CYCLES ON SHIMURA VARIETIES VIA GEOMETRIC SATAKE, EXAMPLE

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Abstract. This is a part of the forthcoming version of [XZ17+], which aims to provide some examples and explicit computations of cycles. It will be subsumed into the new version of the article once it was ready.

Contents

9. Examples

Our main Theorem [XZ17+, Theorem 1.1.4] (and Theorem [XZ17+, Theorem 7.4.6]) was formulated in representation theoretical terms. In this section, we discuss a few concrete examples and interpret some of our results in more classical terms. Hopefully, some of them will be useful for other applications. We also provide a few alternative proofs of some concrete statements, when more direct (and elementary) methods are available. In particular, for a fixed prime p, we will denote by σ_p the arithmetic Frobenius in $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ and $\phi_p = \sigma_p^{-1}$ the geometric Frobenius. For an unramified connected reductive group G over $\mathbb{Q}_p$, let $(\hat{G}, \hat{B}, \hat{T})$ denote the dual group equipped with a Borel and a maximal torus. Then ϕ_p acts on $(\hat{G}, \hat{B}, \hat{T})$.

9.1. **Hilbert modular surfaces and quaternionic Shimura varieties.** The first examples of our results are Hilbert modular surfaces and more generally quaternionic Shimura varieties. This was previously studied by Tian and the first author [TX19].

9.1.1. *Hilbert modular surfaces.* We first discuss Hilbert modular surfaces. Let F be a real quadratic field, and $p > 2$ an *inert* prime of F. Set $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{PGL}_2$.[1] Take a neat open compact subgroup $K = K^p K_p \subseteq G(\mathbb{A}_f)$ with $K_p = \mathrm{PGL}_2(\mathbb{Z}_p)$. We write $\mathbf{Sh}_K(G, X)$ for the Hilbert modular surface over $\mathbb{Q}$, which admits an integral model $\mathscr{S}_{G,K}$ over $\mathbb{Z}_{(p)}$, and let $\overline{\mathscr{S}}_{G,K} := \mathscr{S}_{G,K} \times_{\mathbb{Z}_{(p)}} \mathbb{F}_{p^2}$ denote

Date: June 6, 2020.
2010 *Mathematics Subject Classification.* 11G18 (primary), 14G35 14C25 22E57 (secondary).
Key words and phrases. Tate conjecture, basic loci of Shimura varieties, geometric Satake, generalized Chevalley restriction theorem.
L.X. was partially supported by Simons Collaboration grant #278433 and NSF grant DMS–1502147. X.Z. was partially supported by NSF grant DMS–1303296/1535464 and DMS–1602092 and the Sloan Fellowship.

[1] Rigorously speaking, this does not quite follow from the arXiv version of [XZ17+, Theorem 7.4.6], as G is not of Hodge type. In a forthcoming version of our article, we will extend our main theorem to abelian type.

the special fiber over $\mathbb{F}_{p^2}$ and $\mathrm{Sh}_{G,K} = \overline{\mathscr{S}}_{G,K}^{\mathrm{perf}}$ the perfection. Although $\mathrm{Sh}_{G,K}$ can be defined over $\mathbb{F}_p$, it helps our discussion to view it over $\mathbb{F}_{p^2}$ because the group $G_{\mathbb{Q}_p}$ splits over $\mathbb{Q}_{p^2}$. In this case, the Langlands dual group is $\hat{G} = \mathrm{SL}_2 \times \mathrm{SL}_2$, on which the Frobenius σ_p acts by interchanging the two factors. The weight lattice of $\hat{G}$ may be identified with $\mathbb{Z}^2 = \{(m,n)\}$, and σ_p interchanges m and n.

The Hodge cocharacter for the Hilbert modular surface is $\mu = (1,1)$. Thus $V_{\mu^*} = \mathrm{std}^* \boxtimes \mathrm{std}^*$, where std^* is the dual of the standard representation of SL_2, and

$$V_{\mu^*}^{\mathrm{Tate}} = V_{\mu^*}(\lambda_1) \oplus V_{\mu^*}(\lambda_2), \quad \text{with} \quad \lambda_1 = (-1,1) \text{ and } \lambda_2 = (1,-1).$$

In addition, for each $i = 1,2$ the set of MV-cycles $\mathbb{MV}_{\mu^*}(\lambda_i) = \{\mathbf{b}_i\}$ consists of a single element. We can choose

$$\nu_{\mathbf{b}_1} = (1,0), \quad \nu_{\mathbf{b}_2} = (0,1), \quad \text{and} \quad \tau_{\mathbf{b}_1} = \tau_{\mathbf{b}_2} = (0,0)$$

so that $\sigma(\nu_{\mathbf{b}_i}^*) - \nu_{\mathbf{b}_i}^* = \lambda_i$ and $X_{\mu^*}^{\mathbf{b}_i,\min}(1) = (\mathbb{P}^1)^{\mathrm{perf}}$.

9.1.2. *Description of the cycles via partial Hasse invariants.* In our case of Hilbert modular surfaces, for each $i = 1,2$, $X_{\mathbf{b}_i} := G'(\mathbb{Q})\backslash G'(\mathbb{A}_f) \times X_{\mu^*}^{\mathbf{b}_i,\min}(1)/K \to \mathrm{Sh}_{G,K}$ is a closed embedding, giving a family of $(\mathbb{P}^1)^{\mathrm{perf}}$'s. Here G' denotes the definite group determined by [XZ17+, Lemma 1.1.3] (which is $\mathrm{Res}_{F/\mathbb{Q}} PB^\times$ with B the quaternion algebra over F exactly ramified at the two archimedean places), and we identify $G(\mathbb{A}_f)$ with $G'(\mathbb{A}_f)$ using the isomorphism θ from the inner twist.

The family $X_{\mathbf{b}_i}$ can be alternatively described as follows (even before taking the perfection). Set $G_{\mathrm{PEL}} = \{g \in \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2 \mid \det g \in \mathbb{G}_m\}$. Assume that there is an open compact subgroup $K_{\mathrm{PEL}} \subset G_{\mathrm{PEL}}(\mathbb{A}_f)$ whose image in $G(\mathbb{A}_f)$ is K. The associated Shimura variety $\mathscr{S}_{G_{\mathrm{PEL}},K_{\mathrm{PEL}}}$ is the moduli space of abelian surfaces with a faithful $\mathcal{O}_F$-action, and an $\mathcal{O}_F$-linear principal polarization, and a K_{PEL}-level structure. The Hodge line bundle ω on $\mathscr{S}_{G_{\mathrm{PEL}},K_{\mathrm{PEL}}}$, given by the determinant of the sheaf of invariant differentials of the universal abelian surface over $\mathscr{S}_{G_{\mathrm{PEL}},K_{\mathrm{PEL}}}$. When base changed to $\mathcal{O}_{F,(p)}$, it admits a decomposition

$$\omega = \omega_1 \otimes \omega_2.$$

After restricting to the special fiber $\overline{\mathscr{S}}_{G_{\mathrm{PEL}},K_{\mathrm{PEL}}}$, the usual Hasse invariant $h \in \mathrm{Hom}(\omega, \omega^p)$ can be decomposed as the product of two *partial Hasse invariants* $h = h_1 h_2$, where $h_1 : \omega_1 \to \omega_2^p$ and $h_2 : \omega_2 \to \omega_1^p$. Taking the limit over K_{PEL}^p, the zero loci $Z(h_i)$ for each $i = 1,2$ on the limit $\mathscr{S}_{G_{\mathrm{PEL}},K_{\mathrm{PEL},p}}$ is stable under the action of $\mathscr{A}(G_{\mathrm{PEL},\mathbb{Z}_{(p)}})$, and thus, we may perform the usual operation to transport the subspaces $Z(h_i)$ to subspaces

$$Z_i := Z(h_i) \times^{\mathscr{A}(G_{\mathrm{PEL}})} \mathscr{A}(G) \tag{9.1.1}$$

of $\mathscr{S}_{G,K_p,\overline{\mathbb{F}}_p}$. Then (after possibly switching $i = 1$ and 2), Z_i is precisely the image of $X_{\mathbf{b}_i}$. We also remark that $\overline{\mathscr{S}}_{G,K}$ and $\overline{\mathscr{S}}_{G_{\mathrm{PEL}},K_{\mathrm{PEL}}}$ are in fact defined over $\mathbb{F}_p$, and the nontrivial element in $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the line bundles ω_1 and ω_2 and hence interchanges Z_1 and Z_2.

