Special values of L-functions 12

—— L-factors at infinity and Deligne's conjecture

<u>Recall</u>: For a subring $A \subseteq \mathbb{R}$, an <u>A-Hodge structure</u> is

- a finite projective A-module V
- a decomposition $V \otimes_A \mathbb{C} = \bigoplus_{p,q} V^{pq}$ s.t. $\overline{V^{pq}} = V^{qp}$

An $\mathbb{R}$-Hodge structure on an $\mathbb{R}$-vector space V corresponds to

$$h_{\mathbb{C}} : \mathbb{G}_m \times \mathbb{G}_m \longrightarrow GL(V)_{\mathbb{C}} \quad \text{s.t. } h_{\mathbb{C}}(z,w) \text{ acts on } V^{pq} \text{ by mult}_{z^{-p}w^{-q}}$$

The condition $\overline{V^{pq}} = V^{qp}$ means

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \times V \arrow[r, "act"] \arrow[d] & V \arrow[d, "conj"] & (z, w, \overset{V^{pq}}{\overset{\cup}{v}}) \arrow[r, mapsto] \arrow[d, mapsto] & z^{-p}w^{-q}v \arrow[d, mapsto] \\
\mathbb{G}_m \times \mathbb{G}_m \times V \arrow[r, "act"] & V & (\underbrace{?, ?}_{\text{has to be } \bar{w}, \bar{z}}, \underset{V^{qp}}{\underset{\cap}{\bar{v}}}) \arrow[r, mapsto] & \bar{z}^{-p}\bar{w}^{-q}\bar{v}
\end{tikzcd}$$

From this, $\underset{(z,w) \mapsto (\bar{w},\bar{z})}{\underset{\circlearrowleft}{\mathbb{G}_m \times \mathbb{G}_m}} \longrightarrow \underset{\text{conjugation}}{\underset{\circlearrowleft}{GL(V)_{\mathbb{C}}}}$

Descent to $h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \longrightarrow GL(V)_{\mathbb{R}}$.

<u>Remark on comparison</u>: $\underset{F_\infty}{\underset{\circlearrowleft}{H^n(X(\mathbb{C})^{an}, \mathbb{Z})}} \otimes \underset{c}{\underset{\circlearrowleft}{\mathbb{C}}} \cong H^n_{dR}(X/\mathbb{R}) \otimes_{\mathbb{R}} \underset{c}{\underset{\circlearrowleft}{\mathbb{C}}} \cong \bigoplus_{p+q=n} H^{pq}(X(\mathbb{C}))$

analytic conjugation $1 \otimes c$ $\longleftrightarrow$ sends H^{pq} to H^{qp} i.e. $\overline{H^{pq}} = H^{qp}$

algebraic conjugation $F_\infty \otimes c$ $\longleftrightarrow$ $1 \otimes c$ preserves Hodge filtration, sends H^{pq} to H^{pq}.

<u>Corollary</u>: $F_\infty \otimes 1 : H^{pq}(X(\mathbb{C})) \xrightarrow{\sim} H^{qp}(X(\mathbb{C}))$ $\mathbb{C}$-linear.

Example: A abelian variety$/\mathbb{R}$ of dim d, rank $V = 2d$

$$\overset{=:V}{\boxed{H^1_B(A(\mathbb{C})^{an}, \mathbb{Z})}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^1_{dR}(A/\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

$$\begin{tikzcd} 0 \arrow[r] & H^0(A, \Omega^1_{A/\mathbb{C}}) \arrow[r] \arrow[d, equal] & H^1_{dR}(A/\mathbb{C}) \arrow[r] \arrow[d, equal] & H^1(A, \mathcal{O}_A) \arrow[r] \arrow[d, equal] & 0 \\ & \underbrace{\mathrm{Lie}^\vee_{A/\mathbb{R}} = t^\vee_{A,0}}_{} & V_{\mathbb{C}} & V^{0,1} & \end{tikzcd}$$

$\mathrm{Fil}^1 V_{\mathbb{C}} = V^{10}$

Deligne's choice of weight is that $(z,w) \in \mathbb{C}^\times \times \mathbb{C}^\times$ acts on V^{10} by z^{-1}

b/c it is the dual of tangent space $t_{A,0} \xrightarrow{\exp} A(\mathbb{C})$.

