma V^T \qquad \Sigma \in R^{m\times m}$$

$$K = Y V \Sigma^{-1} U^T \in R^{N \times N} \qquad K^T = U$$

$$\tilde{K} = V \Sigma^{-1} U^T Y \in R^{m \times m}$$

$$\boxed{\begin{aligned} ABu &= \lambda u \\ w^T AB &= \lambda w^T \end{aligned}} \Rightarrow \boxed{\begin{aligned} w^T ABA &= \lambda w^T A \\ BABu &= \lambda Bu \end{aligned}}$$

扩展动力学模态分解

$$\Psi(u) = \begin{pmatrix} \psi_1(u) \\ \psi_2(u) \\ \vdots \\ \psi_D(u) \end{pmatrix} \qquad Y = KX$$

① $w_j^T K = \lambda_j w_j^T$ $\qquad \{\lambda_j, w_j^T \Psi\}$

② $u = \sum_{j=1}^{l} w_j^T \psi \, v_j$ $\qquad \{v_j\}$

$$\tilde{w}_j^T \tilde{K} = \tilde{\lambda}_j \tilde{w}_j^T \qquad \hat{K} = U^T Y V \Sigma^{-1}$$

$$\Leftrightarrow \tilde{w}_j^T U^T Y V \Sigma^{-1} = \tilde{\lambda}_j \tilde{w}_j^T$$

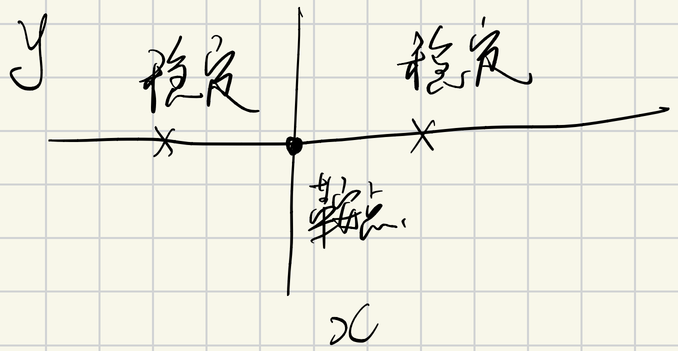
$$\tilde{w}_j^T U^T \underbrace{Y V \Sigma^{-1} U^T}_{K} = \tilde{\lambda}_j \tilde{w}_j^T U^T$$

$$w_j^T = \tilde{w}_j^T U \qquad \sum_{j=1}^{l} w_j^T \psi = \sum \tilde{w}_j^T U^T \psi \, v_j$$

例子

$$\ddot{u} = -\delta \dot{u} + u(\beta + \alpha u^2)$$

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ -\delta \dot{u} + u(\beta + \alpha u^2) \end{pmatrix} \qquad \begin{pmatrix} y \\ -0.5y - x(x^2-1) \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ -2 & -0.5 \end{pmatrix}$$

$$\lambda = -0.25 \pm 1.39\,i$$

使用 10^3 条轨道，每条 11个 t_i，在 xy 平面上的 $r^2 \log r$ radial basis function

$$K^t g(x) = g(F^t x) = e^{\lambda t} g(x)$$

$$g(x) = \begin{cases} 1 & x \text{ 去 } (-1,0) \text{ attractor} \\ -1 & x \text{ 去 } (1,0) \text{ attractor} \end{cases}$$

$$\lambda = 0$$