Special values of L-functions 19

—— Periods of algebraic Hecke characters III

- $e^{\sqrt{163}\pi} = 262537412640768743.99999999999925\cdots \approx 640320^3+744$

$j: X(1) \cong \mathbb{P}^1, \quad j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$

$j\left(\frac{1+\sqrt{-163}}{2}\right) = (-640320)^3, \quad q = e^{2\pi i \tau} = -e^{-\sqrt{163}\pi}$

So, $-640320^3 = -e^{\sqrt{163}\pi} + 744 - 196884e^{-\sqrt{163}\pi} + \cdots$

- For simplicity, let $K/\mathbb{Q}$ be an imaginary quadratic field with class number 1. s.t. $\mathcal{O}_K^\times = \{\pm 1\}$

E.g. $K = \mathbb{Q}(\sqrt{-163}) = \mathbb{Z} \oplus \mathbb{Z}\tau_0$

Fix an embedding $K \hookrightarrow \mathbb{C}$ and write σ for complex conjugation.

Fix $\alpha \geq 1$, $\beta \geq 0$, $\rightsquigarrow$ unramified algebraic "$\mathbb{Q}$-Hecke character"

$$\chi^{\mathbb{Q}}: \mathbb{A}_{K,f}^\times / \hat{\mathcal{O}}_K^\times \longrightarrow K^\times$$

$$\begin{array}{ccc} \rotatebox{90}{$\cong$} & & \\ K^\times/\{\pm 1\} & \gamma \longmapsto \gamma^{-\alpha}\cdot\sigma(\gamma)^\beta \end{array} \qquad (\text{maybe require } 2 \mid \beta-\alpha.)$$

$\rightsquigarrow \chi = \chi_{\alpha,\beta}: K^\times \backslash \mathbb{A}_K^\times / \hat{\mathcal{O}}_K^\times \longrightarrow \mathbb{C}^\times$ s.t. for each principal ideal (γ)

$$\chi((\gamma)) = \gamma^{-\alpha}\cdot\sigma(\gamma)^\beta$$

The associated L-function is
$$L(\chi, s) = \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_K \\ \text{prime}}} \frac{1}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}} = \sum_{\substack{0\neq I=(\gamma)\subseteq\mathcal{O}_K \\ \text{ideal}}} \frac{\chi((I))}{N(I)^s}$$

$$= \frac{1}{2}\sum_{\gamma\in\mathcal{O}_K\setminus\{0\}} \frac{\bar{\gamma}^{\beta-s}}{\gamma^{\alpha+s}} = \frac{1}{2}\sum_{c,d\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{(c\bar{\tau}_0+d)^{\beta-s}}{(c\tau_0+d)^{\alpha+s}}$$

<u>Rmk</u>: when $s\in\mathbb{Z}$, $L(\chi_{\alpha,\beta}, s) = L(\chi_{\alpha+s,\beta-s}, 0)$

<u>Remark</u>: $L(\chi_{\alpha,\beta}, s)$ is critical when $s\in[1-\alpha, \beta]\cap\mathbb{Z}$.

* Will assume $\alpha \geqslant \beta+4$ and $2 \mid \alpha-\beta$ to avoid technical issues.

$k := \alpha-\beta, \quad r := \beta$

- Eisenstein series even weight $k \geqslant 4$, (no level so weight is even)

$$G_k(\tau) = \sum_{(a,b)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(a+b\tau)^k} = 2\zeta(k)\cdot\underbrace{\left(1+\frac{2}{\zeta(1-k)}\sum_{n=1}^{+\infty}\sigma_{k-1}(n)q^n\right)}_{E_k(\tau).}$$

for $\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1}$.

