偏微分方程求解

变分: $\int (-\Delta u - f)\phi \, dx = 0 \quad \forall \phi$

$$\int -\nabla\cdot(\nabla u\, \phi) + \nabla u \cdot \nabla \phi - f\phi \, dx = 0 \quad \forall \phi$$

$$\int -\frac{\partial u}{\partial n}\phi \, d\partial\Omega + \int \nabla u \nabla \phi - f\phi \, dx = 0 \quad \forall \phi$$

$$\int \nabla u \nabla \phi - f\phi \, dx - \int g\,\phi \, d\partial\Omega = 0 \quad \forall \phi$$

$$\min_u \int \frac{1}{2}|\nabla u|^2 - fu \, dx - \int g\, u \, d\partial\Omega$$

变分: $\int (-\Delta u - f)\phi \, dx = 0 \quad \forall \phi \quad \phi|_{\partial\Omega} = 0$

$$\int \nabla u \nabla \phi - f\phi \, dx + \int \frac{\partial u}{\partial n}\phi \, d\partial\Omega = 0$$

$$\Rightarrow \int \nabla u \nabla \phi - f\phi \, dx = 0$$

$\min \int \frac{1}{2}|\nabla u|^2 - f u \, dx$ 且 $u|_{\partial\Omega} = g$

半线性抛物型偏微分方程：

抛物方程：

$$u_t = \sum_{ij} \partial_{x_j}\left(A_{ij}(x,t)\,\partial_{x_i} u\right) + f(u)$$

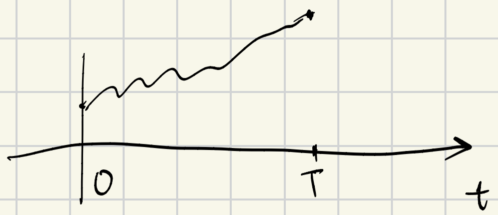
$[A_{ij}]$ 的所有特征值实部均大于 0

热方程 $u_t = \sum_{ij} \partial_{x_j}\left(\nu\, \partial_{x_i} u\right)$

半线性抛物型偏微分方程：

粒子方程：

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$



$$du(t,X_t) = -f\left(t, X_t, u(t,X_t), \sigma(t,X_t)^T \nabla u(t,X_t)\right)dt$$

$$+ \nabla u(t,X_t)^T \sigma(t,X_t)\,dW_t$$

Ito公式：

$$u(t,X_t) = u(0,X_0) + \int_0^t \partial_t u(s,X_s)\,ds$$

$$+ \sum_{j=1} \int_0^t \partial_{x_j} u(s,X_s)\,dx_{js}$$

$$+ \frac{1}{2}\sum_{i,j=1} \int_0^t \partial_{x_i x_j} u(s,X_s)\,dx_{js}\,dx_{is}$$

$E\,dW_t^2 = dt$

$$= u(0, X_0) + \int_0^t \partial_t u(s, X_s)\, ds$$

$$+ \int_0^t \nabla_x u(s, X_s) \cdot \mu(s, X_s)\, ds$$

$$+ \int_0^t \nabla_x u(s, X_s)^T \sigma(s, X_s)\, dW_s$$

$$+ \frac{1}{2}\sum_{ij} \int_0^t \partial_{x_i x_j} u(s, X_s)\ \sigma_{jk} dW_{ks}\ \sigma_{il} dW_{ls}$$

$$\sum_{ij} \int_0^t \partial_{x_i x_j} u(s, X_s)\ \sigma_{jk}\, \sigma_{ik}\, ds$$

$$= \sum \int_0^t \sigma\, \sigma^T : \mathrm{Hess}\, u(s, X_s)\, ds$$

$$\Rightarrow du(t, X_t) = \left(\partial_t u(t, X_t) + \nabla_x u(t, X_t) \cdot \mu(t, X_t)\right) dt$$

$$+ \nabla_x u(t, X_t)^T \sigma(t, X_t)\, dW_t$$

$$+ \frac{1}{2} \sigma\, \sigma^T : \mathrm{Hess}\, u(t, X_t)\, dt$$

方程的解：

$$X_t = X_0 + \int_0^t \mu(s, X_t)\, ds + \int_0^t \sigma(s, X_t)\, dW_s$$

$$u(t, X_t) = u(0, X_0) - \int_0^t f(s, X_s, u(s, X_s), \sigma^T \nabla u)\, ds$$

$$+ \int_0^t \nabla u^T \sigma\, dW_s$$

已知 $g(x) = u(T, x) \Rightarrow u(0, x)$

Hamiton-Jacobian 方程

$$\frac{\partial u}{\partial t}(t,x)+\Delta u(t,x)-\lambda\|\nabla u(t,x)\|^2=0$$

$$u(t,x)=-\frac{1}{\lambda}\ln E\left[\exp\left(-\lambda g\left(x+\sqrt{2}W_{T-t}\right)\right)\right]$$

