Special values of L-functions 10

—— Introduction to motives I

§1. Cohomology of algebraic varieties

Exercise: X / F = number field

Let X be a projective smooth variety of dim $d/\mathbb{Q}$, $n \le 2d$.

Will discuss 3 cohomology theories: de Rham, Betti, étale.

- de Rham cohomology $H^\bullet_{dR}(X/\mathbb{Q}) := H^\bullet(X, \Omega^\bullet_X)$

$$\Omega^\bullet_X = [\mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \cdots]$$

$$\text{deg } 0 \qquad 1 \qquad 2 \qquad \cdots$$

Spectral sequence $E_1^{pq} := H^q(X, \Omega^p) \Rightarrow H^{p+q}_{dR}(X/F)$

$$\begin{array}{llll}
\vdots & & & \\
H^2(X,\mathcal{O}_X) & \to H^2(X,\Omega^1_X) & & \\
H^1(X,\mathcal{O}_X) & \to H^1(X,\Omega^1_X) & \to & \\
H^0(X,\mathcal{O}_X) & \to H^0(X,\Omega^1_X) & \to & H^0(X,\Omega^2_X)
\end{array}$$

Over characteristic zero field, the de Rham spectral sequence degenerates at E_1, i.e. all $d_1 = 0$

E.g. $H^2_{dR}(X/\mathbb{Q})$ consists of 3 parts:

sub: $H^0(X,\Omega^2_X)$ middle: $H^1(X,\Omega^1_X)$, quotient: $H^2(X,\mathcal{O}_X)$.

$$\mathrm{Fil}^2 H^2_{dR}(X/\mathbb{Q}) = H^0(X,\Omega^2_X)$$

$$\mathrm{Fil}^1 H^2_{dR}(X/\mathbb{Q}) = H^0(X,\Omega^2_X) \overset{\text{extension}}{—} H^1(X,\Omega^1_X)$$

$$\mathrm{Fil}^0 H^2_{dR}(X/\mathbb{Q}) = H^2_{dR}(X/\mathbb{Q})$$

Generally for $0 \le i \le n$, $\Omega^{\geq i}_X \hookrightarrow \Omega^\bullet_X$ is a subcomplex

$$\mathrm{Fil}^i H^n_{dR}(X/\mathbb{Q}) = \mathrm{Im}\left(H^n(X,\Omega^{\geq i}_X) \to H^n(X,\Omega^\bullet_X)\right).$$

• <u>Betti cohomology</u> Let $c :=$ complex conjugation

Define $F_\infty : X(\mathbb{C}) \longrightarrow X(\mathbb{C})$ to be the complex conj on points.

$\rightsquigarrow H^n_B(X(\mathbb{C})^{an}, \mathbb{Z})$ same as singular cohomology or cohom of const sheaf $\underline{\mathbb{Z}}_{X(\mathbb{C})^{an}}$

<u>Betti-de Rham comparison</u>: $H^n_B(X(\mathbb{C})^{an}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^n_{dR}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$

(canonical) $\qquad F_\infty \otimes c \longleftrightarrow 1 \otimes c$

<u>Hodge decomposition</u> $H^n_B(X(\mathbb{C})^{an}, \mathbb{Q}) \otimes \mathbb{C} = \bigoplus_{\substack{p+q=n \\ p,q\geq 0}} H^{pq}$ with $H^{pq} = \overline{H^{qp}}$ ← conjugation on coeffs $\mathbb{C}$ <u>not</u> on $X(\mathbb{C})^{an}$.

* Relation to Hodge filtration: $\mathrm{Fil}^i H^n_{dR} \otimes \mathbb{C} = \bigoplus_{p \geq i} H^{p,n-p}$

$$\overline{\mathrm{Fil}^j H^n_{dR} \otimes \mathbb{C}} = \bigoplus_{p \geq j} \overline{H^{p,n-p}} = \bigoplus_{p\geq j} H^{n-p,p}$$

$$\text{So } H^{pq} = \left(\mathrm{Fil}^p H^n_{dR} \otimes \mathbb{C}\right) \cap \overline{\mathrm{Fil}^q H^n_{dR} \otimes \mathbb{C}}$$

• <u>Étale cohomology</u> For any prime $\ell \rightsquigarrow H^n_{et}(X_{\mathbb{Q}^{alg}}, \mathbb{Z}_\ell)$ no $\mathbb{Q}$-analogue!

