Introduction to p-adic Hodge theory 10

—— Origin of (φ,Γ)-modules

Study representations of $\mathrm{Gal}_{\mathbb{Q}_p}$:

① $\mathbb{C}$-coeff $\Rightarrow$ finite image rep'n $\rho: \mathrm{Gal}_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q}_p) \hookrightarrow GL_N(\mathbb{C})$, $K/\mathbb{Q}_p$ finite

② $\mathbb{Q}_\ell$-coeff $(\ell \neq p)$ $\rightsquigarrow$ Grothendieck's monodromy theorem, (r, N)

③ $\mathbb{Q}_p$-coeff complicated

<u>Goal</u>: Use (semi) linear algebra data to express p-adic representations of $\mathrm{Gal}_{\mathbb{Q}_p}$.

§1. <u>mod p representations of Gal_E when char $E = p$</u>

- Let E be a field of char p, <u>not</u> necessarily perfect. e.g. $E_{\mathbb{Q}_p} = \mathbb{F}_p((T))$
- $\mathrm{Gal}_E := \mathrm{Gal}(E^{sep}/E)$
- $\varphi: E \to E \quad x \mapsto x^p$ Frobenius

<u>Definition</u> A <u>φ-module</u> over E is a finite dimensional vector space V over E, together with a non-degenerate E-semilinear homomorphism $\varphi: V \to V$, i.e. $\Phi: V \otimes_{E,\sigma} E \xrightarrow{\sim} E$

$\hookrightarrow$ means $\varphi(av) = \varphi(a)\varphi(v)$ for $a \in E$, $v \in M$.

(Pick a basis $\underline{e}$ of V, φ is given by $P \in GL_d(E)$.

Change to another basis using matrix A, $\rightarrow \underline{e}A$

$$\varphi(\underline{e}A) = \varphi(\underline{e}) \cdot \varphi(A) = \underline{e}P \cdot \varphi(A) = \underline{e}A \cdot A^{-1}P\varphi(A)$$

<u>Theorem</u> We have an equivalence of categories

$$\mathrm{Rep}_{\mathbb{F}_p}(\mathrm{Gal}_E) \longleftrightarrow \varphi\text{-}\mathrm{mod}_E$$

$$V \longmapsto \mathbb{D}(V) = (V \otimes_{\mathbb{F}_p} E^{sep})^{\mathrm{Gal}_E}$$

$$\mathbb{V}(D) = (D \otimes_E E^{sep})^{\varphi=1} \longleftarrow D$$

Moreover, it preserves tensors and duals.

(For $D \in \varphi\text{-}\underline{mod}_E$, define $D^* := \mathrm{Hom}_E(D, E)$ $\quad \varphi(f)(v) = f(\varphi(v))$.)

Proof: ① Let $d = \dim V$. <u>Claim</u>: $V \otimes_{\mathbb{F}_p} E^{sep}$ is isomorphic to $(E^{sep})^d$ as Gal_E-module

Pick a basis of $V \otimes_{\mathbb{F}_p} E^{sep}$ and let $a_g \in GL_d(E^{sep})$ be the matrix for action of $g \in \mathrm{Gal}_E$

Then $\quad \underline{e}\, a_{gh} = gh(\underline{e}) = g(\underline{e}\, a_h) = \underline{e}\, a_g\, g(a_h) \Rightarrow a_{gh} = a_g \cdot g(a_h)$

This is a cocycle in $H^1(\mathrm{Gal}_E, GL_d(E^{sep}))$

By Hilbert 90', $H^1(\mathrm{Gal}_E, GL_d(E^{sep})) = \{1\}$

$$\Rightarrow \exists A \in GL_d(E^{sep}) \text{ s.t. } \forall g \in \mathrm{Gal}_E,\ a_g = A\, g(A)^{-1} \Rightarrow A^{-1} a_g\, \varphi(A) = 1$$

So $V \otimes_{\mathbb{F}_p} E^{sep}$ admits a basis of Gal_E-invariant vectors

$$\Rightarrow \dim_E \big(\underset{\substack{\circlearrowleft \\ \varphi = \mathrm{triv}}}{V} \otimes_{\mathbb{F}_p} \underset{\substack{\circlearrowleft \\ \varphi}}{E^{sep}}\big)^{\mathrm{Gal}_E} = d.$$

$\Rightarrow \mathbb{D}(V)$ admits a φ-semilinear action from E^{sep}.

$$\text{Moreover, } \mathbb{D}(V) \otimes_E E^{sep} \xrightarrow{\sim} V \otimes_{\mathbb{F}_p} E^{sep} \qquad (1)$$

② Let $d = \dim_E D$. We need to show that $\dim_{\mathbb{F}_p} (D \otimes_E E^{sep})^{\varphi=1} = d$.