* Interpretation of weights: Recall that $(z,w) \in \mathbb{C}^\times \times \mathbb{C}^\times$ acts on V^{pq} by $z^{-p} w^{-q}$.

Write $\mathbb{C}^\times \xrightarrow{\Delta} \mathbb{C}^\times \times \mathbb{C}^\times$, then (z,z) acts on V^{pq} by $z^{-p-q} = z^{-n}$

$\updownarrow$

Coming from $\mathbb{G}_m \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \overset{h}{\dashrightarrow} GL_{\mathbb{R}}(V)$

So given $h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow GL_{\mathbb{R}}(V)$

$\rightsquigarrow$ weight map : $w : \mathbb{G}_m \longrightarrow \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow GL_{\mathbb{R}}(V)$

V has pure weight $n \iff w(z) = z^{-n}$ for $z \in \mathbb{G}_m$.

Definition A polarization of a Hodge structure V of pure weight n is a pairing

$$\psi : V \times V \longrightarrow A(-n) = (2\pi i)^{-n} \cdot A$$ (did not ask to be perfect)

s.t. $(x,y) \longmapsto (2\pi i)^n \psi(x, h(i) y)$ is symmetric and positive definite.

Remark: If X is a projective smooth variety of dim d, $X \subseteq \mathbb{P}^N$, say $n \leq d$

$$L := c_1(\mathcal{O}_{\mathbb{P}^N}(1)) \in H^2(X, \mathbb{Q}(1)) = H^2(X, 2\pi i \mathbb{Q})$$

Take $n=2$, $d=3$ as an example:

$$\begin{tikzcd} H^0(X, \mathbb{Q}(-1)) \arrow[dd, bend right, "L"'] \\ H^1(X, \mathbb{Q}) \arrow[dd, bend right, "L"'] \\ H^2(X, \mathbb{Q}) = L \cdot H^0(X, \mathbb{Q}(-1)) \oplus \boxed{H^2_{prim}(X, \mathbb{Q})} =: V \\ {} \end{tikzcd}$$

$$\begin{tikzcd}
H^3(X,\mathbb{Q}(1)) = L\cdot H^1(X,\mathbb{Q}) \oplus H^3_{prim}(X,\mathbb{Q}(1)) \arrow[dd, bend right, "L"'] \\
H^4(X,\mathbb{Q}(1)) = L^2\cdot H^0(X,\mathbb{Q}(-1)) \oplus L\cdot H^2_{prim}(X,\mathbb{Q}) \arrow[dd, bend right, "L"'] \\
H^5(X,\mathbb{Q}(2)) \simeq L^2\cdot H^1(X,\mathbb{Q}) \\
H^6(X,\mathbb{Q}(2))
\end{tikzcd}$$

$$\psi : H^n_{prim}(X(\mathbb{C})^{an},\mathbb{Q}) \times H^n_{prim}(X(\mathbb{C})^{an},\mathbb{Q}) \longrightarrow H^{2d}(X,\mathbb{Q}(d-n)) \simeq \mathbb{Q}(-n)$$

$$(\eta_1 \ , \ \eta_2) \longmapsto \eta_1 \cup \eta_2 \cup c_1(\mathcal{O}_{\mathbb{P}^N}(1))^{d-n}$$

Under the identification $H^n_{prim}(X(\mathbb{C})^{an},\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{pq}$. $h(i)$ on H^{pq} by i^{-p+q}

$$(H^{pq} \oplus H^{qp})^{c=1} \otimes (H^{pq} \oplus H^{qp})^{c=1} \longrightarrow \mathbb{R}$$ is positive definite

$$(\omega, \bar{\omega}) \otimes (\eta, \bar{\eta}) \longmapsto (2\pi i)^n \int_{X(\mathbb{C})^{an}} \omega \wedge i^{-p+q} \bar{\eta} \wedge c_1(\mathcal{O}_{\mathbb{P}^N}(1))^{d-n}$$

$$= (2\pi)^n (-1)^q \int_{X(\mathbb{C})^{an}} \omega \wedge \bar{\eta} \wedge c_1(\mathcal{O}_{\mathbb{P}^N}(1))^{d-n}$$

