Special values of L-functions 13

— Examples of Deligne's conjecture

<u>Correction on L-factor at ∞</u>:

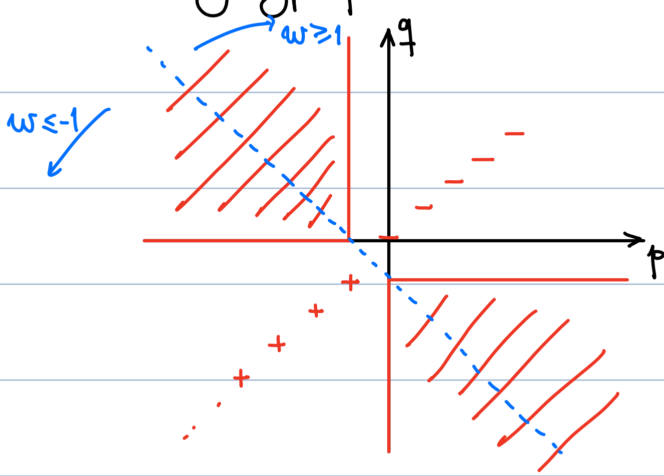
* $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$
* Contribution to $L_\infty(M,s)$ for $H^{pp}(M)^{F_\infty=(-1)^{p+1}}$ is $\Gamma_{\mathbb{R}}(s-p\cancel{-1}+1)^{\dim H^{pp}(M)^{F_\infty=(-1)^{p+1}}}$
  - Way to remember: $L_\infty(M(n),s) = L_\infty(M,s+n)$
  - For a pair of H^{0p} & H^{p0} with $p>0$, $\rightsquigarrow \Gamma_{\mathbb{C}}(s)$ so H^{pq} & $H^{qp} \rightsquigarrow \Gamma_{\mathbb{C}}(s-\min\{p,q\})$
  - For $H^{00}(M)^{F_\infty=1}$ gets $\Gamma_{\mathbb{R}}(s)$ $\rightsquigarrow$ $H^{pp}(M)^{F_\infty=(-1)^p}$ gets $\Gamma_{\mathbb{R}}(s-p)$

    $H^{00}(M)^{F_\infty=-1}$ gets $\Gamma_{\mathbb{R}}(s+1)$ $\rightsquigarrow$ $H^{pp}(M)^{F_\infty=(-1)^{p+1}}$ gets $\Gamma_{\mathbb{R}}(s-p+1)$

Say 0 is <u>critical</u> for M if neither $L_\infty(M,s)$ nor $L_\infty(M^\vee(1),-s)$ has a pole at $s=0$

In particular, 0 is critical for $M \Leftrightarrow$ 0 is critical for $M^\vee(1)$.

<u>Proposition</u>: Allowed Hodge type for a motive M that 0 is critical.



means $F_\infty = 1$

<u>Theorem</u> (Deligne). 0 is critical for M if and only if $\dim_{\mathbb{Q}} M_B^+ = \dim_{\mathbb{Q}} M_{dR}/F^0 M_{dR}$.

Proof: First assume that $p+q = w(M) \leq -1$ (will prove the other half later)

Consider contribution to $\dim_{\mathbb{C}}(M_B\otimes\mathbb{C})^{F_\infty=1} - \dim_{\mathbb{C}} M_{dR,\mathbb{C}}/F^0 M_{dR,\mathbb{C}}$.

For a pair H^{pq} & H^{pq}, $p>q$ $\qquad 1 \quad - \quad 1 \quad = 0 \qquad$ if $p\geq 0$

$$
\begin{array}{lllll}
 & 1 - 2 = -1 & \text{if } p<0 \\
\text{For } H^{pp} \quad p<0 & 1 - 1 = 0 & \text{if } F_\infty = 1 \text{ on } H^{pp} \\
 & 0 - 1 = -1 & \text{if } F_\infty = -1 \text{ on } H^{pp}
\end{array}
$$

When exactly as shown in the above diagram, $\dim_{\mathbb{C}}(M_B\otimes\mathbb{C})^{F_\infty=1} = \dim_{\mathbb{C}} M_{dR,\mathbb{C}}/F^0 M_{dR,\mathbb{C}}$ $\square$

* Consider $\alpha_M : M_B^+\otimes\mathbb{R} = (M_B\otimes\mathbb{C})^{\substack{F_\infty=1\\ c=1}} \hookrightarrow (M_B\otimes\mathbb{C})^{F_\infty\otimes c=1} = M_{dR}\otimes\mathbb{R} \twoheadrightarrow \dfrac{M_{dR}}{F^0M_{dR}}\otimes\mathbb{R}$

Then $\det(\alpha_M)\in \mathbb{R}^\times/\mathbb{Q}^\times$

<u>Deligne's conjecture</u>: When 0 is critical for M, α_M is an isomorphism and $L(M,0)\in\mathbb{Q}^\times\cdot\det(\alpha_M)$.