Take a basis of $D \rightsquigarrow \varphi$-action given by $P \in GL_n(E)$

Want to find $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in (E^{sep})^n = D \otimes_E E^{sep}$ s.t.

$$P \cdot \varphi(\underline{v}) = \underline{v}$$

This is equivalent to finding an E^{sep}-point of $E[v_1, \cdots, v_n] \Big/ \left(\begin{pmatrix} v_1^p \\ \vdots \\ v_n^p \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right)$

But this is an étale E-algebra by Jacobian criterion

$$\left(\partial f_i / \partial v_j \right) = P^{-1} \text{ is invertible}$$

$$\Rightarrow \text{We have } p^d \text{ solutions in } E^{sep} \Rightarrow \dim_{\mathbb{F}_p} \underbrace{(D \otimes_E E^{sep})^{\varphi=1}}_{=: \mathbb{V}(D)} = d$$

$$\text{Moreover, } \mathbb{V}(D) \otimes_{\mathbb{F}_p} E^{sep} \xrightarrow{\sim} D \otimes_E E^{sep} \qquad (2)$$

Wrap up: $\mathbb{D}(\mathbb{V}(D)) = \left((D \otimes_E E^{sep})^{\varphi=1} \otimes_{\mathbb{F}_p} E^{sep} \right)^{Gal_E} \overset{(2)}{\simeq} (D \otimes_E E^{sep})^{Gal_E} = D.$

$\mathbb{V}(\mathbb{D}(V)) = \left((V \otimes_{\mathbb{F}_p} E^{sep})^{Gal_E} \otimes_E E^{sep} \right)^{\varphi=1} \overset{(1)}{\simeq} (V \otimes_{\mathbb{F}_p} E^{sep})^{\varphi=1} = V.$ $\square$

Definition. A Cohen ring for a field E of char p is a (mixed char) CDVR C_E with uniformizer p s.t. $C_E/(p) \simeq E$.

Rmk: (1) When E is perfect, $C_E = W(E)$; in this case, Cohen ring is unique.

(2) A Cohen ring C_E for E is unique (but not up to unique isomorphism)

(3) Frobenius φ lifts to C_E, but not uniquely. (when E is perfect, such lift is unique)

In fact, (C_E, φ), (C_E, φ') determines a unique automorphism

$$\eta : (C_E, \varphi) \longrightarrow (C_E, \varphi')$$

& also a unique embedding $\iota : (C_E, \varphi) \hookrightarrow (W(E^{perf}), \varphi_{W(E^{perf})})$

In example, in $\tilde{\mathbb{E}}_{\mathbb{Q}_p} = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}^\flat \simeq \mathbb{F}_p((\varepsilon - 1))^{perf,\wedge}$

Inside $W(\tilde{\mathbb{E}}_{\mathbb{Q}_p}) = W(\mathbb{F}_p((\varepsilon-1))^{perf,\wedge}) \supseteq \mathbb{Z}_p(([\varepsilon]-1))^\wedge = \mathbb{Z}_p((T))^\wedge$ $\uparrow \varphi(T) = (1+T)^p - 1$

$\cup\!\shortmid$ $\mathbb{Z}_p(([\varepsilon - 1]))^\wedge$, $[\varepsilon - 1] = T'$, $\varphi(T') = T'^p$

They are both Cohen rings; but how φ acts on generators determine the embedding in $\tilde{\mathbb{A}}_{\mathbb{Q}_p}$

Corollary. Let C_E be a Cohen ring of E and pick a Frobenius lift φ on C_E

Then we have an equivalence of tensor categories

$$\underline{Rep}^{(tor)}_{\mathbb{Z}_p}(Gal_E) \xrightarrow{\simeq} \varphi\text{-}\underline{mod}^{(tor),et}_{C_E}$$

$$\underline{Rep}_{\mathbb{Q}_p}(Gal_E) \xrightarrow{\simeq} \varphi\text{-}\underline{mod}^{ét}_{C_E[\frac{1}{p}]}$$

$$V \longmapsto \mathbb{D}(V) = (V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}})^{Gal_E}$$

$$\mathbb{V}(D) = (D \otimes_{C_E} \widehat{C_E^{ur}})^{\varphi=1} \longleftarrow D$$