• <u>Betti-Étale comparison</u> If $\mathbb{Q}^{alg} :=$ algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$,

(canonical only for $\mathbb{Q}^{alg}$) $\quad H^n_{et}(X_{\mathbb{Q}^{alg}}, \mathbb{Z}_\ell) \cong H^n_{et}(X_{\mathbb{C}}, \mathbb{Z}_\ell) \cong H^n_B(X(\mathbb{C})^{an}, \mathbb{Z}) \otimes \mathbb{Z}_\ell$

• <u>Galois action</u> $\quad H^n_{et}(X_{\mathbb{Q}^{alg}}, \mathbb{Z}_\ell) \circlearrowleft \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$

typically does not stabilize $H^n_B(X)$.

The Galois rep'n $\rho : \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q}) \longrightarrow H^n_{et}(X_{\mathbb{Q}^{alg}}, \mathbb{Z}_\ell)$ is "<u>nice</u>".

• If X admits an good reduction at a prime $p \neq \ell$

i.e.
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\text{proj. sm.}"] & \mathfrak{X} \arrow[d] & \mathfrak{X}_{\mathbb{F}_p} \arrow[l] \arrow[d] \\
\mathrm{Spec}\,\mathbb{Q} \arrow[r] & \mathrm{Spec}\,\mathbb{Z}_{(p)} & \mathrm{Spec}\,\mathbb{F}_p \arrow[l]
\end{tikzcd}$$

not canonical, need a choice of embedding $\mathbb{Q}^{alg} \hookrightarrow \mathbb{Q}_p^{alg}$

$$\text{then } H^n_{et}(X_{\mathbb{Q}^{alg}}, \mathbb{Z}_\ell) \simeq H^n_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_\ell) \simeq H^n_{et}(\mathfrak{X}_{\overline{\mathbb{F}}_p}, \mathbb{Z}_\ell)$$

$$\mathrm{Gal}_{\mathbb{Q}} \longleftarrow \mathrm{Gal}_{\mathbb{Q}_p} \longrightarrow\!\!\!\!\rightarrow \mathrm{Gal}_{\mathbb{F}_p} \ni \phi_p \text{ geometric Frobenius.}$$

Deligne's theorem

① Characteristic poly $\det\left(x\cdot\mathbb{1}-\phi_p : H^n_{et}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)\right) \in \mathbb{Z}[x]$

② Every root of the char poly λ is a Weil p^n-number,

i.e. λ is an algebraic number & for any complex embedding ι of $\mathbb{Q}(\lambda)$, $|\iota(\lambda)|_\infty = p^{\frac{n}{2}}$.

③ The characteristic poly is independent of ℓ (if $\ell \neq p$)

Remark: Deligne's observation is that Hodge filtration wt: $n = p+q$ is the same as Frob. weight $|\iota(\lambda)|_\infty = p^{\frac{n}{2}}$

- Away from a finite set S of places, ρ is unramified.

$$\forall p \notin S, \quad L_p(\rho, s) := \frac{1}{\det\left(\mathbb{1} - \rho(\phi_p)\cdot p^{-s}\right)}$$

Corollary. $$L^S(\rho, s) := \prod_{p\notin S} L_p(\rho, s)$$ converges absolutely when $Re(s) > \frac{n+1}{2}$.

Remark: We often encounter the situation that $H^n(X)$ is natural direct sum of pieces.

It is natural to study these pieces independently.

§2. Chow group

Let k be a field. Let $SmProj/k$ denote the category of smooth projective varieties$/k$.