* <u>Fancy version</u>: α_M induces $\underbrace{\Lambda^{top}M_B^+}\otimes\mathbb{R} \xrightarrow{\sim} \Lambda^{top}(M_{dR}/F^0M_{dR})\otimes\mathbb{R}$

two $\mathbb{Q}$-lattices in the same $\mathbb{R}$-line

<u>Deligne's conjecture</u>: $\alpha_M\left(L(M,0)\cdot\Lambda^{top}M_B^+\right) = \Lambda^{top}(M_{dR}/F^0M_{dR})$.

<u>Examples</u>: $M=\mathbb{Q}(n)=\mathbb{Q}\cdot(2\pi i)^n \circlearrowleft F_\infty$ is the complex conjugation pure wt $-2n$

$$M_B^+ = \begin{cases} 1\text{-dim'l} & \text{if } n=\text{even} \\ 0\text{-dim'l} & \text{if } n=\text{odd}\end{cases} \qquad M_{dR}=\mathbb{Q},\ H^{-n,-n}(M)\neq 0$$

When $n>0$, $\alpha_M : M_B^+\otimes\mathbb{R} \xrightarrow{\sim} \underbrace{M_{dR}/F^0M_{dR}\otimes\mathbb{R}}_{1\text{-dim'l}}$

if and only if n is even,

in this case $\alpha_M : \mathbb{Q}(2\pi i)^n\otimes\mathbb{R} \xrightarrow{\sim} \mathbb{Q}\otimes\mathbb{R}$

$$(2\pi i)^n \longmapsto (2\pi i)^n\cdot 1$$

So $L(\mathbb{Q}(n),0)=\zeta(n)\in(2\pi i)^n\cdot\mathbb{Q}^\times$

When $n<0$, $\alpha_M : M_B^+\otimes\mathbb{R} \xrightarrow{\sim} \underbrace{M_{dR}/F^0M_{dR}\otimes\mathbb{R}}_{0\text{-dim'l}}$

if and only if n odd.

In this case $\det(\alpha_M)=1 \Rightarrow L(\mathbb{Q}(n),0)=\zeta(n)\in\mathbb{Q}^\times$ at negative odd integers

* <u>Return to the proof of Deligne's theorem</u>:

When $w(M)\geqslant 0$, $\quad 0$ is critical for M $\qquad$ Same for $\alpha_{M^\vee(1)}$

$\checkmark \Updownarrow \qquad\qquad\qquad\qquad \Updownarrow ??$

0 is critical for $M^\vee(1)$ $\xLeftrightarrow[\checkmark]{w(M^\vee(1))\leqslant -1}$ α_M is a map between v.s. of same dimension

<u>Note</u>: $M_{dR}\otimes\mathbb{R} = (M_{dR}\otimes\mathbb{C})^{1\otimes c=1} = (M_B\otimes\mathbb{C})^{F_\infty\otimes c=1} = M_B^+\otimes\mathbb{R}\oplus M_B^-\otimes\mathbb{R}(1)$

<u>Rewrite</u>:

$$\begin{tikzcd}
0 \arrow[r] & M_B^+\otimes\mathbb{R} \arrow[r] \arrow[drr, "\alpha_M"] & (M_B\otimes\mathbb{C})^{F_\infty\otimes c=1} \arrow[r] \arrow[d, "\wr\ \text{Betti-de Rham comparison}"] & M_B^-\otimes\mathbb{R}\cdot 2\pi i \arrow[r] & 0 \\
0 \arrow[r] & F^0M_{dR}\otimes\mathbb{R} \arrow[r] \arrow[urr, "\alpha^*_{M^\vee(1)}"] & M_{dR}\otimes\mathbb{R} \arrow[r] & (M_{dR}/F^0M_{dR})\otimes\mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