Let $\overline{\mathscr{S}}_{G',K} = \mathscr{S}_{G',K} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_{p^2}$ denote the special fiber of the Shimura set associated to $(G', \{1\})$. In this case, we have two natural correspondences

$$\overline{\mathscr{S}}_{G',K} \xleftarrow{(\mathbb{P}^1)^{\mathrm{perf}}\text{-bundle}} X_{\mathbf{b}_i} := G'(\mathbb{Q})\backslash X_{\mu^*}^{\mathbf{b}_i,\min}(1) \times G'(\mathbb{A}_f)/K \longrightarrow \overline{\mathscr{S}}_{G,K}^{\mathrm{perf}}, \quad i = 1,2.$$

The union of the images of $X_{\mathbf{b}_1}$ and $X_{\mathbf{b}_2}$ in $\overline{\mathscr{S}}^{\mathrm{perf}}_{G,K}$ is exactly the basic locus $\overline{\mathscr{S}}^{\mathrm{perf}}_{G,K,b}$. They induce natural maps (for $i, j \in \{1, 2\}$)

$$\begin{tikzcd}[column sep=small] \mathrm{H}^0_{\mathrm{et}}\big(\overline{\mathscr{S}}_{G',K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\big)[\pi_f] \cong \mathrm{H}^0_{\mathrm{et}}\big(X_{\mathbf{b}_i,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\big)[\pi_f] \arrow[dr, "\mathrm{Gys}_{\mathbf{b}_i}"] & \\ & \mathrm{H}^2_{\mathrm{et}}\big(\overline{\mathscr{S}}_{G,K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\big)[\pi_f] \arrow[dl, "\mathrm{Res}_{\mathbf{b}_j}"] \\ \mathrm{H}^0_{\mathrm{et}}\big(\overline{\mathscr{S}}_{G',K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\big)[\pi_f] \cong \mathrm{H}^2_{\mathrm{et},c}\big(X_{\mathbf{b}_j,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\big)[\pi_f] & \end{tikzcd} \tag{9.1.2}$$

for any automorphic representation π_f of $G(\mathbb{A}_f)$. (The compositions of two horizontal maps in (9.1.2) are precisely $\mathrm{JL}_{\tau_{\mathbf{b}_i},\mu}(\mathbf{a}_{i,\mathrm{in}})$ and $\mathrm{JL}_{\mu,\tau_{\mathbf{b}_j}}(\mathbf{a}_{j,\mathrm{out}})$ in [XZ17$^+$, §7.4.3], respectively.) According to [TX19], the composition of the two maps give rise to the following matrix representing the intersection of irreducible components of the basic locus:

$$\big(\mathrm{Res}_{\mathbf{b}_j} \circ \mathrm{Gys}_{\mathbf{b}_i}\big)_{1 \le i,j \le 2} = \begin{pmatrix} -2p & T_p \\ T_p & -2p \end{pmatrix}, \tag{9.1.3}$$

where $T_p = [K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p]$ is the Hecke operator at p. Let α and β denote the two Satake parameters at p of π_f so that $\alpha\beta = p^2$. The π_f-part of the intersection matrix (9.1.3) is exactly

$$\begin{pmatrix} -2p & \alpha + \beta \\ (\alpha + \beta) & -2p \end{pmatrix}$$

whose determinant is $4p^2 - (\alpha + \beta)^2 = p^2(\frac{\alpha}{\beta} - 1)(\frac{\beta}{\alpha} - 1)$. So when $\alpha \neq \beta$, the map $\mathrm{Gys}_{\mathbf{b}_1} \oplus \mathrm{Gys}_{\mathbf{b}_2}$ is injective on the π_f-component, and is an isomorphism if $\alpha \neq \pm\beta$. This verifies the prediction by the Tate conjecture in this case.

Remark 9.1.3. In the case when the prime p splits in F, the Frobenius σ_p acts trivially on the dual group $\hat{G}$. In this case, $V^{\mathrm{Tate}}_{\mu^*}$ is trivial; correspondingly, one can prove that the basic locus of $\overline{\mathscr{S}}_{G,X}$ is 0-dimensional (see for example [AG04, §8.2]). So this is not in the scope of our main theorem.

9.1.4. *Quaternionic Shimura varieties.* The discussion above extends to quaternionic Shimura varieties. This is essentially treated in [TX19]; we briefly recall a variant here. Let F be a totally real number field of degree g in which a prime $p > 2$ is inert. Let B be a quaternion algebra over F that is split at p, and, for simplicity, split at all the archimedean places. Let $G = \mathrm{Res}_{F/\mathbb{Q}} PB^\times$ be the projective multiplicative group of B; its dual group of G is $\hat{G} \cong (\mathrm{SL}_2)^g$, on which σ_p acts by cyclically permuting the factors. If we identify the weight lattice of $\hat{G}$ with $\mathbb{Z}^g$, the Hodge cocharacter is $\mu = (1, \ldots, 1)$, and $V_{\mu^*} \cong \boxtimes^g \mathrm{std}^*$; so $V_{\mu^*}|_{\Delta(\mathrm{SL}_2)} = (\mathrm{std}^*)^{\otimes g}$ where Δ is the diagonal embedding. As above, choose a neat open compact subgroup $K = K^p K_p \subset G(\mathbb{A}_f)$ with K_p hyperspecial, and let $\mathscr{S}_{G,K}$ denote the canonical integral model of the associated Shimura variety over $\mathbb{Z}_{(p)}$ with special fiber $\overline{\mathscr{S}}_{G,K} := \mathscr{S}_{G,K} \times_{\mathbb{Z}_{(p)}} \mathbb{F}_{p^g}$, which is smooth of dimension g.

When g is odd, $V^{\mathrm{Tate}}_{\mu^*} = 0$; in this case, the basic locus of $\overline{\mathscr{S}}_{G,K}$ has dimension $\frac{g-1}{2}$. Our main theorem does not apply to this case. For the rest of this subsection, we assume that g is even. Then $V^{\mathrm{Tate}}_{\mu^*}$ has dimension $\binom{g}{g/2}$; it is the direct sum $V^{\mathrm{Tate}}_{\mu^*} = \bigoplus_{I \subset \{1,\ldots,g\}} V_{\mu^*}(\lambda_I)$ over subsets $I \subset \{1, \ldots, g\}$ of cardinality of $g/2$, where λ_I is 1 at those places in I, and -1 at those places not in I. Moreover, each $\mathbb{MV}_{\mu^*}(\lambda_I)$ is a singleton $\{\mathbf{b}_I\}$. For each such I, there is a minimal $\nu_I \in \mathbb{X}^*(\hat{T})$ such that $\sigma(\nu_I^*) - \nu_I^* = \lambda_I$; this ν_I corresponds to the "periodic semi-meander" as in [TX19].

The associated irreducible component $X^{\mathbf{b}_I,\min}_{\mu^*}(0)$ is always isomorphic to an iterated $\mathbb{P}^1$-bundle, of dimension $g/2$. Let G' denote the projective multiplicative group of the quaternion algebra over F which is locally isomorphic to B at all finite places and is ramified at all archimedean places.

