Introduction to p-adic Hodge theory 7

——— Perfectoid fields and tilting process.

Last time: Defined Big Witt Ring & p-typical Witt vectors $W(R)$

- If $\mathrm{char}\, R = p$, there's a Teichmüller lift $[\cdot]: R \longrightarrow W(R)$ **multiplicative**

$$a \longmapsto (a, 0, 0, \cdots)$$

If $\mathrm{char}\, R = p$, then $(a_0, a_1^p, a_2^{p^2}, \cdots) = [a_0] + p[a_1] + p^2[a_2] + \cdots$

- There's a Frobenius $F: W(R) \longrightarrow W(R)$, $F(a_0, a_1, \cdots) = (a_0^p, a_1^p, \cdots)$

If R is perfect ($x \mapsto x^p$ is bijective), then $W(R)/pW(R) \simeq R$.

Theorem. If A is a ring complete for p-adic topology and if R is a perfect ring in char p.
then a map $f: R \to A/pA$ lifts uniquely to $\tilde{f}: W(R) \to A$.

Proof: For $x \in R$, denote $x^{(n)} := p^n$th root of x.

First pick any set-theoretic lift $\hat{f}: R \to A$ of f.

Define $\tilde{f}([x]) := \lim_{m \to +\infty} \hat{f}(x^{(m)})^{p^m}$ (and $\hat{f}(x^{1/p^m})^{p^m} \bmod p^{m+1}$ is well-defined.)

This converges b/c $\hat{f}(x^{(m+1)})^p \equiv \hat{f}(x^{(m)}) \bmod p$ implies that $\hat{f}(x^{(m+1)})^{p^{m+1}} \equiv \hat{f}(x^{(m)}) \bmod p^{m+1}$.

In general, define for $a = \sum_{n \geq 0} p^n [a_n] \in W(R)$: $\tilde{f}(a) = \sum_{n \geq 0} p^n \hat{f}([a_n])$.

Clearly, $\hat{f}([a]) \cdot \hat{f}([b]) = \hat{f}([ab])$

We need to show $\tilde{f}([a]) + \tilde{f}([b]) = \tilde{f}([a] + [b])$ (*) (this proof reveals the defn of Witt vectors)

For this, we consider the universal case $R_{univ} = \mathbb{Z}[x^{1/p^\infty}, y^{1/p^\infty}] \twoheadrightarrow \bar{R}_{univ} = \mathbb{F}_p[x^{1/p^\infty}, y^{1/p^\infty}]$

we have $[x] + [y] = [S_0(x,y)] + p \cdot [S_1(x,y)] + p^2 \cdot [S_2(x,y)] + \cdots$

- Proof of (*) mod p: $\hat{f}(x) + \hat{f}(y) = \hat{f}(S_0(x,y)) \Rightarrow S_0(x,y) \equiv x + y \bmod p$

• Proof of (*) mod p^2: $\hat{f}(x^{1/p})^p + \hat{f}(y^{1/p})^p = \hat{f}\left(S_0(x, y^{1/p})\right)^p + p\hat{f}(S_1(x,y)) \mod p^2$

But $(x^{1/p})^p + (y^{1/p})^p = (S_0(x^{1/p}, y^{1/p}))^p + pS_1(x,y) \mod p^2$ is known.

(This can be seen as $\mathbb{Z}[x^{1/p}, y^{1/p}] \xrightarrow{\hat{\hat{f}}} A$ (pick any lift), with $\mathbb{Z}[x^{1/p}, y^{1/p}] \to \mathbb{F}_p[x^{1/p}, y^{1/p}] \to R \to A/pA$

$$\Rightarrow \hat{\hat{f}}(x^{1/p})^p + \hat{\hat{f}}(y^{1/p})^p = \hat{\hat{f}}(S_0(x^{1/p}, y^{1/p}))^p + p\hat{\hat{f}}(S_1(x,y))$$

But (*) mod p^2 does not depend on $\hat{f}$. So okay.)