- <u>Non-holomorphic version</u>:

$$E_{k,r}(\tau,s) := (\tau-\bar{\tau})^{-r}(2\pi i)^{-(k+r)}\cdot\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0,0\}}\left(\frac{c\bar{\tau}+d}{c\tau+d}\right)^r (c\tau+d)^{-k}\,|c\tau+d|^{-2s}$$

<u>note</u>: The additional factor of $2\pi i$ is to keep rationality (e.g. when $r=0$)

Weight of $E_{k,r}$? For $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$, $\operatorname{Im}(\gamma\tau) = \dfrac{\operatorname{Im}(\tau)}{|c\tau+d|^2}$

$$\gamma(C\tau+D) = C\cdot\frac{a\tau+b}{c\tau+d}+D = \frac{(C\ D)\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\tau\\ 1\end{pmatrix}}{c\tau+d}$$

So the weight factor of $E_{k,r}$ is $\quad E_{k,r}\left(\frac{a\tau+b}{c\tau+d},s\right) = ?\ E_{k,r}(\tau,s)$

$$? = |c\tau+d|^{2r}\cdot\left(\frac{c\bar{\tau}+d}{c\tau+d}\right)^{-r}(c\tau+d)^k\,|c\tau+d|^{2s} = (c\tau+d)^{k+2r}\cdot|c\tau+d|^{2s}$$

When $s=0$, $E_{k,r}(\tau,0)$ is a non-holomorphic modular form of wt $k+2r$.

Then for $\tau=\tau_0$, $k=\alpha-\beta$, $r=\beta$, we have

$$E_{k,r}(\tau_0,0) = \underbrace{(2\operatorname{Im}\tau_0)^{-\beta}}_{\sqrt{163}^{-\beta}}\cdot(2\pi i)^{-\alpha}\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0,0\}}\frac{(c\bar{\tau}+d)^r}{(c\tau+d)^{r+k}} = \sqrt{163}^{-\beta}\cdot(2\pi i)^{-\alpha}\cdot L(\chi_{\alpha,\beta},0)$$

"Correction" from last time (rationality statement)

- $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau \quad \mathcal{E}_U$ The elliptic curve E_τ has the following algebraic equation

$$\begin{array}{cc} \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau & \mathcal{E}_U \\ | & | \\ \tau \in & U \end{array}$$

$$X := \wp(z) = \frac{1}{z^2} + \sum_{\lambda\in(\mathbb{Z}\oplus\mathbb{Z}\tau)\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

$$Y := \wp'(z) = -2\sum_{\lambda\in(\mathbb{Z}\oplus\mathbb{Z}\tau)} \frac{1}{(z-\lambda)^3}$$

Equation for $\mathcal{E}_U$ is $Y^2 = 4X^3 - 60\,G_4(\tau)\,X - 140\,G_6(\tau)$

But $G_4 = 2\zeta(4)E_4 = \frac{\pi^4}{45}E_4$, $G_6 = 2\zeta(6)E_6 = \frac{2\pi^6}{945}E_6$ <u>Upshot</u>: E_4 and E_6 are rational.

Put $Y = (2\pi i)^3 y$, $X = (2\pi i)^2 x \rightsquigarrow y^2 = * x^3 + * E_4(\tau) x + * E_6 \leftarrow$ defined $/\mathbb{Q}$

$$dz = \frac{dX}{Y} = (2\pi i)^{-1}\frac{dx}{y}$$

So $2\pi i\, dz$ is defined over $\mathbb{Q}$

- <u>Kodaira-Spencer isomorphism</u>

$$0 \to \omega_{\bar{\mathcal{E}}} \to \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \to \text{quot} \to 0$$

$$\text{quot} = \underbrace{\Lambda^2 \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X)}_{} \otimes \omega_{\bar{\mathcal{E}}}^{-1}$$

$\uparrow$ maybe it's better to keep as is

Then
$$\begin{tikzcd} \omega_{\bar{\mathcal{E}}} \subseteq \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \arrow[r, "\nabla_{GM}"] \arrow[dr, "K\text{-}S"'] & \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \otimes \Omega^1_X(\log C) \arrow[d] \\ & \Lambda^2 \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \otimes \omega_{\bar{\mathcal{E}}}^{-1} \otimes \Omega^1_X(\log C) \end{tikzcd}$$

So $\Omega^1_X(\log C) \cong \omega_{\bar{\mathcal{E}}}^{\otimes 2} \otimes \Lambda^2 \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X)^{\otimes -1}$ more canonical this way.