Thus one has morphisms

$$\overline{\mathscr{S}}_{G',K} \xleftarrow{\text{iterated } (\mathbb{P}^1)^{\mathrm{perf}}\text{-bundle}} X_{\mathbf{b}_I} := G'(\mathbb{Q})\backslash X_{\mu^*}^{\mathbf{b}_I,\min}(0) \times G'(\mathbb{A}_f)/K \longrightarrow \overline{\mathscr{S}}_{G,K}^{\mathrm{perf}}.$$

From this, we construct the Gysin and restriction maps as in (9.1.2). The determinant of the composition matrix

$$\big(\mathrm{Res}_{\mathbf{b}_J} \circ \mathrm{Gys}_{\mathbf{b}_I}\big)_{I,J\subseteq\{1,\dots,g\}} \tag{9.1.4}$$

was computed in [TX19] using the combinatorics of periodic semi-meanders that was often discussed in the context of mathematical physics. Up to a nonzero constant, The determinant is equal to $(T_p^2 - 4p^g)^\eta$, where

$$\eta = \sum_{r>0} \dim\big((\mathrm{std})^{\otimes g}\big)(r) = \binom{g}{0} + \cdots + \binom{g}{\frac{g}{2}-1} = 2^{g-1} - \frac{1}{2}\binom{g}{\frac{g}{2}}.$$

Theorem [XZ17$^+$, Theorem 7.4.6] (and particularly Theorem [XZ17$^+$, Theorem 1.4.1]) gives a different proof of this computation. As a corollary of the computation of determinant, if the two Satake parameters of π_f at p are distinct, the π_f-component of the cycle classes generated by the irreducible components of the basic locus $\overline{\mathscr{S}}_b$ are linearly independent. One verifies the Tate conjecture for the π_f-component if the Satake parameters are strongly general with respect to V_{μ^*} in the sense of [XZ17$^+$, Definition 1.4.2].

9.2. **Unitary Shimura varieties.** In this and the next two subsections, we give a detailed discussion of our results in the case of $\mathrm{U}(1,2r)$-Shimura varieties, in particular Picard modular surfaces ($r=1$), and in the case of $\mathrm{U}(2,s)$-Shimura varieties, at an inert prime. In the $\mathrm{U}(1,2r)$ case, the description of the basic (=supersingular) locus was essentially contained in [Vo10, VW11], but the computation of the intersection of irreducible components of the basic locus is new. In the $\mathrm{U}(2,s)$ case, the description of basic locus is contained in [HP14] when $s=2$, and is new for $s\geq 3$. As we shall see, compared with all previous related works on the basic locus of Shimura varieties, new phenomenon appears. Namely, when $s\geq 3$, we encounter certain generalization of Deligne–Lusztig varieties admitting actions of $\mathrm{U}(\mathbb{Z}_p/p^2\mathbb{Z}_p)$ which does not factor through $\mathrm{U}(\mathbb{F}_p)$.

9.2.1. *Satake isomorphism for unitary groups.* We first give explicit formulas for the Satake isomorphism for unitary groups, and discuss basic properties of the dual group of unitary groups.

Let $\mathcal{V}_p$ be an n-dimensional Hermitian space over $\mathbb{Q}_{p^2}$. We write $r = \lfloor \frac{n}{2} \rfloor$. Let $G = \mathrm{U}(\mathcal{V}_p)$ denote the unitary group. Although σ_p and ϕ_p are equal in $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, we distinguish them as they play different roles in our general theory.

We first recall the representation theory on the dual group side. For the dual group $\hat{G} = \mathrm{GL}_n$, we take the Cartan subgroup $\hat{T}$ and the Borel subgroup $\hat{B}$ to be the group of diagonal matrices and the group of upper triangular matrices. For an index $i \in \{1,\dots,n\}$, we write $i^\vee := n+1-i$, and for a subset $I \subset \{1,\dots,n\}$, write $I^\vee := \{i^\vee | i \in I\}$. Then $\sigma_p = \phi_p \in \mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ acts on $\hat{G}$ by sending $A \mapsto J(A^{\mathrm{t}})^{-1}J^{-1}$, where A^{t} is the transpose of A and $J = (J_{ij})_{1\leq i,j\leq n}$ is the anti-diagonal matrix with alternating 1's and -1's, i.e. $J_{i,j} = 0$ if $j \neq i^\vee$ and $J_{i,i^\vee} = (-1)^{i-1}$.

The group $\mathbb{X}^\bullet(\hat{T})$ admits a basis $\varepsilon_1,\dots,\varepsilon_n$, where ε_i is the character of $\hat{T}$ given by evaluating the (i,i)-entry of $\hat{T}$. Then σ_p and ϕ_p act on $\mathbb{X}^\bullet(\hat{T})$ by

$$\phi_p(\varepsilon_i) = \sigma_p(\varepsilon_i) = -\varepsilon_{i^\vee} \text{ for } i = 1,\dots,n.$$

The Bruhat partial order on $\mathbb{X}^\bullet(\hat{T})$ is generated by $\varepsilon_i \succ \varepsilon_j$ whenever $i<j$. Put $\hat{S} := \hat{T}/(\sigma_p - 1)\hat{T}$ so that

$$\mathbb{X}^\bullet(\hat{S}) = \mathbb{X}^\bullet(\hat{T})^{\sigma_p} = \bigoplus_{i=1}^{r} \mathbb{Z}(\varepsilon_i - \varepsilon_{i^\vee}),$$

where $r=\lfloor \frac{n}{2}\rfloor$ as introduced above. The absolute Weyl group $W\cong S_n$ of $\hat G$ permutes $\varepsilon_1,\dots,\varepsilon_n$; and the relative Weyl group is

$$W_0=W^{\sigma_p}=\langle (i,i^\vee),(ij)(i^\vee,j^\vee)\mid i,j=1,\dots,r\rangle=(\mathbb{Z}/2\mathbb{Z})^r\rtimes S_r.$$

For a character η of $\mathbb{X}^\bullet(\hat T)$, we write e^η for the corresponding function on $\hat T$. If we write $\mathfrak{S}_i$ for the ith elementary symmetric power in $e^{\varepsilon_1-\varepsilon_{1^\vee}}+e^{\varepsilon_{1^\vee}-\varepsilon_1},\dots,e^{\varepsilon_r-\varepsilon_{r^\vee}}+e^{\varepsilon_{r^\vee}-\varepsilon_r}$, then

$$\boldsymbol{J}=\boldsymbol{J}_{\hat G}:=\overline{\mathbb{Q}}_\ell[\hat G\phi_p]^{\hat G}\cong\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat S)]^{W_0}\cong\overline{\mathbb{Q}}_\ell[\mathfrak{S}_1,\dots,\mathfrak{S}_r]$$

is a subring of $\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat T)]=\overline{\mathbb{Q}}_\ell[e^{\pm\varepsilon_1},\dots,e^{\pm\varepsilon_n}]$. When $r=1$ (that is $n=2,3$),

$$\boldsymbol{J}=\overline{\mathbb{Q}}_\ell[e^{\varepsilon_1-\varepsilon_{1^\vee}}+e^{\varepsilon_{1^\vee}-\varepsilon_1}].$$

For a σ_p-invariant dominant weight $\lambda\in\mathbb{X}^\bullet(\hat T)^{\sigma_p}=\mathbb{X}^\bullet(\hat S)$, restricting the character function of the highest weight $\hat G$-representation V_λ to the coset $\hat G\phi_p$ gives an element in the above ring $\boldsymbol{J}$.

Example 9.2.2. Let $M=\mathrm{M}_{3,3}$ denote the space of 3×3-matrices equipped with the adjoint representation of GL_3. The associated character is $\chi_M=e^{\varepsilon_1-\varepsilon_{1^\vee}}+e^{\varepsilon_{1^\vee}-\varepsilon_1}+1$.