凸神经网络：

$f(x;\theta)$ 关于 x 是凸函数

引理：凸函数之和仍为凸函数

凸函数 g 与单增凸函数 h 的复合仍为凸函数。

$$h\circ g(\lambda x+(1-\lambda)y)$$ h 单增，g 凸

$$\leq h(\lambda g(x)+(1-\lambda)g(y))$$ h 凸

$$\leq \lambda h\circ g(x)+(1-\lambda)h\circ g(y)$$

$f_i(x)$ 的每一维均为凸函数（归纳法）

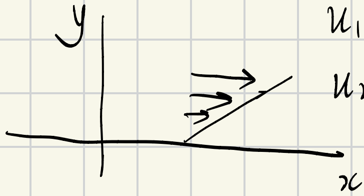
f_1 ： $W^{(1)}x+\tilde{W}^{(1)}x+b^{(1)}$ 凸函数

$h^{(1)}$ 为单增凸函数

f_2: $W^{(2)} f(x)$ 为凸函数，$W^{(2)} x$ 为凸函数

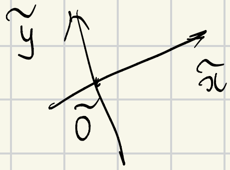
$h^{(2)}$ 为单增凸函数

伽利略原理:



$u_1 = y$

$u_2 = 0$



$$[\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3] = [e_1\ e_2\ e_3]\, Q$$

$$Q^T Q = I \qquad q_{ki}\, q_{kj} = \delta_{ij}$$

点:

$$O\tilde{O} = [e_1\ e_2\ e_3]\, b$$

$$OA = [\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3]\,\tilde{x} + O\tilde{O} = [e_1\ e_2\ e_3]\, Q\tilde{x} + [e_1\ e_2\ e_3]\, b$$

$$x = Q\tilde{x} + b$$

向量: (速度场)

$$[\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3]\,\tilde{u}(\tilde{x}) = [e_1\ e_2\ e_3]\, Q\,\tilde{u}(\tilde{x}) := [e_1\ e_2\ e_3]\, u(x)$$

$$u(x) = Q\,\tilde{u}(Q^T(x-b)) = Q\,\tilde{u}(\tilde{x})$$

矩阵: (∇u)

$$\frac{\partial u_i}{\partial x_j} e_i \otimes e_j = \frac{\partial\, q_{im}\,\tilde{u}_m(\tilde{x})}{\partial x_j}\ \tilde{e}_k q_{ik} \otimes \tilde{e}_l q_{jl}$$

$$= q_{im}\frac{\partial \tilde{u}_m(\tilde{x})}{\partial \tilde{x}_s}\frac{\partial \tilde{x}_s}{\partial x_j}\ \tilde{e}_k q_{ik} \otimes \tilde{e}_l q_{jl}$$

$$= \underline{q_{im}}\frac{\partial \tilde{u}_m(\tilde{x})}{\partial \tilde{x}_s}\ \underline{q_{js}}\ \tilde{e}_k \underline{q_{ik}} \otimes \tilde{e}_l \underline{q_{jl}}$$

$$= \delta_{mk}\,\delta_{sl}\frac{\partial \tilde{u}_m(\tilde{x})}{\partial \tilde{x}_s}\ \tilde{e}_k \otimes \tilde{e}_l$$

$$= \frac{\partial \tilde{u}_k(\tilde{x})}{\partial \tilde{x}_l}\ \tilde{e}_k \otimes \tilde{e}_l$$

$$\frac{\partial u_i}{\partial x_j} = q_{im} \frac{\partial \tilde{u}_m(\tilde{x})}{\partial \tilde{x}_s} q_{js}$$

$$\nabla_x u = Q \nabla_{\tilde{x}} \tilde{u} \, Q^T$$

如果输出是标量 $\tilde{f} = f$，$f(\tilde{u}, \underline{\underline{\nabla_{\tilde{x}} \tilde{u}}}) = f(u, \nabla_x u)$

$\lambda(\nabla_x u)$ 不变量

如果输出是二阶张量

$Q \tilde{\tau} Q^T = \tau$ $\tau = \nabla_x u + \nabla_x u^T + \nabla_x u \, \nabla_x u^T \cdots$

可以乘上不变量。