- If we want to trivialize $\Lambda^2 \mathcal{H}^1_{dR}(\mathcal{E}/Y) \otimes \mathbb{C} \simeq \Lambda^2 \mathcal{H}^1_B(\mathcal{E}/Y) \otimes \mathbb{C}$

At each point $y \in Y$, $\underbrace{\Lambda^2 H^1_{dR}(\mathcal{E}_y)}_{\mathbb{Q}(-1)_{dR}} \otimes \mathbb{C} \simeq \underbrace{\Lambda^2 H^1_B(\mathcal{E}_y)}_{\mathbb{Q}(-1)_B} \otimes \mathbb{C}$

$\eta \longmapsto 2\pi i\, S \wedge T \quad \| \quad 2\pi i\, dz \longmapsto 2\pi i\,(S + \tau T)$

Recall the differential operator $\mathbb{D}^{(r)} := \mathbb{D}_{k+2r-2}\circ\cdots\circ\mathbb{D}_{k+2}\circ\mathbb{D}_k$

$$\omega^k \subseteq \mathrm{Sym}^k H^1_{dR}(\mathcal{E}/Y) \longrightarrow \mathrm{Sym}^{k+2r} H^1_{dR}(\mathcal{E}/Y) \twoheadrightarrow \omega^{k+2r}$$

— Hodge projection

$$f(\tau)\otimes(2\pi i\,dz)^k \longmapsto (2\pi i)^{-r}\left(\frac{\partial}{\partial\tau}+\frac{k+2r-2}{\tau-\bar\tau}\right)\circ\cdots\circ\left(\frac{\partial}{\partial\tau}+\frac{k+2}{\tau-\bar\tau}\right)\circ\left(\frac{\partial}{\partial\tau}+\frac{k}{\tau-\bar\tau}\right)(f)\otimes(2\pi i\,dz)^{k+2r}$$

<u>Claim</u>: $(2\pi i)^{-1}\left(\frac{\partial}{\partial\tau}+\frac{k+2r+s}{\tau-\bar\tau}\right)\left(E_{k,r}(\tau,s)\right) = (r+k+s)\cdot E_{k,r+1}(\tau,s)$

Proof: $$(2\pi i)^{-1}\left(\frac{\partial}{\partial\tau}+\frac{k+2r+s}{\tau-\bar\tau}\right)\left((\tau-\bar\tau)^{-r}(2\pi i)^{-(k+r)}\cdot\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0\}}\left(\frac{c\bar\tau+d}{c\tau+d}\right)^{r}(c\tau+d)^{-k}|c\tau+d|^{-2s}\right)$$

$$=(2\pi i)^{-(k+r+1)}\left(\frac{\partial}{\partial\tau}+\frac{k+2r+s}{\tau-\bar\tau}\right)\left((\tau-\bar\tau)^{-r}\cdot\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0\}}\frac{(c\bar\tau+d)^{r-s}}{(c\tau+d)^{r+k+s}}\right)$$

$$=(2\pi i)^{-(k+r+1)}\Bigg[(k+2r+s)\cdot(\tau-\bar\tau)^{-r-1}\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0\}}\frac{(c\bar\tau+d)^{r-s}}{(c\tau+d)^{r+k+s}}+(-r)\cdot(\tau-\bar\tau)^{-r-1}\cdot\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0\}}\frac{(c\bar\tau+d)^{r-s}}{(c\tau+d)^{r+k+s}}$$

$$+(\tau-\bar\tau)^{-r}\cdot\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0\}}-(r+k+s)\cdot\frac{(c\bar\tau+d)^{r-s}}{(c\tau+d)^{r+k+s+1}}\cdot \underbrace{c}_{c=\frac{(c\tau+d)-(c\bar\tau+d)}{\tau-\bar\tau}}\Bigg]$$