- $\mathrm{length}_{\mathbb{Z}_p}(V) = \mathrm{length}_{C_E}(D)$, $\dim_{\mathbb{Q}_p}(V) = \dim_{C_E[\frac{1}{p}]}(D)$

Here we need the concept of étale φ-modules

- An étale φ-module over C_E is a finite C_E mod D together w/ an isom $\Phi : \varphi^* D \xrightarrow{\sim} D$

An étale φ-module over $C_E[\frac{1}{p}]$ is a finite dim'l $C_E[\frac{1}{p}]$-mod D with a semilinear φ-action s.t. $\exists$ a φ-stable C_E-lattice D° s.t. D° is an étale φ-module.

(In particular, we cannot take φ to be multiplication by p)

Story of (φ,Γ)-modules

$K_0 = W(k)[\frac{1}{p}]$ for k perfect of char $p>0$

$$\begin{array}{ccccccc}
K_0^{alg} & & \widehat{K_0^{alg,b}} = \widetilde{\mathbb{E}} & = & k((T))^{alg,\wedge} & \supseteq & k((T))^{sep} = \mathbb{E} \\
| & \rightsquigarrow & | \qquad | & & | & & | \quad \Big) H_K \\
K_\infty = K_0(\mu_{p^\infty}) & & \widehat{K_\infty^b} = \widetilde{\mathbb{E}}_{K_0} & = & k((T))^{perf,\wedge} & \supseteq & k((T)) = \mathbb{E}_{K_0} \\
| \Big) \text{ Galois group } \Gamma \cong \mathbb{Z}_p^\times & & & & \overset{\shortparallel}{\varepsilon - 1} & & \\
K_0 & & & & & &
\end{array}$$

For $\gamma \in \Gamma$, $\gamma(T) = (1+T)^\gamma - 1$.

<u>Theorem</u> We have an equivalence of categories

$$\mathrm{Rep}_{\mathbb{F}_p}(\mathrm{Gal}_{K_0}) \xrightarrow{\simeq} (\varphi,\Gamma)\text{-mod}/\mathbb{E}_{K_0}$$

$$V \longmapsto \mathbb{D}(V) = (V \otimes_{\mathbb{F}_p} \mathbb{E}_{K_0}^{sep})^{H_K} \circlearrowleft \varphi, \Gamma$$

$$\mathbb{V}(D) = (D \otimes_{\mathbb{E}_{K_0}} \mathbb{E}_{K_0}^{sep})^{\varphi=1} \longleftarrow D$$

$\circlearrowleft \mathrm{Gal}_{K_0} \twoheadrightarrow \Gamma$ $\qquad \circlearrowleft \mathrm{Gal}_{K_0}$

- More generally,
  - coefficients: $A_{K_0} := W(k)((T))^{\wedge, p\text{-adic}} \subseteq \widetilde{A}_{K_0} = W(k((T))^{perf,\wedge})$

    $B_{K_0} = A_{K_0}[\frac{1}{p}]$,

T stands for $[\varepsilon]-1 \rightsquigarrow \gamma(T) = (1+T)-1$, $\varphi(T) = (1+T)^p - 1$.

* General field K : CDVF of mixed char, k_K = residue fied.

$$\begin{tikzcd} & & K^{alg} \arrow[d, dash, "H_K"] & & & & \mathbb{E} \arrow[d, dash, "H_K"] \\ K_0(\mu_{p^\infty}) \arrow[urr, dash] \arrow[dd, dash] & & K(\mu_{p^\infty}) \arrow[d, dash, "\Gamma_K \subseteq \Gamma"] \arrow[dl, dash] & & \rightsquigarrow & \mathbb{E}_{K_0} \arrow[r, dash] & \mathbb{E}_K \arrow[loop right, "\Gamma_K"] \\ & K_0' = W(k')[\frac{1}{p}] \arrow[r, dash] & K & & & & \\ K_0 = W(k_K)[\frac{1}{p}] \arrow[ur, dash] \arrow[urr, dash] & & & & & & \end{tikzcd}$$

$\uparrow$ residue field k'

Abstractly, $\mathbb{E}_K \simeq k'((\pi_K))$ for some uniformizer π_K

$$\begin{array}{l} \mathbb{E}_K \simeq k'((\pi_K)) \\ \mid \\ \mathbb{E}_{K_0} \simeq k_K((T)) \end{array} \quad \nwarrow \text{ finite separable ext'n}$$

- Note: $\mathrm{Gal}(\mathbb{A}_{K_0}^{ur}/\mathbb{A}_{K_0}) \simeq \mathrm{Gal}(\mathbb{E}_{K_0}^{sep}/\mathbb{E}_{K_0})$

So $\exists!$ $\mathbb{A}_K$ unram. ext'n of $\mathbb{A}_{K_0}$ corresponding to $\mathbb{E}_K$ & φ extends naturally.