Theorem. There is a 1-1 correspondence between

$$\left\{\begin{array}{l}\text{Polarized } \mathbb{Z}\text{-Hodge structure} \\ \text{of pure weight } -1\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Abelian varieties} \\ \text{with a polarization} /\mathbb{C}\end{array}\right\}$$

$$\underset{\text{rank } 2g}{V_{\mathbb{Z}}} \longmapsto \underset{\mathbb{Z}^{2g}}{\overset{}{V_{\mathbb{Z}}}} \rightarrow \underset{\mathbb{C}^g}{\underbrace{V_{\mathbb{C}}/F^0}} \rightsquigarrow V_{\mathbb{Z}} \backslash V_{\mathbb{C}}/F^0 \text{ is a complex torus}$$

$V \otimes V \rightarrow \mathbb{Z}(1)$ gives the data for a polarization

$$V = H_1(A,\mathbb{Z}) := H^1_B(A(\mathbb{C})^{an},\mathbb{Z})^\vee \longleftarrow\!\!| \ A$$

§ L-function attached to a motive

For $M \in Mot/\mathbb{Q}$, assume that M is pure of weight $w = i - 2m$ (i.e. a summand of $H^i(X)(m)$)

(Despite we do not have Künneth for motives, we have it for all realizations.)

$\rightsquigarrow$ $M_{ét,\ell}$ is a $\mathbb{Z}_\ell$-mod, equipped with $Gal_{\mathbb{Q}}$-action

- * There's a finite set S of primes s.t.

  $\forall p \in S$, $M_{\bar{e}t,\ell}$ is unramified at p, and for any ϕ_p-eigenval λ, $|\iota(\lambda)| = p^{\frac{w}{2}}$ for any complex embedding

Define $L_p(M,s) := \dfrac{1}{\det\left(\mathbb{1} - \phi_p \cdot p^{-s} ; M_{et,\ell}^{I_{\mathbb{Q}_p}}\right)}$ if it makes sense.

(This is conjectured to be independent of ℓ.)

Put $L(M,s) := \prod_p L_p(M,s)$. expected to converge when $\mathrm{Re}\, s > \frac{w+1}{2}$

* <u>L-factor at ∞ a là Tate</u>:

<u>Recall</u>: $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$, $\Gamma_{\mathbb{C}}(s) := \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$.

$L_\infty(M,s)$ is a product of $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$ determined by the following recipe.

* $M_B \otimes \mathbb{C} = \bigoplus_{p,q} H^{pq}(M)$ s.t. $\overline{H^{pq}(M)} = H^{qp}(M)$

- For $p \neq q$, $H^{pq}(M)$ and $H^{qp}(M)$ contribute $\Gamma_{\mathbb{C}}(s - \min\{p,q\})^{\dim H^{pq}(M)}$
- For $p = p$, $F_\infty \curvearrowright M_B \otimes \mathbb{C} \supseteq H^{pp}(M)$ stable under F_∞-action

  contribute $\Gamma_{\mathbb{R}}(s-p)^{\dim H^{pp}(M)^{F_\infty = (-1)^p}} \cdot \Gamma_{\mathbb{R}}(s-p+1)^{\dim H^{pp}(M)^{F_\infty = (-1)^{p+1}}}$

<u>Conjectural functional equation</u>: Put $\Lambda(M,s) := L(M,s) \cdot L_\infty(M,s)$

Then it is conjecture that $\Lambda(M,s)$ admits a meromorphic continuation, and satisfies a functional equation $\Lambda(M,s) = \varepsilon(M,s) \cdot \Lambda(M^\vee(1), -s)$

<u>Remark</u>: If M has weight w and is polarizable, i.e. $M \simeq M^\vee(-w)$

then function equation is $\Lambda(M,s) = \varepsilon(M,s) \cdot \Lambda(M(w+1), s)$

$$= \varepsilon(M,s) \cdot \Lambda(M, w+1-s)$$

i.e. the central line is $\mathrm{Re}(s) = \frac{w+1}{2}$.