The roots in the relative root system $\Phi^\vee_{\mathrm{rel}}\subset\mathbb{X}^\bullet(\hat S)=\mathbb{X}^\bullet(\hat T)^{\sigma_p}$ consist of the W_0-orbits of $\varepsilon_1-\varepsilon_{1^\vee}$ and the W_0-orbits of $\varepsilon_1-\varepsilon_{1^\vee}-\varepsilon_2+\varepsilon_{2^\vee}$ (when $r\geq 2$). Correspondingly, there are two discriminant functions (only one if $r=1$) in $\boldsymbol{J}$:

$$\mathrm{disc}_1:=\prod_{1\leq i\leq r}\Big((e^{\varepsilon_i-\varepsilon_{i^\vee}}-1)(e^{\varepsilon_{i^\vee}-\varepsilon_i}-1)\Big)=\sum_{i=0}^r(-1)^{r-i}2^i\mathfrak{S}_{r-i};\tag{9.2.1}$$

$$\mathrm{disc}_2:=\prod_{1\leq i<j\leq r}\Big((e^{\varepsilon_i-\varepsilon_{i^\vee}}-e^{\varepsilon_j-\varepsilon_{j^\vee}})(e^{\varepsilon_i-\varepsilon_{i^\vee}}-e^{\varepsilon_{j^\vee}-\varepsilon_j})(e^{\varepsilon_{i^\vee}-\varepsilon_i}-e^{\varepsilon_j-\varepsilon_{j^\vee}})(e^{\varepsilon_{i^\vee}-\varepsilon_i}-e^{\varepsilon_{j^\vee}-\varepsilon_j})\Big).$$

Lemma 9.2.3. *Set $\mathfrak{S}_0=1$. Let $V_a:=\wedge^a\mathrm{std}\otimes(\wedge^a\mathrm{std})^*$ with $0\leq a\leq r$.*

(1) *For each $0\leq a\leq r$, we have*

$$\chi_{V_a}=\begin{cases}\mathfrak{S}_a+\binom{r-a+2}{1}\mathfrak{S}_{a-2}+\cdots+\binom{r-a+2j}{j}\mathfrak{S}_{a-2j}+\cdots & n=2r\\ \mathfrak{S}_a+\mathfrak{S}_{a-1}+(r-a+2)\mathfrak{S}_{a-2}+\cdots+\binom{r-a+j}{\lfloor j/2\rfloor}\mathfrak{S}_{a-j}+\cdots & n=2r+1.\end{cases}$$

(2) *When $n=2r+1$, have*

$$\mathrm{disc}_1=\sum_{a=0}^r(-1)^{r-a}(2a+1)\chi_{V_{r-a}}.$$

In particular, when $r=1$, $\mathrm{disc}_1=3-\chi_{\mathrm{std}\otimes\mathrm{std}^*}$.

Proof. (1) Let $b_1,\dots,b_n$ denote the standard basis of std, and $b_1^*,\dots,b_n^*$ the dual basis in std^*. For a subset $I\subseteq\{1,\dots,n\}$, we write $\varepsilon_I=\sum_{i\in I}\varepsilon_i$ and $\langle I\rangle=\sum_{i\in I}i$; if the elements in $I=\{i_1,\dots,i_s\}$ is in increasing order, we write $b_I:=b_{i_1}\wedge\cdots\wedge b_{i_a}$ and $b_I^*:=b_{i_1}^*\wedge\cdots\wedge b_{i_a}^*$. There is a natural map $\phi_p:\mathrm{std}\to\mathrm{std}^*$ that sends b_i to $(-1)^ib^*_{i^\vee}$, and thus $\phi_p(b_I)=(-1)^{\langle I\rangle+a(a-1)/2}b_I^*$.

For $\hat t\in\hat T$, $\hat t\phi_p$ acts on $\wedge^a\mathrm{std}\otimes(\wedge^a\mathrm{std})^*$ as

$$b_I\otimes b_J^*\xrightarrow{\phi_p}(-1)^{\langle I\rangle+\langle J^\vee\rangle}b^*_{I^\vee}\otimes b_{J^\vee}\xrightarrow{\hat t}(-1)^{\langle I\rangle+\langle J^\vee\rangle}e^{-\varepsilon_{I^\vee}+\varepsilon_{J^\vee}}(\hat t)\cdot b^*_{I^\vee}\otimes b_{J^\vee},$$

where $I,J\subseteq\{1,\dots,n\}$ are subsets of cardinality a. This term contributes to the trace if and only if $I=J^\vee$, in which case the contribution is $e^{-\varepsilon_{I^\vee}+\varepsilon_I}(\hat t)$. Summing over those I such that $\#(I\cap I^\vee)=j$, we obtain $\binom{r-a+j}{\lfloor j/2\rfloor}\mathfrak{S}_{a-j}$ with the condition that j must be even if n is even; this is because $(I\backslash I^\vee)\cup(I^\vee\backslash I)$ consists of $a-j$ pairs of distinct numbers $\{i,i^\vee\}$, and when fixing that, we are left with $\binom{r-a+j}{\lfloor j/2\rfloor}$ choices for the set $I\cap I^\vee$ (which, when n is odd, must contain $\frac{n+1}{2}$ if j is odd, and must not contain $\frac{n+1}{2}$ if j is even). Summing this over all j gives the formula in the lemma.

(2) Using part (1), we deduce that

$$\begin{aligned}\sum_{a=0}^{r}(-1)^{r-a}(2a+1)\chi_{V_{r-a}} &= \sum_{a=0}^{r}\sum_{0\le i\le r-a}(-1)^{r-a}(2a+1)\binom{r-(r-a)+(r-a-i)}{\lfloor\frac{r-a-i}{2}\rfloor}\mathfrak{S}_i\\ &= \sum_{i=0}^{r}\sum_{0\le a\le r-i}(-1)^{r-a}(2a+1)\binom{r-i}{\lfloor\frac{r-a-i}{2}\rfloor}\mathfrak{S}_i.\end{aligned}$$

Comparing to the formula for disc_1 in (9.2.1), we need to show the coefficient of $\mathfrak{S}_i$ above is equal to $(-1)^i2^{r-i}$. Rewrite $k=r-i$ and $\ell=k-a$ for simplicity; so we need to show that

$$\sum_{0\le\ell\le k}(-1)^{k-\ell}(2k-2\ell+1)\binom{k}{\lfloor\ell/2\rfloor}=(-2)^k.$$

But the left hand side is equal to

$$\begin{aligned}&\binom{k}{0}\big((-1)^k(2k+1)+(-1)^{k-1}(2k-1)\big)+\binom{k}{1}\big((-1)^{k-2}(2k-3)+(-1)^{k-3}(2k-5)\big)+\cdots\\ =&\binom{k}{0}(-1)^k\cdot2+\binom{k}{1}(-1)^k\cdot2+\binom{k}{2}(-1)^k\cdot2+\cdots.\end{aligned}$$

Some easy analysis by the parity of k shows that the above expression is just $(-1)^k$ times the binomial expansion of $(1+1)^k$. Our lemma is proved. □