Definition For $X \in SmProj/k$ pure of dim d, and $0 \leq n \leq d$,

$$Z^n(X) = \{\text{group of codim d cycles}\} = \left\{\text{finite sum} \sum_{\substack{Z_i \subseteq X \\ \text{codim } n}} a_i [Z_i] \; ; a_i \in \mathbb{Z}\right\} =: Z_{d-n}(X)$$

"dim $d-n$ cycles"

If $W \subseteq X$ is irreducible of codim $d-1$, and if $f \in k(W)^\times$,

then $$\operatorname{div}(f) = \sum_{\substack{Z_i \subseteq X \\ \text{codim } n}} \operatorname{ord}_{Z_i}(f)\cdot[Z_i] \in Z^n(X)$$

$CH^n(X) := Z^n(X)/\langle \mathrm{div}(f) ;\ f \in k(W)^\times \rangle$ is the codim d Chow group of X

If $Z, Z' \in Z^n(X)$ have same image in $CH^n(X)$, say Z is rationally equiv to Z', denoted $Z \sim Z'$.

Operations on Chow groups

① Intersection. $CH^i(X) \times CH^j(X) \xrightarrow{\ \cdot\ } CH^{i+j}(X)$

For $Z_1 \in CH^i(X)$ and $Z_2 \in CH^j(X)$ such that Z_1 and Z_2 intersect properly, i.e. $Z_1 \cap Z_2$ is pure of dim $i+j$.

For each irred component W of $Z_1 \cap Z_2$, put $A := \mathcal{O}_{X,W}$

put $i(Z_1, Z_2 ; W) := \sum_r (-1)^r \mathrm{length}_A \left(\mathrm{Tor}_r^A \left(A/I(Z_1), A/I(Z_2) \right) \right)$

Define $Z_1 \cdot Z_2 := \sum_W i(Z_1, Z_2 ; W) [W]$

This definition depend only on rational equivalent class of Z_1 and Z_2

(Moving lemma) Given $Z \in Z^i(X)$, $W_1, \cdots, W_\ell \in Z^j(X)$, then $\exists Z' \sim Z$ s.t.

Z and each W_j intersect properly.

This makes $CH^\bullet(X) := \bigoplus_{i \geq 0} CH^i(X)$ a ring, called the Chow ring

② Pullback : $f : X \to Y \rightsquigarrow \Gamma_f = \text{graph of } f = \{ (x, f(x)) ;\ x \in X \} \subseteq X \times Y$

We define pullback $f^* : CH^i(Y) \to CH^i(X)$ by

$$\begin{tikzcd}
X & X \times Y \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
\Gamma_f \arrow[u, "p_f"', "\cong"] \arrow[ur, "\subseteq"'] & Y
\end{tikzcd}$$

for $Z \in CH^i(Y)$, define

$$f^*(Z) := p_f^{-1} \left(\Gamma_f \cdot (X \times Z) \right)$$

③ Pushforward : $f : X \to Y$ (automatically proper)

$\rightsquigarrow f_* : CH_\ell(X) \to CH_\ell(Y)$ defined by :

for irreducible $Z \in CH_\ell(X)$, write $W := f(Z)$ for the reduced image,

$$\text{define } f_*(Z) := \begin{cases} [k(Z) : k(W)] & \text{if } \dim W = \ell \\ 0 & \text{if } \dim W < \ell . \end{cases}$$

Realizations of Chow cycles:

(1) Betti + de Rham realization

For $X \in SmProj/k$ with char $k=0$, there is a de Rham class map

$$cl_{dR} : CH^i(X) \longrightarrow Fil^i H^{2i}_{dR}(X/k)$$

For $X \in SmProj/\mathbb{C}$, there's a Betti map

$$cl_B : CH^i(X) \longrightarrow H^{2i}_B(X(\mathbb{C})^{an}, \mathbb{Q})$$

Hodge Conjecture For $X \in SmProj/\mathbb{C}$, the cycle map

as $H^{2i}(\mathbb{Q})$ is invariant under complex conj.

$$cl : CH^i(X) \otimes \mathbb{Q} \longrightarrow H^{2i}_B(X(\mathbb{C})^{an}, \mathbb{Q}) \cap Fil^i H^{2i}_{dR}(X/\mathbb{C}) = H^{2i}_B(X(\mathbb{C})^{an}, \mathbb{Q}) \cap H^{i,i}$$ is surjective.