Indeed, take $(\ )^\vee(1)$ of the above diagram gives the diagram for $M^\vee(1)$

<u>Corollary</u>: $\dim\alpha_M := \dim\operatorname{coker}\alpha_M - \dim\ker\alpha_M = \dim\alpha^*_{M^\vee(1)} = -\dim\alpha_{M^\vee(1)}$

If we write $\delta(M)$ for $\det\left((M_B\otimes\mathbb{C})^{F_\infty\otimes c=1}\xrightarrow{\sim} M_{dR}\otimes\mathbb{R}\right)$,

then $\delta(M)=\det(\alpha_M)\cdot\det(\alpha^*_{M^\vee(1)})^{-1} = \det(\alpha_M)\cdot\det(\alpha_{M^\vee(1)})^{-1}$

dual means transpose, not inverse.

<u>Compatibility of Deligne's conjecture with functional equation</u>:

Write $\alpha\sim\beta$ if $\alpha/\beta\in\mathbb{Q}^\times$

Then Deligne conjectured: $L(M,0)\sim\det(\alpha_M)$

$$L(M^\vee(1),0)\sim\det(\alpha_{M^\vee(1)})$$

<u>Functional equation</u>: $L(M,0)\cdot L_\infty(M,0)=\varepsilon(M,0)\cdot L(M^\vee(1),0)\,L_\infty(M^\vee(1),0)$

$$\delta(M):=\det\left((M_B\otimes\mathbb{C})^{F_\infty\otimes c=1}\longrightarrow M_{dR}\right)$$

$$=(2\pi i)^{-\dim M_B^-}\cdot\det(M_B\otimes\mathbb{C}\longrightarrow M_{dR}\otimes\mathbb{C})$$

$$=(2\pi i)^{-\dim M_B^-}\cdot\det\left(\Lambda^{top}M_B\otimes\mathbb{C}\longrightarrow\Lambda^{top}M_{dR}\otimes\mathbb{C}\right)$$ total weight $=w\cdot\dim M_B$

same as $\mathbb{Q}\left(-\frac{w\cdot\dim M}{2}\right)\otimes\mathbb{C}\longrightarrow\mathbb{C}$

$$=(2\pi i)^{-\dim M_B^- - w\cdot\dim M/2}$$

<u>Archimedean computation</u>: $\Gamma_{\mathbb{R}}(s) \sim \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \sim \begin{cases} \pi^{-\frac{s}{2}} & \text{if } s \text{ even} \\ \pi^{\frac{1-s}{2}} & \text{if } s \text{ odd} \end{cases}$ $\quad (\Gamma(\tfrac{1}{2}) = \sqrt{\pi})$

$$\Gamma_{\mathbb{C}}(s) \sim \pi^{-s}$$

So $L_\infty(M,0) \sim \prod_{p<q} \pi^{p \cdot \dim H^{pq}} \cdot \prod_p \left(\Gamma_{\mathbb{R}}(-p)^{\dim H^{pp, F_\infty = (-1)^p}} \cdot \Gamma_{\mathbb{R}}(-p+1)^{\dim H^{pp, F_\infty = (-1)^{p+1}}} \right)$

$$L_\infty(M^\vee(1),0) \sim \prod_{p<q} \pi^{-(1+q)\cdot \dim H^{pq}} \prod_p \left(\Gamma_{\mathbb{R}}(p+1)^{\dim H^{pp, F_\infty = (-1)^p}} \Gamma_{\mathbb{R}}(p+2)^{\dim H^{pp, F_\infty = (-1)^{p+1}}} \right)$$

$$\Rightarrow \frac{L_\infty(M,0)}{L_\infty(M^\vee(1),0)} \sim \prod_{p<q} \pi^{(p+q+1)\cdot \dim H^{pq}} \prod_p \begin{cases} p \text{ odd} & \pi^{(p+1)\dim H^{pp, F_\infty=(-1)^p}} \cdot \pi^{p \cdot \dim H^{pp, F_\infty = (-1)^{p+1}}} \\ p \text{ even} & \pi^{p \cdot \dim H^{pp, F_\infty = (-1)^p}} \cdot \pi^{(p+1)\cdot \dim H^{pp, F_\infty = (-1)^{p+1}}} \end{cases}$$

$$\sim \pi^{\dim(M)\cdot w/2 + \dim M_B^-}$$

<u>Expect</u>: $\varepsilon(M,0) = \dfrac{L(M,0)\, L_\infty(M,0)}{L(M,0)\, L_\infty(M^\vee(1),0)} \sim i^{\dim M \cdot w/2 + \dim M_B^-}$

