Introduction to p-adic Hodge theory 3

—— Analytic functions on p-adic annuli, Ax-Tate-Sen Theorem

Let K be a complete valued field. $v(-): K^\times \to \mathbb{R}$

<u>Theorem</u> (Vertex factorization). Let $f(t) = \sum_{n\in\mathbb{Z}} a_n t^n$ be a formal Laurent series

Suppose that $NP(f)$ has a vertex P_0, then we have a decomposition $f(t) = g(t)\cdot h(t)$,

where $NP(g)$ = part of $NP(f)$ to the left of P_0

$NP(h)$ = part of $NP(f)$ to the right of P_0.

<u>Theorem</u>. The norm on K extends uniquely to K^{alg}.

Proof: <u>Existence</u>: For $\alpha \in K^{alg}$, take a finite ext'n L of K containing α (e.g. $L = K(\alpha)$)

$$\text{define } v(\alpha) := \frac{1}{[L:K]} v\left(N_{L/K}(\alpha)\right)$$

(1) This is independent of L.

(2) $v(\alpha\beta) = v(\alpha) + v(\beta)$. b/c $N_{L/K}(\alpha\beta) = N_{L/K}(\alpha) + N_{L/K}(\beta)$

(3) $v(\alpha+\beta) \geq \min\{v(\alpha), v(\beta)\}$.

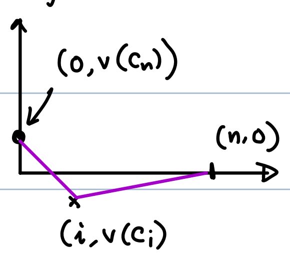
<u>Pf</u>: WLOG, $v(\alpha) \geq v(\beta)$. Suffices $v\left(\frac{\alpha}{\beta}+1\right) \geq 0$ (if $v\left(\frac{\alpha}{\beta}\right) \geq 0$)

Put $\gamma = \frac{\alpha}{\beta}$. Write $P(t) = t^n + c_1 t^{n-1} + \cdots + c_n \in K[t]$ for min poly of γ over K.

$$c_n = \pm N_{K(\gamma)/K}(\gamma)$$

<u>Claim</u>: $v(c_i) \geq 0$ for all i

If not,



$\Rightarrow P(t)$ is reducible. $\divideontimes$

So min poly for $\gamma+1$ is $P(t-1) \in \mathcal{O}_K[t] \Rightarrow v(\gamma+1) \geq 0$.

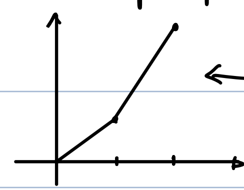
<u>Uniqueness</u>: left as an exercise.

Theorem. Let $f(t) = 1 + a_1 t + a_2 t^2 + \cdots \in K[[t]]$ be a power series.

Then as multisets, $\left\{ -v\left(\underset{(\text{in } K^{alg})}{\text{zeros of } f}\right) \right\} = \left\{ \text{slopes in } NP(f) \right\}$ $\quad (*)$

E.g. $f(x) = (1+pt)(1+p^2t)$ has zeros $t = -\frac{1}{p}, -\frac{1}{p^2}$, valuations $-1, -2$

$= 1 + (p+p^2)t + p^3t^2$



$\leftarrow$ slopes are 1 and 2.

Proof: Suppose that $f(t) = (1-\alpha_1 t)(1-\alpha_2 t)(1-\alpha_3 t) \cdots$ with $\alpha_i \in K^{alg}$

with $v(\alpha_1) \leq v(\alpha_2) \leq \cdots \quad \Rightarrow$ LHS of $(*) = \{ v(\alpha_1), v(\alpha_2), \ldots \}$

$\Rightarrow v(a_1) = v\left(\sum \alpha_i\right) \geq v(\alpha_1)$

$v(a_2) = v\left(\sum_{i<j} \alpha_i \alpha_j\right) \geq v(\alpha_1) + v(\alpha_2)$

$\ldots\ldots$

The equality for $v(a_i)$ holds if $v(\alpha_i) < v(\alpha_{i+1})$. $\square$.