(2) Étale realization:

* For X a variety $/k$, there are two versions of étale cohomology ($\ell \neq$ char k)
  - $H^n_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell) \circlearrowleft Gal(\bar{k}/k)$
  - $H^n_{et}(X, \mathbb{Z}_\ell)$ "absolute étale cohomology"

Hoshchild-Serre spectral sequence:

$$H^i(Gal(\bar{k}/k), H^j_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell)) \Rightarrow H^{i+j}_{ét}(X, \mathbb{Z}_\ell)$$

- Now, if $X \in SmProj/k$, we have cycle map

$$cl : CH^i(X) \longrightarrow H^{2i}_{ét}(X, \mathbb{Z}_\ell(i)) \longrightarrow H^{2i}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^{Gal_k}$$

Tate Conjecture: If k is a finitely generated field ($\ell \neq$ char k)

then $cl_{ét} : CH^i(X) \otimes \mathbb{Q}_\ell \longrightarrow H^{2i}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^{Gal_k}$ is surjective.

Moreover, $CH^i(X)_0 := \ker\left(CH^i(X) \otimes \mathbb{Q} \longrightarrow H^{2i}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^{Gal_k}\right)$ is independent of ℓ.

- Abel-Jacobi map: $AJ : CH^i(X)_0 \longrightarrow H^1(Gal_k, H^{2i-1}(X_{\bar{k}}, \mathbb{Q}_\ell(i)))$

(When k is a number field, this is a very important map.)

§3 Correspondence

Definition. For $X, Y \in SmProj/k$, we define the group of correspondences from X to Y to be

$$\mathrm{Corr}^m(X,Y) := CH^{m+d_X}(X \times Y)$$

* Every correspondence $C \in \mathrm{Corr}^m(X,Y)$ induces natural maps

$$\begin{tikzcd} & C \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

① $CH^i(X) \longrightarrow CH^{i+m}(Y)$

$$T \longmapsto [C]_*(T) := p_{Y*}(p_X^{-1}(T))$$

② $[T]_*$ $H^i_{et}(X, \mathbb{Z}_\ell) \longrightarrow H^{i+2m}_{et}(Y, \mathbb{Z}_\ell(m))$ or the over $\bar{k}$ version

$$\begin{tikzcd} H^i_{et}(X,\mathbb{Z}_\ell) \arrow[rr] \arrow[dr, "p_X^*"'] & & H^{i+2m}_{et}(Y,\mathbb{Z}_\ell(m)) \\ & H^i_{et}(C,\mathbb{Z}_\ell) \arrow[ur, "\text{Gysin map along } p_Y\text{, ``dual to } p_Y^*\text{''}"'] & \end{tikzcd}$$

③ If $\mathrm{char}\, k = 0$, $[T]_* : \mathrm{Fil}^a H^i_{dR}(X/k) \longrightarrow \mathrm{Fil}^{a+m} H^i_{dR}(Y/k)$

④ If $k = \mathbb{C}$, $[T]_* : H^i_B(X(\mathbb{C})^{an}, \mathbb{Z}) \to H^i_B(Y(\mathbb{C})^{an}, \mathbb{Z})$

* Transpose of a correspondence $C \in \mathrm{Corr}^m(X,Y)$ is denoted ${}^tC \in \mathrm{Corr}^{d_Y - d_X + m}(Y,X)$

Example: If $f : X \to Y$ is a morphism in $SmProj/k$, then

$$\Gamma_f \in \mathrm{Corr}^{d_Y - d_X}(X,Y) \text{ corresponds to } f_!$$

$${}^t\Gamma_f \in \mathrm{Corr}^0(Y,X) \text{ corresponds to } f^*$$

* Composition : $\mathrm{Corr}^m(X,Y) \times \mathrm{Corr}^n(Y,Z) \longrightarrow \mathrm{Corr}^{m+n}(X,Z)$

$$C \in \mathrm{Corr}^m(X,Y), \quad D \in \mathrm{Corr}^n(Y,Z)$$

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

$$C \circ D := p_{13*}(p_{12}^{-1}(C) \times p_{23}^{-1}(D))$$

* identity correspondence : $\mathrm{id}_X := [\Delta_X] \in \mathrm{Corr}^0(X,X)$

* In particular, $\mathrm{Corr}^0(X,X)$ is a ring.