But $\varepsilon(M,0) = \varepsilon(\underbrace{\Lambda^{top} M}_{\text{of weight } \dim M \cdot w}, 0)$

E.g. for $\eta : \mathrm{Gal}_{\mathbb{Q}} \to \{\pm 1\}$, $\varepsilon(\mathbb{Q}(n)\cdot\eta, 0) = i^{-n} \cdot \eta(\text{cplx conj})$. <u>later</u>

§ <u>Motives with coefficients</u>

$E :=$ a number field. Two ways to define motives over k with coefficients in E: $\quad \mathrm{Mot}_E/k$

<u>Option A</u>: A motive $X \in \mathrm{Mot}/k$ together with $E \to \mathrm{End}(X)$

<u>Option B</u>: Start with SmProj/k, but consider morphism as $\mathrm{Corr}^0(X,X) \otimes E$.

Then take "Karoubian closure" i.e. objects (X,e) for $e \in \mathrm{Corr}^0(X,X)\otimes E$ s.t. $e^2 = e$.

Finally invert $\mathbb{Z}(-1)$

In fact, they gave the same category:

<u>Option B $\Rightarrow$ Option A</u>: For $M \in Mot/k$, and V = finite dim'l $\mathbb{Q}$-v.s.

define Serre tensor $M \otimes V$ = $\dim V$ copies of M

then $\mathrm{Hom}(Y, X \otimes V) := \mathrm{Hom}(Y, X) \otimes V$

We have $Mot_E^B/k \longrightarrow Mot_E^A/k$

$$M_E \longmapsto M \otimes E$$

<u>Option A $\Rightarrow$ Option B</u>: If $M \in Mot_E^A/k$, $E \xrightarrow{act} \mathrm{End}_{Mot/k}(M)$ $\quad E \circlearrowleft act$

then viewing M naïvely in Mot_E^B/k, $\underbrace{\mathrm{End}_{Mot_E^B/k}(M)}_{\text{This is an } E\otimes E\text{-module}} = \mathrm{End}_{Mot/k}(M) \otimes E$

$\rightsquigarrow$ e idempotent in $E \otimes E$ cuts out $E \otimes E \xrightarrow{mult} E$ $\quad (\mathrm{Spec}\, E \overset{\Delta}{\hookrightarrow} \mathrm{Spec}\, E \otimes E)$

The correct object in Mot_E^B/k is (M, e).

If $M \in Mot_E/\mathbb{Q}$, there are realizations

- <u>étale realization</u>: For each ℓ-adic place λ of E

$$Mot_E/\mathbb{Q} \longrightarrow E_\lambda[Gal_{\mathbb{Q}}]\text{-Mod}$$

$$M \longmapsto M_{et,\lambda}$$

Put $M_{et,\ell} = \bigoplus_{\lambda \mid \ell} M_{et,\lambda} \circlearrowleft E \otimes \mathbb{Q}_\ell$

To define L-functions, we hope to consider

$$\forall p \neq \ell \qquad L_p(\sigma, M, s) := \frac{1}{\sigma\left(\det\left(1 - \phi_p \cdot p^{-s};\, M_{et,\lambda}^{I_{\mathbb{Q}_p}}\right)\right)}$$

where $\det(1 - \phi_p \cdot T; M_{et,\lambda}^{I_{\mathbb{Q}_p}}) \in E[T]$, $\sigma: E \to \mathbb{C}$ is an embedding.

i.e. $\forall \sigma: E \hookrightarrow \mathbb{C}$ embedding, we can define

$$L(\sigma, M, s) := \prod_p L_p(\sigma, M, s)$$

Collectively, $L(M,s) := \left(L(\sigma,M,s)\right)_{\sigma\in\mathrm{Hom}(E,\mathbb{C})} \in E\otimes\mathbb{C}$.