Theorem If K is a complete valued field, then $\widehat{K^{alg}}$ is algebraically closed.

Proof: First prove that $\widehat{K^{alg}}$ is perfect if $\operatorname{char} K = p > 0$.

$\left(\forall \alpha \in \widehat{K^{alg}}\right.$ is the limit of $\alpha_n \in K^{alg}$, $v(\alpha_n^{1/p} - \alpha_{n+1}^{1/p}) = \frac{1}{p} v(\alpha_n - \alpha_{n+1})$

So $\alpha_n^{1/p}$ converges to $\alpha^{1/p}$. $\left.\right)$

Suppose that $\hat{P}(x) = x^d + \hat{a}_1 x^{d-1} + \cdots + \hat{a}_d$ is an irreducible separable polynomial in $\widehat{K^{alg}}[x]$ $\quad d \geq 2$

- Since $NP(\hat{P})$ is a straight line, may assume $v(\hat{a}_d) = 0$ & all zeros of $\hat{P}$ has valuation 0

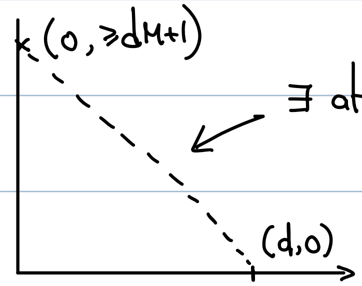
Let $M = \max\{ v(\hat{\alpha} - \hat{\beta})$ for two distinct zeros $\hat{\alpha}$ and $\hat{\beta}$ of $\hat{P} \}$

Now, take $a_i \in K^{alg}$ s.t. $v(a_i - \hat{a}_i) \geq dM + 1$

Consider $Q(x) = \hat{P}(x+\alpha) = (x+\alpha)^d + \hat{a}_1 (x+\alpha)^{d-1} + \cdots + \hat{a}_d$

$$= x^d + \cdots \quad + \hat{P}'(\alpha)\cdot x + \big(\underbrace{\alpha^d + \hat{a}_1\alpha^{d-1} + \cdots + \hat{a}_d}_{\substack{\| \\ P(\alpha) + (-a_1+\hat{a}_1)\alpha^{d-1} + \cdots + (-a_d+\hat{a}_d)}}\big)$$

$\leftarrow$ valuation $\geq dM+1$

$\Rightarrow NP(Q) =$ 

$\exists$ at least one slope $> M$.

But no two zeros have valuations both $\geq M$.

$\Rightarrow NP(Q)$ has exactly one slope $> M \Rightarrow Q$ reducible. Contradiction!

Theorem. Suppose char $K = p > 0$. Then K^{sep} is dense in K^{alg}

Proof: Given $\alpha \in K^{sep}$, want to approximate α^{1/p^m} by elements in K^{sep}.

WLOG $v(\alpha) \geq 0$ (say $v(\pi) = 1$)

Consider $\beta_n \in K^{sep}$ a zero of $x^{p^m} - \pi^n x = \alpha$ (for $n \gg 0$ so that the NP is a straight line)

$\Rightarrow v(\beta_n) = \frac{1}{p^m} v(\alpha)$.

$(\beta_n - \alpha^{1/p^m})^{p^m} = \pi^n \beta_n \Rightarrow v(\beta_n - \alpha^{1/p^m}) = \frac{1}{p^m}\left(n + \frac{1}{p^m} v(\alpha)\right) \to \infty$ as $n \to \infty$

Theorem (Ax–Sen–Tate) If F is a complete valued field and $K \subseteq F^{alg}$,

then $(\widehat{F^{alg}})^{Gal_K} = \widehat{K^{perf}}$, where K^{perf} = perfection of K.