···· By induction, we proved the theorem. (and we see "why Witt vector is defined as this.") □

Perfectoid fields.

Definition A complete valued field K is called a perfectoid field if $\exists$ an element $\varpi \in \mathfrak{m}_K$ s.t. (1) $\varpi^p \mid p$.

(2) The Frobenius $\sigma: \mathcal{O}_K/\varpi \xrightarrow{\sim} \mathcal{O}_K/\varpi^p$ is an isomorphism.

Remarks: ① (2) is equivalent to $\sigma: \mathcal{O}_K/\varpi^p \twoheadrightarrow \mathcal{O}_K/\varpi^p$ is surjective.

b/c it automatically factors through $\mathcal{O}_K/\varpi$

② K perfectoid $\Rightarrow v(K^\times) \subseteq \mathbb{R}$ must be dense! (so certainly not a CDVF.)

b/c keep taking inverses of σ, we get elements of norm $\frac{1}{p^m} v(\varpi)$.

Examples:

① $k((t, t^{1/p^\infty})) = K$. k perfect of char $p>0$ $\qquad \dfrac{k[\![t^{1/p^\infty}]\!]}{(t)} \xrightarrow[\simeq]{\sigma} \dfrac{k[\![t^{1/p^\infty}]\!]}{(t^p)}$

In fact, for any complete valued field K of char $p>0$

K perfect $\Leftrightarrow$ K perfectoid.

② $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})} = K_\infty$ $\quad \dfrac{\mathcal{O}_{K_\infty}}{(\zeta_p-1)\mathcal{O}_{K_\infty}} = \bigcup_{n\geq 1} \dfrac{\mathbb{Z}_p[\zeta_{p^n}-1]}{(\zeta_p-1)\mathbb{Z}_p[\zeta_{p^n}-1]} \xrightarrow{\text{setting } x_n=\zeta_{p^n}-1} \bigcup_{n\geq 1} \dfrac{\mathbb{F}_p[x_n]}{(x_n^{p^{n-1}})}$

Yet note that $x_{n+1} = \zeta_{p^{n+1}}-1$; $x_n = (1+x_{n+1})^p - 1 \equiv x_{n+1}^p \bmod p$

so $\mathcal{O}_K/(\zeta_p-1)\mathcal{O}_{K_\infty} \simeq \mathbb{F}_p[x_1^{1/p^\infty}]/(x_1)$ (as rings)

By exactly same argument above, ✓

③ $K/\mathbb{Q}_p$ finite. ϖ = uniformizer, $f \in \mathcal{F}_\varpi$, $\mathcal{F}_f$, ϖ_n primitive root of $[\varpi^n](X)$

We have $x_n = [\varpi]_f(x_{n+1}) \equiv x_{n+1}^q \bmod \varpi$

$$\frac{\mathcal{O}_{K_\infty}}{x_1\mathcal{O}_{K_\infty}} \simeq \mathbb{F}_q[x_1^{1/q^\infty}]/(x_1)$$

Tilting process: Define the tilt of $\mathcal{O}_K$ & K to be

$$\mathcal{O}_{K^\flat} := \varprojlim_{x\mapsto x^p} \frac{\mathcal{O}_K}{\varpi} \ni (0, \varpi^{1/p}, \varpi^{1/p^2}, \cdots) =: \varpi^\flat$$

$\rightsquigarrow$ admits a ring structure (a choice of p^{th} power roots)

$$K^\flat = \mathcal{O}_{K^\flat}\left[\frac{1}{\varpi^\flat}\right].$$

Properties.