Next, we discuss the Satake isomorphism. We can assume that there is a basis $\{e_1,\dots,e_n\}$ of the hermitian space $\mathcal{V}_p$ such that the hermitian form is given by $h(e_i,e_{j^\vee})=\delta_{ij}$. We identify $G(\mathbb{Q}_p)$ as a subgroup of $\mathrm{GL}_n(\mathbb{Q}_{p^2})$ via this basis. Then the stabilizer K_p of the lattice $\Lambda=\bigoplus_{i=1}^n\mathbb{Z}_{p^2}e_i$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. We choose

$$S=\big\{\,\mathrm{diag}\{a_1,\dots,a_n\}\in G(\mathbb{Q}_p)\mid a_1a_{1^\vee}=\cdots=a_ra_{r^\vee}=a_{(n+1)/2}=1\,\big\}$$

as a maximal split torus of $G_{\mathbb{Q}_p}$, where when n is even, we omit the last term $a_{(n+1)/2}$. Its dual group is exactly $\hat{S}$ and its cocharacter group is $\mathbb{X}_\bullet(S)=\mathbb{X}^\bullet(\hat{T})^{\sigma_p}$ is given as above. For each $\lambda\in\mathbb{X}_\bullet(S)$ we write $\lambda(p)$ for the image of p under the natural map $\mathbb{Q}_p^\times\xrightarrow{\lambda}G(\mathbb{Q}_p)$. Explicitly, for a tuple $\underline{m}=(m_1,\dots,m_r)\in\mathbb{Z}^r$, we write

$$\lambda_{\underline{m}}:=m_1(\varepsilon_1-\varepsilon_{1^\vee})+\cdots+m_r(\varepsilon_r-\varepsilon_{r^\vee})\in\mathbb{X}_\bullet(S)$$

for the associated cocharacter. Then

$$\lambda_{\underline{m}}(p)=\mathrm{diag}\big\{p^{m_1},\dots,p^{m_r},1,p^{-m_r},\dots,p^{-m_1}\big\}\in G(\mathbb{Q}_p),$$

where when n is even, we omit the term 1 in the middle. The character $\lambda_{\underline{m}}$ is dominant if and only if $m_1\ge\cdots\ge m_r\ge0$. We will use often the following dominant characters for $i=0,\dots,r$:

$$\lambda_i=\lambda_{(1^i,0^{n-2i},(-1)^i)}=\varepsilon_1+\cdots+\varepsilon_r-\varepsilon_{r^\vee}-\cdots-\varepsilon_{1^\vee}\in\mathbb{X}_\bullet(S).$$

Let $\mathrm{Sph}_p:=H(G(\mathbb{Q}_p),K_p)$ be the spherical (a.k.a unramified) Hecke algebra with $\overline{\mathbb{Q}}_\ell$-coefficients. Let

$$T_{\underline{m}}:=T_{\lambda_{\underline{m}}}:=1_{K_p\lambda_{\underline{m}}(p)K_p}\in H(G(\mathbb{Q}_p),K_p)$$

be the Hecke operator corresponding to the characteristic function of $K_p\lambda_{\underline{m}}(p)K_p$. Then

$$\big\{T_{\underline{m}}\mid(m_1,\dots,m_r)\in\mathbb{Z}^r,m_1\ge\cdots\ge m_r\ge0\big\}$$

form a $\overline{\mathbb{Q}}_\ell$-basis of Sph_p, by the Cartan decomposition. In particular, for $0 \le i \le r$, we write $T_{p,i}$ for T_{λ_i}, which together generate Sph_p as a $\overline{\mathbb{Q}}_\ell$-algebra. In particular, when $r = 1$, Sph_p as an algebra is generated by Hecke operators: $T_{p,1} = 1_{K_p \mathrm{diag}\{p,1,p^{-1}\} K_p}$.

We fix a square root $p^{1/2} \in \overline{\mathbb{Q}}_\ell$. Then we have the usual Satake isomorphism

$$\mathrm{CT}_p : \mathrm{Sph}_p \cong H(S(\mathbb{Q}_p), K_p \cap S(\mathbb{Q}_p))^{W_0} \simeq \overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]^{W_0}.$$

On the other hand, the geometric Satake correspondence associates every representation V of ${}^L G_{\mathbb{F}_p}$ a perverse sheaf $\mathrm{Sat}(V)$ on Gr, whose trace function defines an element $h_V \in \mathrm{Sph}_p$. It follows from the compatibility of geometric Satake and the classical Satake ([XZ17$^+$, Proposition 3.5.5]) that $\mathrm{CT}_p(h_V) = \chi_V$. As $\{T_{\underline{m}}\}$ is a basis of the Hecke algebra, every h_V can be written as the linear combination of these characteristic polynomials. The coefficients can be expressed by the so-called Lusztig–Kato polynomials[2]. For our later application, we obtain a formula for h_{V_a} for $V_a := \wedge^a \mathrm{std} \otimes (\wedge^a \mathrm{std})^*$, $0 \le a \le r$ directly as follows. Recall the v-analogue of the binomial coefficients

$$[0]_v = 1, \quad [n]_v = \frac{v^n - 1}{v - 1}, \quad [n]_v! = [n]_v [n-1]_v \cdots [1]_v, \quad \begin{bmatrix} n \\ m \end{bmatrix}_v = \frac{[n]_v!}{[n-m]_v! [m]_v!}.$$

Lemma 9.2.4. *For $0 \le a \le r$, we have*

$$h_{V_a} = p^{-a(n-a)} \sum_{i=0}^{a} \begin{bmatrix} n-2i \\ a-i \end{bmatrix}_v T_{p,i}, \tag{9.2.2}$$

with $v = -p$. In particular,

$$h_{V_1} = p^{1-n} \Big(T_{p,1} + \frac{(-p)^n - 1}{-p-1} \Big).$$

Proof. We sketch a proof. Let $\mathrm{Sat} : \mathrm{Rep}(\hat{G}) \to \mathrm{P}_{L^+G \otimes \overline{\mathbb{F}}_p}(\mathrm{Gr} \otimes \overline{\mathbb{F}}_p)$ denote the geometric Satake equivalence. Note that $G \otimes \mathbb{Z}_{p^2} \simeq \mathrm{GL}_n$. Therefore, there is a natural isomorphism $\mathrm{Gr}_G \otimes \mathbb{F}_{p^2} \cong \mathrm{Gr}_{\mathrm{GL}_n}$, which we identify with lattices in $\mathcal{V}_p$ as usual. Under this isomorphism, the Frobenius endomorphism $F := \mathrm{Frob}_{\mathrm{Gr}_G} \otimes \mathrm{id}_{\mathbb{F}_{p^2}}$ of $\mathrm{Gr}_G \otimes \mathbb{F}_{p^2}$ sends, for a perfect $\mathbb{F}_{p^2}$-algebra R, a lattice $L \subset \mathcal{V}_p \otimes_{\mathbb{Z}_{p^2}} W(R)$ to $\mathrm{Frob}_R^*(L)^\vee$, where $\mathrm{Frob}_R : W(R) \to W(R)$ is the morphism induced by the Frobenius of R, and $\mathrm{Frob}_R^*(L)^\vee$ is the lattice dual to $\mathrm{Frob}_R^*(L)$ with respect to the hermitian form on $\mathcal{V}_p$.

Since $V_a = \wedge^a \mathrm{std} \otimes \wedge^a \mathrm{std}^*$,

$$\mathrm{Sat}(V_a) = \mathrm{Sat}(\wedge^a \mathrm{std}) \star \mathrm{Sat}(\wedge^a \mathrm{std}^*),$$

here $\star$ stands for the convolution product of the Satake category. Note that $\mathrm{Sat}(\wedge^a \mathrm{std})$ and $\mathrm{Sat}(\wedge^a \mathrm{std}^*)$ are constant sheaves (up to shift and twist) supported on the corresponding minuscule Schubert varieties $\mathrm{Gr}_{\wedge^a \mathrm{std}}$ and $\mathrm{Gr}_{\wedge^a \mathrm{std}^*}$ defined over $\mathbb{F}_{p^2}$, which are switched by F. The Weil structure on $\mathrm{Sat}(V_a)$ then is given by

$$\begin{aligned} F^*(\mathrm{Sat}(V_a)) &= F^* \mathrm{Sat}(\wedge^a \mathrm{std}) \star F^* \mathrm{Sat}(\wedge^a \mathrm{std}^*) \\ &\cong \mathrm{Sat}(\wedge^a \mathrm{std}^*) \star \mathrm{Sat}(\wedge^a \mathrm{std}) \cong \mathrm{Sat}(\wedge^a \mathrm{std}) \star \mathrm{Sat}(\wedge^a \mathrm{std}^*), \end{aligned}$$

where the last isomorphism comes from the commutativity constraints.