(Note: This works for K/F infinite.

The perfection is needed b/c Gal_K cannot distinguish K from K^{perf}

also changing $\widehat{F^{alg}}$ to $\widehat{F^{sep}}$ doesn't help b/c F^{sep} is dense in F^{alg})

Proof: Will only treat the case when F is an extension of $\mathbb{Q}_p$.

If $\alpha \in F^{alg}$, put $\Delta_K(\alpha) = \inf_{g \in Gal_K} \left(v(g\alpha - \alpha) \right)$

<u>Lemma</u>. Suppose $[K(\alpha):K] = n$. Let $m_0 = \max\{m \text{ s.t. } p^{m+1} \leq n\}$

There exists $\delta \in K$ such that $v(\alpha - \delta) \geq \Delta_K(\alpha) - \sum_{k=0}^{m_0} \frac{1}{p^k(p-1)} > \Delta_K(\alpha) - \frac{p}{(p-1)^2}$.

(Note: <u>Lemma</u> $\Rightarrow$ <u>Thm</u>. Given $\alpha \in \left(\widehat{F^{alg}}\right)^{Gal_K}$, write $\alpha = \lim \alpha_n$ with $\alpha_n \in F^{alg}$, $r_n := v(\alpha - \alpha_n) \to +\infty$

Then $\Delta_K(\alpha_n) \geq r_n \Rightarrow \exists\, \delta_n \in K$ s.t. $v(\alpha_n - \delta_n) \geq r_n - \frac{p}{(p-1)^2} \Rightarrow \lim \delta_n = \alpha$.)

<u>Proof of lemma</u>. Proof by induction on $n = [K(\alpha):K]$.

Let $Q(x) :=$ minimal poly of α over K (of deg n)

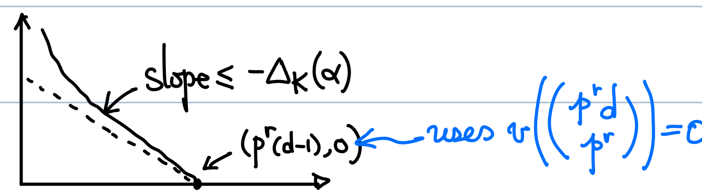
Consider $P(x) := Q(x+\alpha)$, its zeros are $\sigma(\alpha) - \alpha \Rightarrow$ valuation $\geq \Delta_K(\alpha)$

So $NP(P)$ has slope $\leq -\Delta_K(\alpha)$.

<u>Case 1</u>. $n = p^r \cdot d$ for $p \nmid d$ yet $d > 1$.

Consider $\frac{1}{p^r!}\left(\frac{d}{dx}\right)^{p^r}(P) =: P^{(p^r)}(x) = Q^{(p^r)}(x+\alpha)$

On the one hand, $NP(P^{(p^r)}) =$



$\Rightarrow \beta$ a zero with $v(\beta) \geq \Delta_K(\alpha)$ and $v(\text{difference of conj of } \beta) \geq \Delta_K(\alpha)$

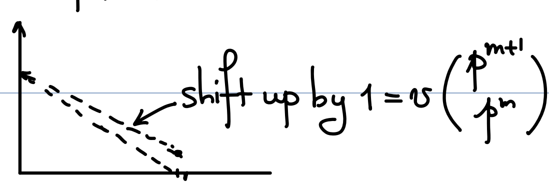
$\Rightarrow \alpha + \beta$ is a zero of $Q^{(p^r)}(x)$, a polynomial of degree $< n$

& $\Delta_K(\alpha+\beta) = \min_{\sigma \in Gal_K} \{ v(\sigma(\alpha+\beta) - (\alpha+\beta)) \} \geq \min_{\sigma \in Gal_K} \{ v(\sigma\alpha - \alpha), v(\beta), v(\sigma(\beta)) \}$

Okay by induction. $\geq \Delta_K(\alpha)$

<u>Case 2</u>: $n = p^{m+1}$, Consider $P^{(p^m)} := \frac{1}{p^m!}\left(\frac{d}{dx}\right)^{p^m}(P)$ instead

$NP(P^{(p^m)}) =$



So we can get β with $v(\beta) \geq \Delta_K(\alpha) - \frac{1}{p^m(p-1)}$.