- <u>de Rham realization</u>: $\mathrm{Mot}_E/\mathbb{Q} \longrightarrow \text{Fil-Mod}/\mathbb{Q}\otimes E$

$$M \longmapsto M_{dR}$$

(This is in fact rather subtle, suppose consider Mot_E/k for $k \supseteq E^{Gal}$.

then M_{dR} is a filtered module $/ k\otimes E \simeq \prod_{\sigma: E\to k} k$

$M_{dR} \simeq \bigoplus M_{dR,\sigma}$ ← filtration on each $M_{dR,\sigma}$ is quite independent.

But if M comes from $\mathrm{Mot}_E/\mathbb{Q}$; $h^{pq}(M_{dR,\sigma})$ should be independent of σ.)

- <u>Betti realization</u> M_B is an E-vector space

$\circlearrowleft$ F_∞ E-linear map.

<u>Interesting lemma</u>. $L_\infty(\sigma, M, s)$ is independent of $\sigma\in\mathrm{Hom}(E,\mathbb{C})$!

This is because $h^{pq}(M_{dR,\sigma})$ is independent of σ

and $h^{pp^+}(M_{dR,\sigma}) = \dim M^+_{B,\sigma} - \sum_{q<p} h^{pq}(M_{dR,\sigma})$. $\square$

Can define $\alpha_M: M_B^+\otimes\mathbb{R} \longrightarrow (M_B\otimes\mathbb{C})^{F_\infty\otimes c = 1} \cong M_{dR}\otimes\mathbb{R} \longrightarrow M_{dR}/F^0M_{dR}\otimes\mathbb{R}$

Then $\det\alpha_M \in (E\otimes\mathbb{R})^\times/E^\times$

<u>Conjecture</u> (Deligne) Say 0 is <u>critical</u> for M if $\dim_E M_B^+ = \dim_E M_{dR}/F^0M_{dR}$

In this case, we expect. $L(M,0)\in\det\alpha_M\cdot E^\times$

Example from (nontrivial) primitive Dirichlet characters $\eta: (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow E^\times$ $\quad E\subseteq\mathbb{Q}^{cyc}$

Put $$f_\eta(t) := \frac{\sum_{n=1}^{N-1}\eta(n)\cdot e^{-nt}}{1-e^{-Nt}} \in E[[t]]$$

When $n\in\mathbb{Z}_{\geq 0}$, $\eta(-1) = (-1)^{n+1}$, $L(\eta,-n) = (-1)^n f_\eta^{(n)}(0)\in E^\times$.

(This corresponds to $\alpha_M: M_B^+ \longrightarrow M_{dR}/F^0M_{dR}$ is $0\longrightarrow 0$).

Functional equation $\Rightarrow$ for $n \in \mathbb{Z}_{\geq 0}$, $\eta(-1) = (-1)^n$,

$$\frac{L(\eta, n)}{(2\pi i)^n} \in \underbrace{G(\eta)}_{\text{why } G(\eta)?} \cdot E^\times$$

$M = E \cdot \eta$ is part of $\operatorname{Spec} \mathbb{Q}(\mu_N) \otimes E$ where $(\mathbb{Z}/N\mathbb{Z})^\times$ acts by η

$$\operatorname{Spec} \mathbb{Q}(\mu_N) \circlearrowleft (\mathbb{Z}/N\mathbb{Z})^\times$$

(First assume $E = \mathbb{Q}$)

$$M_B = H^0(\operatorname{Spec} \mathbb{Q}(\mu_N) \otimes \mathbb{C}, \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} M_{dR} = H^0(\operatorname{Spec} \mathbb{Q}(\mu_N), \mathcal{O}) \otimes \mathbb{C}$$

$\mathbb{Q}$-basis comes from a subset of ζ_N^i

$$= \bigoplus_{\sigma_j : \mathbb{Q}(\mu_N) \to \mathbb{C}} \mathbb{C} \cdot [\sigma_j] \qquad\qquad \simeq \prod_{\sigma_j : \mathbb{Q}(\mu_N) \to \mathbb{C}} \mathbb{C}$$

$\cup\!\mathrm{I}$ η-part

$$\sum_{j \in (\mathbb{Z}/N\mathbb{Z})^\times} \eta(j) [\sigma_j] \cdot \mathbb{C} \longmapsto \Big(\sum_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \eta^{-1}(i) \zeta_N^{ij} \cdot \mathbb{C} \Big)_{\sigma_j}$$

$= \eta(j) \cdot G(\eta^{-1})$

$$\sigma_j \longmapsto 1 \text{ in } \sigma_j \text{ but } 0 \text{ otherwise}$$