[2] Strictly speaking, Lusztig–Kato polynomials are only defined and studied for (residually) split groups in literature. But the generalization to unramified groups is not difficult, using the results and methods in [Zhu15].

Over $\mathbb{F}_{p^2}$, we are working with the affine Grassmannian of GL_n; it is well known that the Poincaré polynomial of the stalk cohomology $\mathrm{Sat}(V_a)_{\lambda_i(p)}$ of $\mathrm{Sat}(V_a)$ at $\lambda_i(p)$ is

$$\sum_{j\geq 0}\dim\Big(\mathcal{H}^{-2j-\langle\lambda_i,2\rho\rangle}_{\lambda_i(p)}\big(\mathrm{Sat}(V_a)\big)\Big)v^j=\begin{bmatrix}n-2i\\ a-i\end{bmatrix}_v.$$

Here v denotes an indeterminate. On the other hand, from the construction of the commutativity constraints from [Zhu17a] (in particular Section 2.4.5), we see that for $0\leq i\leq a$,

$$\mathrm{tr}\big(\phi_p\mid \mathrm{Sat}(V_a)_{\lambda_i(p)}\big)=p^{-i(n-i)}\begin{bmatrix}n-2i\\ a-i\end{bmatrix}_v,\quad v=-p^{-1},$$

which is exactly what the lemma claims. □

Using Lemmas 9.2.3 and 9.2.4, and the notation therein, we may express the preimage of disc_1 under the Satake isomorphism

$$\mathrm{CT}_p:\mathrm{Sph}_p\cong\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]^{W_0}.$$

$$\mathrm{CT}_p^{-1}(\mathrm{disc}_1)=\sum_{i=0}^{r}(-1)^i(2i+1)h_{V_{r-i}}=\sum_{i=0}^{r}(-1)^i(2i+1)p^{-(r-i)(r+i+1)}\sum_{j=0}^{r-i}\begin{bmatrix}2r+1-2j\\ r-i-j\end{bmatrix}_{v=-p}T_{p,j}.$$

Let ${}^LG_{\mathbb{F}_p}:=\hat{G}\rtimes\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ denote the unramified Langlands dual group, regarded as a pro-algebraic group. As explained in [XZ17$^+$, Remark 3.5.3], there are two maps from the representation ring $R({}^LG_{\mathbb{F}_p})$ of ${}^LG_{\mathbb{F}_p}$ to $\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]^{W_0}$. One is given by restricting a character function of a representation V to the coset $\hat{G}\phi_p$, and the other one is given by restricting to the coset $\hat{G}\sigma_p$. They in general differ by an involution of $\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]^{W_0}$ induced by multiplication by (-1) on $\mathbb{X}^\bullet(\hat{S})$. In our case the canonical isomorphism

$$\mathrm{inv}:\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]^{W_0}\cong\overline{\mathbb{Q}}_\ell[\hat{G}\sigma_p]^{\hat{G}}\xrightarrow{g\sigma_p\mapsto\phi_p(g^{-1})\phi_p}\overline{\mathbb{Q}}_\ell[\hat{G}\phi_p]^{\hat{G}}\cong\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]^{W_0}\tag{9.2.3}$$

turns out to be the identity map. So $\mathrm{inv}(\mathrm{disc}_1)=\mathrm{disc}_1$.

9.3. **Picard modular surfaces and $\mathrm{U}(1,2r)$-Shimura varieties.** We first describe our main theorem in a more concrete way in this case. We will start with the computation on the representation side, and then translate the supposedly intersection matrix in terms of the Satake isomorphism. Then we rewrite our results in more classical way, explaining how they lead to intersection numbers of cycles. When possible, we make explicit the computation in the case of Picard modular surfaces.

9.3.1. *PEL type moduli problem of* $\mathrm{U}(1,2r)$*-Shimura varieties.* Let E be an imaginary quadratic field and let $p>2$ be an inert prime. Let $\mathcal{V}$ be a $(2r+1)$-dimensional hermitian space over E of signature $(1,2r)$ at the infinity and is unramified at p. Let $G=\mathrm{U}(\mathcal{V})$ be the corresponding unitary group. We choose the level structure to be $K=K_pK^p\subset G(\mathbb{A}_f)$ with K_p the hyperspecial subgroup fixing a self-dual $\mathbb{Z}_{p^2}$-lattice $\Lambda\subset\mathcal{V}_p:=\mathcal{V}\otimes_E\mathbb{Q}_{p^2}$. Following [RSZ17$^+$, §3], we realize the Shimura variety $\mathbf{Sh}_K(G,X)$ as a PEL moduli problem as follow. We choose and fix a CM elliptic curve A_0 with a full $\mathcal{O}_E$-action and a principal polarization λ_0 in the prime-to-p isogeny category that is defined over the Hilbert class field E_{H} of E. Then the integral model of the Shimura variety $\mathscr{S}_{G,K,\mathcal{O}_{E_{\mathrm{H}},(p)}}$ classifies triples (A,λ,η) over an $\mathcal{O}_{E,(p)}$-scheme S, where

- A is an abelian variety of dimension $2r+1$ over S with a faithful action by $\mathcal{O}_E$ satisfying the Kottwitz' determinant condition of signature $(1,2r)$,
- $\lambda:A\to A^\vee$ is a principal polarization in the category of abelian varieties up to prime-to-p isogenies such that the Rosati involution induces complex conjugation on $\mathcal{O}_E$, and

- η is a section of

$$\mathrm{Isom}_{\mathrm{Herm}}\big(\mathcal{V}\otimes\mathbb{A}_f^p,\ \mathrm{Hom}_{\mathcal{O}_E}(\mathcal{V}_{\mathrm{et}}(A_0)^p,\mathcal{V}_{\mathrm{et}}(A)^p)\big)/K^p$$

of the étale sheaf parametrizing K^p-orbits of hermitian isomorphisms, where the hermitian structure on the latter is given by

$$\text{for } x,y\in\mathrm{Hom}_{\mathcal{O}_E}(\mathcal{V}_{\mathrm{et}}(A_0)^p,\mathcal{V}_{\mathrm{et}}(A)^p),\quad \langle x,y\rangle:=\lambda_0\circ y^\vee\circ\lambda\circ x\in\mathrm{End}_{\mathcal{O}_E}(\mathcal{V}_{\mathrm{et}}(A_0)^p)=\mathbb{A}_{E,f}^p.$$

When $r=1$, $\mathscr{S}_{G,K,\mathcal{O}_{E_{\mathrm{H}},(p)}}$ is essentially a Picard modular surface. We point out that, by taking appropriate quotient, one can define $\mathscr{S}_{G,K}$ over $\mathcal{O}_{E,(p)}$, but we content with our definition here because p must split in E_{H}/E; so taking the special fiber at a place v of E_{H} above p, we recover the special fiber $\overline{\mathscr{S}}_{G,K}:=\mathscr{S}_{G,K,\mathcal{O}_{E_{\mathrm{H}},(p)}}\otimes_{\mathcal{O}_{E_{\mathrm{H}}}}k(v)$ over $\mathbb{F}_{p^2}$. We also note that $\overline{\mathscr{S}}_{G,K}$ admits a canonical smooth compactification $\widetilde{\overline{\mathscr{S}}}_{G,K}$ by adding a (CM) abelian variety of dimension $2r-1$ at every cusp.