The rest of the argument is the same as above except that we get

$$\Delta_K(\alpha+\beta) \geq \Delta_K(\alpha) - \frac{1}{p^m(p-1)}.$$

§ Analytic functions on p-adic annuli

Fix a complete valued field K.

Let $I \subseteq \mathbb{R} \cup \{\infty\}$ be an interval. Put

$$A(I) := \{ z \in \widehat{K^{alg}} \,;\, v(z) \in I \}$$

A formal Laurent series $f(t) = \sum_{n \in \mathbb{Z}} a_n t^n$ converges on $A(I)$

iff $\forall r \in I$, $v(a_n) + nr \to +\infty$ as $n \to \pm\infty$

For each $r \in I$, define the r-Gauss valuation of $f = \sum a_n t^n$ with coeffs in K to be

$$v_r(f) = \inf_{n \in \mathbb{Z}} \left(v(a_n) + n \cdot r \right)$$

Fact: If $r \in I \cap v(K^{alg,\times})$, then

$$v_r(f) = \min_{\substack{z \in \widehat{K^{alg}} \\ v(z) = r}} v(f(z)) = \text{min. valuation on the annulus } A([r,r])$$

Define $\mathcal{H}(I) :=$ ring of p-adic analytic functions on $A(I)$

$\mathcal{H}(I)$ is a Fréchet space complete for all valuations $v_r(\cdot)$ with $r \in I$

$\cup\!\mathsf{I}$

$\mathcal{B}(I) =$ ring of bounded functions $= \left\{ f \in \mathcal{H}(I) \,\middle|\, v_r(f) \text{ is uniformly bounded below for } r \in I \right\}$

Example $I = (0, +\infty]$, $A(I) =$ open unit disc.

$f(x) = \log(1+X) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} X^n \in \mathcal{H}(I)$ but does not belong to $\mathcal{B}(I)$

$f(x) = \frac{1}{1-X} = 1 + X + X^2 + \cdots \in \mathcal{B}(I)$ but does not belong to $\mathcal{H}([0, +\infty])$

If I is a closed interval $[a,b]$, then $\mathcal{H}(I)=\mathcal{B}(I)$ is a Banach space

Example $I=(0,+\infty]$, $A(I)=$ open unit disc.

$f(X)=\log(1+X)=\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}X^n \in \mathcal{H}(I)$ but does not belong to $\mathcal{B}(I)$

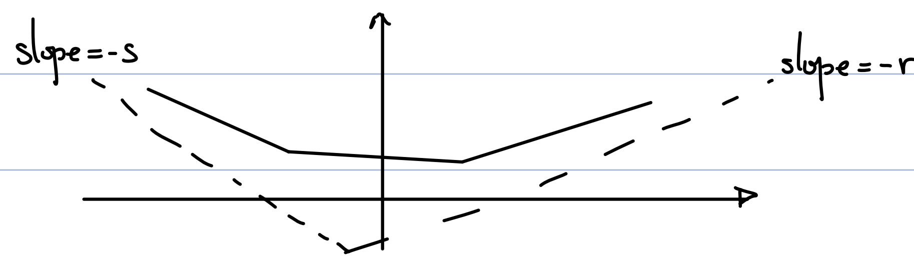
$f(X)=\frac{1}{1-X}=1+X+X^2+\cdots \in \mathcal{B}(I)$ but does not belong to $\mathcal{H}([0,+\infty])$

Lemma. If $f(X)\in\mathcal{H}(I)$, then f has finitely many zero on $A(I) \Rightarrow f\in\mathcal{B}(I)$

If K is discretely valued or I is closed, then $f\in\mathcal{B}(I)\Rightarrow f$ has finitely many zeros on $A(I)$.