Note: $k+2r+s-r+(-(r+k+s))=0$

$$=(2\pi i)^{-(k+r+1)}\cdot(r+k+s)\cdot(\tau-\bar\tau)^{-r-1}\sum_{(c,d)\in\mathbb{Z}^2\setminus\{0\}}\frac{(c\bar\tau+d)^{r+1-s}}{(c\tau+d)^{r+k+s+1}}$$

$$=(r+k+s)\cdot E_{k,r+1}(\tau,s)$$

<u>Corollary</u>: $\mathbb{D}^{(r)}\left(E_k(\tau,s)\otimes(2\pi i dz)^{\otimes k}\right)=(k+s)(k+1+s)\cdots(k+(r-1)+s)\cdot E_{k,r}(\tau,s)\otimes(2\pi i dz)^{\otimes(k+r)}$

• <u>Conclusion</u>: $$\omega^k \subseteq \mathrm{Sym}^k H^1_{dR}(\mathcal{E}/Y)\xrightarrow{\mathbb{D}^{(r)}}\mathrm{Sym}^{k+2r}H^1_{dR}(\mathcal{E}/Y)\xrightarrow{\mathrm{eval}_{\tau_0}}\mathrm{Sym}^{k+2r}H^1_{dR}(\mathcal{E}_{\tau_0})$$

$E_k(\tau)\otimes(2\pi i dz)^{\otimes k} \in \omega^k$

Take eigenspace

$$\begin{tikzcd} \omega_{\mathcal{E}}^{k+2r} \arrow[r, "\mathrm{eval}_{\tau_0}"] & \omega_{\mathcal{E}_{\tau_0}}^{k+2r} \\ (*)\, E_{k,r}(\tau)\otimes(2\pi i dz)^{\otimes(k+2r)} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (*)\, E_{k,r}(\tau_0)\otimes(2\pi i dz)^{\otimes(k+2r)} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

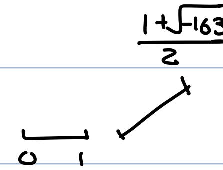
for $\mathbb{Q}(\sqrt{-163})$-action

By construction, $\dfrac{L(\chi_{\alpha,\beta},0)}{(2\pi i)^{k+r}} \underset{\overline{\mathbb{Q}}^\times}{\sim} E_{k,r}(\tau_0)$

$r=\beta,\ k+r=\alpha$

Let ω_0 denote a $\underset{= \mathbb{Q}(\sqrt{-163})}{K(P_{\tau_0})}$-basis of $H^0(\mathcal{E}_{\tau_0}, \omega_{\mathcal{E}_{\tau_0}})$

$H^1(\mathcal{E}_{\tau_0}, \mathbb{Q}) = \mathbb{Q}(\sqrt{-163}) = \mathbb{Q}S \oplus \mathbb{Q}T$ for S, T dual to



Then $\omega_0 = \Omega S$ i.e. $\Omega := \int_0^1 \omega_0$

The rationality condition says $E_{k,r}(\tau_0)\otimes(2\pi i dz)^{\otimes(k+2r)} \underset{\overline{\mathbb{Q}}^\times}{\sim} \omega_0^{\otimes k+2r}$

integration along $\int_0^1 \Rightarrow E_{k,r}(\tau_0)\cdot(2\pi i)^{k+2r} \underset{\overline{\mathbb{Q}}^\times}{\sim} \Omega^{k+2r}$

So we have $L(\chi_{\alpha,\beta},0) \underset{\overline{\mathbb{Q}}^\times}{\sim} (2\pi i)^{k+r} E_{k,r}(\tau_0) \underset{\overline{\mathbb{Q}}^\times}{\sim} \dfrac{\Omega^{k+2r}}{(2\pi i)^r}$ $\square$

Serre's new construction of p-adic zeta function

Slogan: To construct p-adic L-functions attached to algebraic Hecke character of a totally real field, can use Eisenstein Hilbert modular forms.