Lemma 9.3.2. *The Shimura datum (G,X) determines the Hodge character $\mu=\varepsilon_1\in\mathbb{X}^\bullet(\hat{T})$. We have $V_{\mu^*}^{\mathrm{Tate}_p}=V_{\mu^*}(\lambda)$, where $\lambda=-\varepsilon_{r+1}$; it is one-dimensional. In particular, the set of MV cycles $\mathbb{MV}_{\mu^*}^{\mathrm{Tate}}=\mathbb{MV}_{\mu^*}(\lambda)=\{\mathbf{b}\}$ is a singleton. One can write $\lambda=\sigma(\nu^*)-\nu^*+\tau^*$, where $\nu=\nu_{\mathbf{b}}=\varepsilon_1+\cdots+\varepsilon_r$ and $\tau=\tau_{\mathbf{b}}=\varepsilon_1+\cdots+\varepsilon_{2r+1}\in\mathbb{X}_\bullet(Z_G)$.*

Proof. Go through the definitions. $\square$

Corresponding to the MV cycle $\mathbf{b}$, we have the Satake cycle $\mathbf{a}\in\mathbb{S}_{(\nu^*,\mu^*)|\lambda+\nu^*}$ such that

$$\mathrm{Gr}^{0,\mathbf{a}}_{(\nu^*,\mu^*)|\lambda+\nu^*}\cap S_{\nu^*,\lambda}=(S_{\nu^*}\cap\mathrm{Gr}_{\nu^*})\tilde{\times}(S_\lambda\cap\mathrm{Gr}_{\mu^*})^{\mathbf{b}}$$

(see [XZ17+, Lemma 3.2.7]). Via [XZ17+, Proposition 3.1.10] and the geometric Satake, it gives two nonzero morphisms of $\hat{G}$-representations

$$\mathbf{a}_{\mathrm{in}}:V_{\sigma(\nu)}\otimes V_\tau\otimes V_{\nu^*}\to V_\mu\quad\text{and}\quad\mathbf{a}_{\mathrm{out}}:V_{\sigma(\nu^*)}\otimes V_\mu\otimes V_\nu\to V_\tau. \tag{9.3.1}$$

These two morphisms give rise to elements $\Xi_\nu(\mathbf{a}_{\mathrm{in}})\in\mathrm{Hom}(\widetilde{V_\tau},\widetilde{V_\mu})$ and $\Xi_{\nu^*}(\mathbf{a}_{\mathrm{out}})\in\mathrm{Hom}(\widetilde{V_\mu},\widetilde{V_\tau})$ (per recipe [XZ17+, (6.2.2)] and [XZ19, (4.4.2)]), whose restrictions to the fiber $g\sigma_p\in\hat{G}\sigma_p$ are

$$\Xi_\nu(\mathbf{a}_{\mathrm{in}})(g\sigma_p):V_\tau\xrightarrow{\mathrm{id}\otimes\delta_\nu}V_\nu\otimes V_\tau\otimes V_{\nu^*}\xrightarrow{g\sigma_p\otimes1\otimes1}V_{\sigma(\nu)}\otimes V_\tau\otimes V_{\nu^*}\xrightarrow{\mathbf{a}_{\mathrm{in}}}V_\mu,\quad\text{and} \tag{9.3.2}$$

$$\Xi_{\nu^*}(\mathbf{a}_{\mathrm{out}})(g\sigma_p):V_\mu\xrightarrow{\mathrm{id}\otimes\delta_{\nu^*}}V_{\nu^*}\otimes V_\mu\otimes V_\nu\xrightarrow{g\sigma_p\otimes1\otimes1}V_{\sigma(\nu^*)}\otimes V_\mu\otimes V_\nu\xrightarrow{\mathbf{a}_{\mathrm{out}}}V_\tau. \tag{9.3.3}$$

Lemma 9.3.3. *Up to a non-zero constant in $\overline{\mathbb{Q}}_\ell$,*

$$\mathfrak{D}:=\Xi_{\nu^*}(\mathbf{a}_{\mathrm{out}})\circ\Xi_\nu(\mathbf{a}_{\mathrm{in}})=\prod_{i=1}^r\big(e^{\varepsilon_i-\varepsilon_{i^\vee}}+e^{\varepsilon_{i^\vee}-\varepsilon_i}-2\big)=(-1)^r\mathrm{disc}_1, \tag{9.3.4}$$

as elements in $\mathrm{Hom}(\widetilde{V_\tau},\widetilde{V_\tau})=\overline{\mathbb{Q}}_\ell[\hat{G}\sigma_p]^{\hat{G}}=\boldsymbol{J}$. In particular, when $r=1$, $\mathfrak{D}=2-e^{\varepsilon_1-\varepsilon_{1^\vee}}-e^{\varepsilon_{1^\vee}-\varepsilon_1}$.

Proof. This follows from [XZ19, Theorem 1.0.2], specialized to this case. The explicit calculation is also carried out in [XZ19, Example 6.4.2]. $\square$

Next we describe the relevant affine Deligne–Lusztig varieties and the basic locus of $\overline{\mathscr{S}}_{G,K}$. These results are essentially contained in [Vo10, VW11], although we take a different approach.

The Satake cycle $\mathbf{a}\in\mathbb{S}_{(\nu^*,\mu^*)|\tau^*+\sigma(\nu^*)}$ determines a cohomological correspondence supported on $\mathrm{Gr}^{0,\mathbf{a}}_{(\nu^*,\mu^*)|\tau^*+\sigma(\nu^*)}$. Recall that Λ is the self-dual $\mathbb{Z}_{p^2}$-lattice of the hermitian space $\mathcal{V}_p$.

Lemma 9.3.4. *Over $\mathbb{F}_{p^2}$, the correspondence*

$$\mathrm{Gr}^{0,\mathbf{a}}_{(\nu^*,\mu^*)|\tau^*+\sigma(\nu^*)} \subset \mathrm{Gr}_{\nu^*} \times \mathrm{Gr}_{\tau^*+\sigma(\nu^*)}$$

is L^+G-equivariantly isomorphic to

$$H = \{(\ell_1, \ell_2) \in \mathrm{Gr}(r, 2r+1) \times \mathrm{Gr}(r+1, 2r+1) \mid \ell_1 \subset \ell_2\} \subset \mathrm{Gr}(r, 2r+1) \times \mathrm{Gr}(r+1, 2r+1), \tag{9.3.5}$$

where $\mathrm{Gr}(i, 2r+1)$ *classifies i-dimensional subspaces in $\Lambda/p\Lambda$. The map $\mathrm{Gr}_{\nu^*} \to \mathrm{Gr}_{\tau+\sigma(\nu^*)}$ given by $g \mapsto \varpi^\tau\sigma(g)$ is identified with the map sending $\ell \subset \Lambda/p\Lambda$ to* $\mathrm{Frob}(\ell)^\perp$.

Proof. Recall the natural isomorphism $\mathrm{Gr}_G \otimes \mathbb{F}_{p^2} \cong \mathrm{Gr}_{\mathrm{GL}_n}$. For $\nu^* = -\varepsilon_{1^\vee} - \cdots - \varepsilon_{r^\vee}$, we have an identification

$$\begin{array}{rcl} \mathrm{Gr}_{\nu^*}(R) = \big\{\Lambda \otimes W(R) \subset L \subset \frac{1}{p}(\Lambda \otimes W(R)) \mid \mathrm{rk}_R\, L/(\Lambda \otimes W(R)) = 1\big\} & \xrightarrow{\cong} & \mathrm{Gr}(r, 2r+1)(R) \\ L & \longmapsto & L/(\Lambda \otimes W(R)). \end{array}$$

In a similar way, we have $\mathrm{Gr}_{\tau^*+\sigma(\nu^*)} \cong \mathrm{Gr}(r+1, 2r+1)$. The map $\mathrm{Gr}_{\nu^*} \to \mathrm{Gr}_{\tau^*+\sigma(\nu^*)}$ sends a lattice L as above to $\frac{1}{p}\,\mathrm{Frob}_R^*(L)^\vee$. Therefore, after modulo $\Lambda \otimes W(R)$, it is identified with the map $\mathrm{Gr}(r, 2r+1) \to \mathrm{Gr}(r+1, 2r+1)$ as described in the statement of the lemma.