Proof: Say $I=[r,s]$, $(r,s]$, or (r,s).

$$\text{Then } f(X)\in\mathcal{B}(I) \Leftrightarrow \begin{cases} \exists M_1, & v(a_n)+nr \geq M_1 \\ \exists M_2, & v(a_n)+ns\geq M_2 \end{cases} \text{ for all } n$$



f has finitely many zeros on $A(I) \Leftrightarrow NP(f)$ has finitely many segments of slopes in $-I$.

$\Downarrow$

$\mathcal{B}(I)$ is clear.

Conversely, if K is discretely valued. each segment with slopes in (r,s) brings NP closer to the bound by $v(\varpi)$.

Lemma If $f(X)\in\mathcal{H}(I)$ has no zeros in $A(I)$, then $f(X)\in\mathcal{B}(I)^\times$.

Proof: Condition $\Rightarrow f(X)\in\mathcal{H}(I)^\times$

But previous lemma $\Rightarrow f(X) \in B(I)$; so $f(x) \in B(I)^\times$ $\square$

<u>Proposition</u> Assume that K is discretely valued or I is closed. Then $B(I)$ is a PID

Proof: Let $J \subseteq B(I)$ be an ideal.

For each $P(X) \in J$, we can write $P(X) = P_f(X) \cdot f^\times(X)$

for $P_f(X) \in K[x]$ accounts for zeros on $A(I)$, and $f^\times(X) \in B(I)^\times$

All such $P_f(X)$ in $K[x]$ generate a principal ideal $(P_J(x))$.

Then $J = (P_J(x))$. $\square$

• <u>Definition</u>. A <u>divisor</u> D on $A(I)$ is a (Gal_K-stable) multisubset D of $A(I)$ s.t.

for any closed interval $J \subseteq I$, $D \cap A(J)$ is finite ($[a, +\infty]$ is considered closed)

<u>Facts</u> (1) If $f \in \mathcal{H}(I)$, $\mathrm{div}(f)$ is a divisor on $A(I)$.

(b/c # of elements in $\mathrm{div}(f)$ are controlled by $NP(f)$.)

(2) Conversely, if D is a divisor on $A(I)$, then $\exists f \in \mathcal{H}(I)$ s.t. $\mathrm{div}(f) = D$

(Fix $s_0 \in I$. For each $s \in I$ can define a polynomial $P_s(x) \in K[x]$,

so that $\mathrm{div}(P_s(X)) = D \cap A([s,s])$ and $\begin{cases} P_s(0) = 1 & \text{if } s < s_0 \\ P_s(x) \text{ monic} & \text{if } s \geq s_0 \end{cases}$

Define $f = \prod_{s < s_0} P_s(x) \cdot \prod_{s \geq s_0} \frac{P_s(x)}{x^{\deg P_s(x)}}$.

It's like $f(x) = \prod_{n \geq 1} \underbrace{(1 - p^n x)}_{s = -n \text{ terms}} \underbrace{(1 - p^n x^{-1})}_{s = n \text{ terms}}$)

(3) Let D be a divisor on $A(I)$ and suppose we have $P_s(x)$ as above for each $s \in I$

Suppose that we have $Q_s(x) \in K[x]$ s.t. $\deg Q_s(x) < \deg P_s(x)$.

Then $\exists f \in \mathcal{H}(I)$ s.t. $f - Q_s(x)$ is divisible by $P_s(x)$.

(4) $\mathcal{H}(I)$ is a Bézout domain, i.e. any finitely generated ideal is principal.

(b/c (f, g) is the same as the function with $\mathrm{div} = \mathrm{div}(f) \cap \mathrm{div}(g)$.)