(0) If char $K = p$, $K = K^\flat$. So for the rest of this lecture, assume K is of mixed char $(0,p)$

(1) There's a valuation $v_{K^\flat} : K^\flat \longrightarrow \mathbb{R}\cup\{\infty\}$

$$(\bar{a}_0, \bar{a}_1, \cdots) \longmapsto \lim_{n\to+\infty} p^n v_K(\bar{a}_n)$$ (when $n \gg 0$, this limit stabilizes)

(easy to check the conditions of valuation)

(2) $\mathcal{O}_{K^\flat}$ is a complete valued ring and $K^\flat$ its fraction field, both in char p & perfect.

Perfectness: $(\bar{a}_0, \bar{a}_1, \cdots) = (\bar{a}_1, \bar{a}_2, \cdots)^p$

(3) The natural map $\varprojlim_{x\mapsto x^p} \mathcal{O}_K \longrightarrow \varprojlim_{x\mapsto x^p} \mathcal{O}_K/\varpi$ is a bijection & compatible with multiplication

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Define a section: $(\tilde{a}_0, \tilde{a}_1, \dots) \mapsto (a_0, a_1, \dots)$

s.t. $a_n = \lim_{m\to\infty} (\tilde{a}_{n+m})^{p^m}$ for $\tilde{a}_{n+m}$ a lift of a_{n+m} to $\mathcal{O}_K$.

the limit exists because $(\tilde{a}_{n+m})^{p^m} \bmod p^m\cdot\varpi$ is well-defined

(4) We may instead define $v_{K^\flat}(x) := v_K(\psi(x)_0)$

Examples:

① $K_\infty = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, $\dfrac{\mathcal{O}_{K_\infty}}{(\zeta_p - 1)\mathcal{O}_{K_\infty}} = \dfrac{\mathbb{F}_p[x_1^{1/p^\infty}]}{(x_1)}$ for $x_1 = \zeta_p - 1$

Then $\mathcal{O}_{K_\infty}^\flat = \varprojlim_{a\mapsto a^p} \dfrac{\mathbb{F}_p[x_1^{1/p^\infty}]}{(x_1)} = \mathbb{F}_p[\![\zeta^{1/p^\infty}]\!]$ with $\zeta = (x_1, x_1^{1/p}, x_1^{1/p^2}, \dots)$

$\Rightarrow K_\infty^\flat \cong \mathbb{F}_p((\zeta^{1/p^\infty}))$

② $K = K_\pi$ = Lubin-Tate extension of some $F/\mathbb{Q}_p$, $K_\pi^\flat \simeq \mathbb{F}_p((y_1^{1/p^\infty}))$.

* More on ①: Write $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \varprojlim \mathcal{O}_{K_\infty}$

By looking at the image in $\varprojlim \mathcal{O}_{K_\infty}/p\mathcal{O}_{K_\infty}$, we see $\varepsilon - 1 = (0, x_1, x_2, \dots) = \zeta^p$

So $K_\infty^\flat \cong \mathbb{F}_p((\varepsilon-1))^{\text{perf},\wedge}$

Later: Put $\overline{T} := \varepsilon - 1$. Then $\mathrm{Gal}_{K_\infty^\flat} \simeq \mathrm{Gal}_{\mathbb{F}_p((\overline{T}))}$ canonical!

Moreover, $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ acts on $\mathbb{F}_p((\overline{T}))$ by

$\mathbb{Z}_p^\times \ni \gamma$ ($\mathbb{Z}_p^\times$ is) $\quad \gamma(\overline{T}+1) = (\overline{T}+1)^\gamma$ so $\gamma(\overline{T}) = (1+\overline{T})^\gamma - 1$.

Theorem There is a natural surjective ring homomorphism

$$\theta : W(\mathcal{O}_{K^\flat}) \longrightarrow \mathcal{O}_K$$

induced by the map $\mathcal{O}_{K^\flat} \to \mathcal{O}_K/\mathfrak{p}$ $\quad x \longmapsto \bar{x}_0$

Explicitly, $\theta\left(\sum_{n\geq 0} \mathfrak{p}^n [x^{(n)}]\right) = \sum_{n\geq 0} \mathfrak{p}^n \cdot \psi(x^{(n)})_0$

Note: θ is surjective b/c it is so modulo $\mathfrak{p}$.