§ Deligne's conjecture

Recall that $\Gamma(s)$ for $s\in\mathbb{C}$ has no zero but poles exactly at $s\in\mathbb{Z}_{\le 0}$

Definition Let M be a motive pure of weight w.

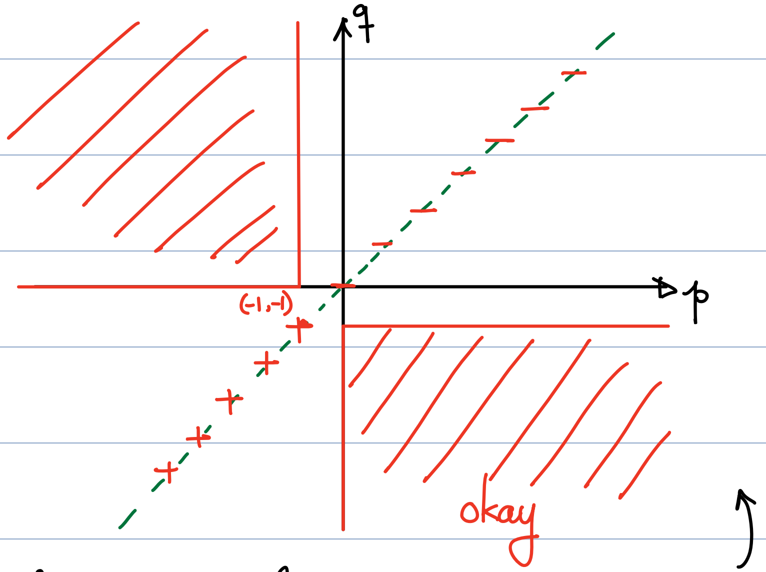
We say $n\in\mathbb{Z}$ is a critical value for M if

neither $L_\infty(M,s)$ nor $L_\infty(M^\vee,1-s)$ has a pole at $s=n$

Note : $\Lambda(M(n),s)=\Lambda(M,s+n)$; $\Lambda(M(n)^\vee,1-s)=\Lambda(M^\vee,1-(s+n))$

* So n is critical value for M $\Longleftrightarrow$ 0 is critical value for $M(n)$.

* 0 is critical for M $\Longleftrightarrow$ 0 is critical for $M^\vee(1)$.



If $s=0$ is critical for M, then allowed H^{pq}

- $H^{pq}(M)\neq 0$ with $p>q \rightsquigarrow \Gamma_{\mathbb{C}}(s-q)$ not okay if $q\geq 0$

  $\Downarrow$

  $H^{-1-p,-1-q}(M^\vee(1))\neq 0 \rightsquigarrow \Gamma_{\mathbb{C}}(-s+(1+p))$ not okay if $p\leq -1$

  (p,q symmetry $\Rightarrow$ the case with $p<q$.)

- $H^{pp}(M)^{F_\infty=(-1)^p}\neq 0 \rightsquigarrow \Gamma_{\mathbb{R}}(s-p)=\left(\frac{\pi}{2}\right)^{-\frac{s-p}{2}}\Gamma\left(\frac{s-p}{2}\right)$ not okay if p even, $p\geq 0$

  $\Downarrow$

  $H^{-1-p,-1-p}(M^\vee(1))^{F_\infty=(-1)^{p+1}}\neq 0 \rightsquigarrow \Gamma_{\mathbb{R}}(-s+(1+p))$ not okay if p odd, $p\leq -1$

- $H^{pp}(M)^{F_\infty=(-1)^{p+1}}\neq 0 \rightsquigarrow \Gamma_{\mathbb{R}}(s-p+1)$ not okay if p odd, $p\geq 1$