Theorem There are equivalence of tensor categories

$$\mathrm{Rep}_{\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p}(\mathrm{Gal}_K) \longleftrightarrow (\varphi, \Gamma_K)\text{-mod}^{et}_{\mathbb{E}_K, \mathbb{A}_K, \mathbb{B}_K} \qquad \mathbb{B}_K = \mathbb{A}_K[\tfrac{1}{p}]$$

Caveat: Abstractly, $\mathbb{A}_K = \mathbb{Z}_p((\pi_K))^\wedge$, but there's no "canonical" lift π_K

& unlike $K = K_0$ case, no guarantee that $\mathbb{A}_K^+ = \mathbb{Z}_p[\![\pi_K]\!]$ is stable under φ, Γ_K.

But $\exists$ some good cases, $K \subseteq K_0^{ur}(\mu_{p^\infty})$, we can still use the "standard" lift π_{K_0}.

From K_0 to an unramified extension $\rightsquigarrow$ change $k_{K_0}((\pi_{K_0}))$ to $k_K((\pi_{K_0}))$

If $K_0 \subset K \subset K_0(\mu_{p^\infty})$, $\mathbb{E}_{K_0} = \mathbb{E}_K$ yet $\Gamma_K \subseteq \Gamma_{K_0} \simeq \mathbb{Z}_p^\times$.

- Compatibility with field extensions L/K

$$\begin{tikzcd} \mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_L) \arrow[r, "\mathbb{D}_L(-)"] \arrow[d, shift left, "\mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_K}(-)"] & (\varphi, \Gamma_L)\text{-mod}^{et}_{\mathbb{A}_L} \arrow[d, shift left, "D \mapsto \mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D"] \\ \mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_K) \arrow[r, "\mathbb{D}_K(-)"] \arrow[u, shift left, "\mathrm{Res}"] & (\varphi, \Gamma_K)\text{-mod}^{et}_{\mathbb{A}_K} \arrow[u, shift left, "D \mapsto D \otimes_{\mathbb{A}_K} \mathbb{A}_L"] \end{tikzcd}$$

Example: 1-dim'l rep'n of $G_{\mathbb{Q}_p}$

$$G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{Q}_p}^{ab} \simeq \widehat{\mathbb{Q}}_p^\times \xrightarrow{\eta} \mathbb{Z}_p^\times \rightsquigarrow \begin{cases} \eta(p) = \alpha \\ \eta|_{\mathbb{Z}_p^\times} \end{cases}$$

geom Frob $\leftrightarrow p$

<u>Claim</u>: $\mathbb{D}(\eta) = \mathbb{A}_{\mathbb{Q}_p} \cdot e$ s.t. $\varphi(e) = \alpha \cdot e$. $\gamma(e) = \eta(\gamma) \cdot e$.

Proof: Write the rep'n as $\mathbb{Z}_p \cdot \mathbb{e}$, s.t. $\forall g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $g(\mathbb{e}) = \eta(g)(\mathbb{e})$

By Hilbert 90, $\exists\, x \in \widehat{\mathbb{A}_{\mathbb{Q}_p}^{ur}}$ s.t. for the arithmetic Frobenius $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p(\mu_{p^\infty}))$

$\sigma(x)/x = \alpha$ (note $\eta(\sigma) = \alpha^{-1}$)

<u>Claim</u>: $e := x\mathbb{e}$ is fixed by $H_{\mathbb{Q}_p}$, b/c $\sigma(x\mathbb{e}) = \alpha x \cdot \alpha^{-1} \mathbb{e} = x\mathbb{e}$

So $\mathbb{D}(\eta) = \mathbb{A}_{\mathbb{Q}_p} \cdot e \circlearrowleft \gamma(e) = \eta(\gamma) \cdot e$

$\varphi(e) = \varphi(\mathbb{e} \otimes x) = \mathbb{e} \otimes \varphi(x) = \mathbb{e} \otimes \alpha x = \alpha \cdot e$. $\square$