Weak topology on $W(\mathcal{O}_{K^\flat})$: A sequence $y_1, y_2, \cdots$ with $y_i = (y_i^{(0)}, y_i^{(1)}, \cdots)$ $y_i^{(j)} \in \mathcal{O}_{K^\flat}$ converges if $\forall m \geq 1, \varepsilon > 0, \exists N$ s.t. if $n > N$,

$\forall j \in \{0, \cdots, m\}$, $\left| y_n^{(j)} - y_{n+1}^{(j)} \right|_{K^\flat} \leq \varepsilon$

Definition An element $\xi \in W(\mathcal{O}_{K^\flat})$ is called distinguished if

(1) $\xi \in \ker\theta$

(2) $\xi = [\varpi_0] + \mathfrak{p}[u]$ for $\varpi_0 \in \mathcal{O}_K^\flat$ s.t. $v_{K^\flat}(\varpi_0) = v_K(p)$

$u \in W(\mathcal{O}_{K^\flat})^\times$ a unit.

Theorem (1) There exists a distinguished element (very much not unique)

(2) Any distinguished element ξ generates $\ker\theta$.

In particular, $\ker\theta$ is principal.

every two distinguished elements differ by a unit in $W(\mathcal{O}_{K^\flat})^\times$

Heuristic version: $W(\mathcal{O}_{K^\flat}) \longleftrightarrow \mathcal{O}_K[\![T]\!]$

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$\mathfrak{p} \longleftrightarrow T$

$[a] \longleftrightarrow a$

Then $\theta: W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K$ is like quotient by some power series $P(T)$

The condition on $P(T)$ is: $NP(P(T))$ is 

By Weierstrass preparation $P(T) = (T-\alpha) \cdot$ unit.

With vector version "p cannot be moved" + "no canonical choice of distinguished elements"

Proof: (1) First, $\exists\ \bar{\varpi}_0 \in \mathcal{O}_K/\varpi$ with $v_K(\bar{\varpi}_0) = \frac{1}{N} < v_K(\varpi)$

then perfectoid $\Rightarrow$ extends to $\bar{\varpi} = (\bar{\varpi}_n) \in \varprojlim \mathcal{O}_K/\varpi$

$\downarrow s\ \psi$

$\psi(\bar{\varpi}) = (\tilde{\varpi}_n) \in \varprojlim \mathcal{O}_K$

Define $\varpi_0 := (\bar{\varpi})^N$, then $v_K(\varpi_0) = v_K(p)$.

Next, write $\varpi_0 = p \cdot (-v)$, with $v \in \mathcal{O}_K^\times$

By previous lemma and its proof, $\exists\ u \in W(\mathcal{O}_K{}^\flat)^\times$ s.t. $\theta(u) = v$.

Then $\xi := [\varpi_0] + p \cdot u \in \ker\theta$. is a distinguished elements

(2) If $\xi = [\varpi_0] + p \cdot u$ is a distinguished element,

we will show by a variant of "Weierstrass division" that $\ker\theta = (\xi)$.

If $a = [a^{(0)}] + p[a^{(1)}] + \cdots \in \ker(\theta) \subseteq W(\mathcal{O}_K{}^\flat)$

$\hookrightarrow \psi(a^{(0)})_0 + p\psi(a^{(1)})_0 + p^2 \cdots$

$\Rightarrow v_K{}^\flat(\psi(a^{(0)})_0) \geq v_K(p)$.

So can write $a^{(0)} = \varpi_0 \cdot b^{(0)}$ for some $b^{(0)} \in \mathcal{O}_K{}^\flat$

Consider $a - \xi \cdot [b^{(0)}] = p \cdot \boxed{\text{////////}} \in \ker\theta$

denoted c

$\Rightarrow c \in \ker\theta$. Continue this process, $\cdots$.