  $\Downarrow$

  $H^{-1-p,-1-p}(M^\vee(1))^{F_\infty=(-1)^p}\neq 0 \rightsquigarrow \Gamma_{\mathbb{R}}(-s+(1+p)+1)$ not okay if p even $p\leq -2$

$$\text{So } p \geq 0, \begin{cases} \text{even can have } H^{pp}(M)^{F_\infty=(-1)^{p+1}} \\ \text{odd can have } H^{pp}(M)^{F_\infty=(-1)^p} \end{cases} \Rightarrow \text{can have } \underbrace{H^{pp}(M)^{F_\infty=-1}}_{\text{denoted } H^{pp}(M)^-}$$

$$p<0 \begin{cases} \text{even can have } H^{pp}(M)^{F_\infty=(-1)^{p}} \\ \text{odd can have } H^{pp}(M)^{F_\infty=(-1)^{p+1}} \end{cases} \Rightarrow \text{can have } \underbrace{H^{pp}(M)^{F_\infty=1}}_{\text{denoted } H^{pp}(M)^+}$$

<u>Deligne's key observation</u>: Assume that M is pure of weight w.

$$0 \text{ is a critical value of } M$$

$$\iff \dim M_B^+ = \dim M_B^{F_\infty=1} = \dim M_{dR}/F^0 M_{dR}$$

Proof: As 0 critical for $\underbrace{M}_{wt=w} \iff 0$ critical for $\underbrace{M^\vee(1)}_{wt=-2-w}$

First assume $w \leq -1$. $\Rightarrow$ all $p+q \leq -1$.

Compute $\dim (M_B \otimes \mathbb{C})^{F_\infty=1} - \dim M/F^0M$ $(*)$

So for each pair $H^{pq}(M)$, $H^{qp}(M)$ with $p>q$.

$$F_\infty : H^{pq}(M) \overset{\simeq}{\rightleftarrows} H^{qp}(M), \text{ so contribution to } (*) \text{ is } \begin{cases} 1-1 & \text{if } p \geq 0 \\ 1-2 & \text{if } p<0 \end{cases}$$

$$\text{for } H^{pp}(M), \text{ with } p \leq -1, \text{ contribution to } (*) \text{ is } \begin{cases} 1 \;-1 & \text{if } F_\infty \text{ acts by } 1 \\ 0-1 & \text{if } F_\infty \text{ acts by } -1 \end{cases}$$

* Now in general, we need $\dim M_B^+ = \dim M_{dR}/F^0 M_{dR}$

$$\iff \dim \underbrace{(M^\vee(1))_B^+}_{\substack{\shortparallel \\ (M_B^-)^*}} = \dim \big(\underbrace{M^\vee(1)_{dR}/F^0(M^\vee(1))_{dR}}_{\substack{\shortparallel \\ (F^0M_{dR})^*}}\big)$$

$$0 \to F^0 M_{dR} \longrightarrow M_{dR} \longrightarrow M_{dR}/F^0 M_{dR} \longrightarrow 0$$

$$\text{dual to } \quad 0 \to F^0(M^\vee(1))_{dR} \longrightarrow M^\vee(1)_{dR} \longrightarrow M^\vee(1)_{dR}/F^0(M^\vee(1))_{dR} \longrightarrow 0$$

But $\dim M_B = \dim M_{dR}$ $\square$.

Deligne's conjecture: Consider $\alpha_M : \underbrace{M_B^+}_{\substack{\uparrow \\ \text{has a } \mathbb{Q}\text{-basis} \\ \circlearrowleft \\ F_\infty = 1\otimes c}} \otimes \mathbb{C} \subseteq M_B \otimes \mathbb{C} \simeq M_{dR} \otimes \mathbb{C} \twoheadrightarrow \underbrace{M_{dR}/F^0 M_{dR}}_{\substack{\text{has a } \mathbb{Q}\text{-basis} \\ \circlearrowleft \\ 1\otimes c}} \otimes \mathbb{C}$

so α_M def'd over $\mathbb{R}$

Then $\exists\ c^+(M) := \det(\alpha_M) \in \mathbb{R}^\times / \mathbb{Q}^\times$ w.r.t. the given $\mathbb{Q}$-basis

Conjecture: $L(M,0) \in \mathbb{Q}^\times \cdot c^+(M)$.