We will only explain this for the case of ζ-functions

Observe the expression of Eisenstein series

$$E_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{+\infty} \sigma_{k-1}(n) q^n \in M_k(\Gamma(1))$$

$$E_k^{(p)}(\tau) := E_k(\tau) - p^{k-1} E_k(p\tau)$$

Note: $\zeta(1-k) = \zeta^{(p)}(1-k)\cdot \dfrac{1}{1-p^{-(1-k)}} = \zeta^{(p)}(1-k)\cdot\dfrac{1}{1-p^{k-1}}$

$$\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1}$$

For $n = n_0 p^r$ ($r \in \mathbb{Z}_{\geq 0}$), the q^n-coeff. of $E_k^{(p)}(\tau)$ is

$$\sum_{d|n} d^{k-1} - \underbrace{p^{k-1} \cdot \sum_{d' | \frac{n}{p}} d'^{k-1}}_{= \sum_{p|d|n} d^{k-1}} = \sum_{\substack{d|n \\ p\nmid d}} d^{k-1} =: \sigma_{k-1}^{(p)}(n)$$

$$E_k^{(p)}(\tau) = \frac{\zeta^{(p)}(1-k)}{2} + \sum_{n=1}^{+\infty} \sigma_{k-1}^{(p)}(n)\, q^n$$

"Think of $k-1$ as a character/function" $x^{k-1} : \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p^\times$

Define $\mathcal{E}^{(p)}(\tau) \in \mathcal{D}(\mathbb{Z}_p^\times, \mathbb{Z}_p[\![q]\!])$ as

viewed as an element in $\mathcal{D}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, delta measure.

$$\mathcal{E}^{(p)}(\tau) = \frac{\zeta_p^{KL,new}}{2} + \sum_{n=1}^{+\infty} q^n \cdot \Big(\sum_{\substack{d|n \\ p\nmid d}} \mu_d \Big) \qquad \int x^{k-1} \mu_d = d^{k-1}$$

<u>Upshot</u>: Except the first term, all other terms belong to $\mathcal{D}(\mathbb{Z}_p^\times, \mathbb{Z}_p)[\![q]\!]$ obvious.

This forces the first term to be p-adically interpolatable.

<u>Picture</u>: $X_{\mathbb{F}_p}^{ord} \subseteq \mathfrak{X}_{\mathbb{F}_p} \rightsquigarrow \mathfrak{X}_{\mathbb{Z}_p}^{ord} \subseteq \mathfrak{X}_{\mathbb{Z}_p} / \mathbb{Z}_p$ ordinary part of modular curve.

$$\begin{array}{ccc}
\exists\, \mathcal{E}is^{(p)} \in H^0(\mathfrak{X}_{\mathbb{Z}_p}^{ord}, \mathcal{O} \hat{\otimes} \mathcal{D}(\mathbb{Z}_p^\times, \mathbb{Z}_p))^{U_p\text{-ord}} & \longrightarrow & \mathcal{D}(\mathbb{Z}_p^\times, \mathbb{Z}_p)[\![q]\!] \\
\big\downarrow \prod_k \int x^k & & \big\downarrow \\
E_k^{(p)} \in \prod_k H^0(\mathfrak{X}_{\mathbb{Z}_p}^{ord}, \omega^k)^{U_p\text{-ord}} & \longrightarrow & \prod_k \mathbb{Z}_p[\![q]\!]
\end{array}$$

The constant term of $\mathcal{E}is^{(p)}$ is ζ_p^{KL}.

<u>Philosophy/Dream</u>: There are two types of "cusps" of modular curves

(1) cusps: $E \rightsquigarrow$ nodal curve

(2) CM pts: $E \rightsquigarrow$ CM elliptic curves.

<u>Expect</u>: Restricting $\mathcal{E}is^{(p)}$ to CM elliptic curves gives p-adic L-functions for CM Hecke characters