Now we show that $\mathrm{Gr}^{0,\mathbf{a}}_{(\nu^*,\mu^*)|\tau^*+\sigma(\nu^*)}$ is given by the correspondence H in (9.3.5). This follows from the fact that the G-orbits of $\mathrm{Gr}(r, 2r+1) \times \mathrm{Gr}(r+1, 2r+1)$ are parameterized by an integer $a \in \{0, \dots, r\}$, characterized as requiring $\dim \ell_1 \cap \ell_2 = a$, and that the correspondence H (corresponding to $a = r$) is the unique (closed) orbit of dimension $r(r+1) + r$. So $\mathrm{Gr}^{0,\mathbf{a}}_{(\nu^*,\mu^*)|\tau^*+\sigma(\nu^*)} \subset \mathrm{Gr}_{\nu^*} \times \mathrm{Gr}_{\tau^*+\sigma(\nu^*)}$ must be equal to H. $\square$

The relevant affine Deligne–Lusztig variety $X_{\mu^*}(\tau^*)$ is defined as

$$X_{\mu^*}(\tau^*) := \{g \in \mathrm{Gr} \mid g^{-1}p^{\tau^*}\sigma(g) \in L^+G \cdot p^{\mu^*} L^+G\}.$$

By [XZ17+, Theorem 4.4.5], an irreducible component of the affine Deligne–Lusztig is given by the following Cartesian diagram

$$\begin{tikzcd} X^{\mathbf{b},\min}_{\mu^*}(\tau^*) \arrow[r] \arrow[d] & \mathrm{Gr}^{0,\mathbf{a}}_{(\nu^*,\mu^*)|\tau^*+\sigma(\nu^*)} \arrow[d, "\mathrm{pr}_1 \times m"] \\ \mathrm{Gr}_{\nu^*} \arrow[r, "1\times p^{\tau^*}\sigma_p"] & \mathrm{Gr}_{\nu^*} \times \mathrm{Gr}_{\tau^*+\sigma(\nu^*)}. \end{tikzcd} \tag{9.3.6}$$

More precisely, what we proved in [XZ17+, Theorem 4.4.5] is that the closure in $X_{\mu^*}(\tau^*)$ of the unique maximal dimensional irreducible component of $X^{\mathbf{b},\min}_{\mu^*}(\tau^*)$ is an irreducible component of $X_{\mu^*}(\tau^*)$. But the explicit description of Lemma 9.3.4 implies that the fiber product (9.3.6) is already irreducible and proper, and it is equal to

$$X^{\mathbf{b},\min}_{\mu^*}(\tau^*) = \{\ell \in \Lambda/p\Lambda \mid \ell \subset \mathrm{Frob}(\ell)^\perp\}. \tag{9.3.7}$$

This is a Deligne–Lusztig variety for the unitary group $\mathrm{GU}_{2r+1}(\mathbb{F}_p)$, defined over $\mathbb{F}_{p^2}$. When $r = 1$, this is a Fermat curve: under an explicit coordinate of the hermitian space $\mathcal{V}_p$ above,

$$X^{\mathbf{b},\min}_{\mu^*}(\tau^*) = \big\{(x, y, z) \in \mathbb{P}^2 \mid x^{p+1} + y^{p+1} + z^{p+1} = 0\big\}.$$

The action of K_p on $X^{\mathbf{b},\min}_{\mu^*}(\tau^*)$ factors through the quotient $K_p \to \mathrm{GU}_{2r+1}(\mathbb{F}_p)$. By [XZ17+, Theorem 4.4.14], we have a $G(\mathbb{Q}_p)$-equivariant surjective map

$$G(\mathbb{Q}_p) \times^{K_p} X^{\mathbf{b},\min}_{\mu^*}(\tau^*) \to X^{\mathbf{b}}_{\mu^*}(\tau^*) = X_{\mu^*}(\tau^*)$$

that induces a bijection on irreducible components.

Let $\mathcal{V}'$ denote the $(2r+1)$-dimensional hermitian space over E, which is isomorphic to $\mathcal{V}$ at all finite places and is of signature $(2r+1,0)$ at the infinity place. Let $G' := U(\mathcal{V}')$ denote the associated unitary group. Then the Rapoport–Zink uniformization [XZ17$^+$, Corollary 7.2.16] induces a natural surjective map

$$G'(\mathbb{Q})\backslash\big(G(\mathbb{Q}_p)\times^{K_p} X_{\mu^*}^{\mathbf{b},\min}(\tau^*)\times G'(\mathbb{A}_f^p)/K^p\big)\to\mathcal{N}_b \tag{9.3.8}$$

onto the basic locus $\mathcal{N}_b\subseteq\overline{\mathscr{S}}_{G,K}^{\mathrm{perf}}$, which induces a bijection on the set of irreducible components. This then defines the cycle class map (the Gysin map)

$$\mathrm{cl}(\mathbf{b}): C(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K,\overline{\mathbb{Q}}_\ell)\to \mathrm{H}^{2r}_{\mathrm{et},c}(\overline{\mathscr{S}}_{G,K,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell(r)).$$

There is also the dual of the cycle class map (the restriction map)

$$\mathrm{cl}^\vee(\mathbf{b}): \mathrm{H}^{2r}_{\mathrm{et},c}(\overline{\mathscr{S}}_{G,K,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell(r))\to C(G'(\mathbb{Q})\backslash G'(\mathbb{A})/K,\overline{\mathbb{Q}}_\ell).$$

Then [XZ17$^+$, Theorem 7.4.6] specialized to this case as follows.

Theorem 9.3.5. *The composition*

$$\mathrm{cl}^\vee(\mathbf{b})\circ\mathrm{cl}(\mathbf{b}): C(G'(\mathbb{Q})\backslash G'(\mathbb{A})/K,\overline{\mathbb{Q}}_\ell)\to C(G'(\mathbb{Q})\backslash G'(\mathbb{A})/K,\overline{\mathbb{Q}}_\ell)$$

is given by the Hecke operator

$$h=\sum_{i=0}^{r}(-1)^i(2i+1)p^{i(i+1)}\sum_{j=0}^{r-i}\begin{bmatrix}2r+1-2j\\ r-i-j\end{bmatrix}_{v=-p}T_{p,j}\in H(G(\mathbb{Q}_p),K_p), \tag{9.3.9}$$

which encodes the intersection matrices of the cycles in $\mathcal{N}_b$. More precisely, for $G'(\mathbb{Q})gK\in G'(\mathbb{A}_f)$, let X_g be the corresponding irreducible component of $\mathcal{N}_b$ by (9.3.8)*. Then if g and g' has the same prime-to-p-component, then the intersection number $X_g\cdot X_{g'}$ is equal to $h(g_p^{-1}g_p')$.*

In particular, if $r=1$, then

$$h=T_{p,1}+(1-p-2p^2).$$

More concretely, if $g,g'\in G'(\mathbb{A}_f)$ have the same prime-to-p part,

$$X_g\cdot X_{g'}=\begin{cases}1 & \text{if } g_p\in g'_pK_p\lambda_{1,0}K_p\\ 1-p-2p^2 & \text{if } g_p\in g'_pK_p\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By [XZ17$^+$, Theorem 7.4.6] or more precisely [XZ17$^+$, Lemma 7.4.5], and the previous discussions (Lemmas 9.2.3 and 9.2.4), we know that h is a multiple of

$$\mathrm{CT}_p^{-1}(\mathfrak{D})=p^{-r(r+1)}T_{p,r}+\cdots.$$

On the other hand, by the discussion later in §9.3.9, when $g,g'\in G'(\mathbb{A}_f)$ have the same prime-to-p part and $\mathrm{Inv}(gK_p,g'K_p)=\lambda_{1^r,0}=\lambda_r$, X_g and X_g' intersect properly at one point. It follows that $h=T_{p,r}+\cdots=p^{r(r+1)}\cdot\mathrm{CT}_p^{-1}(\mathfrak{D})$. The proposition follows. ∎

9.3.6. *Interpretation in classical terms.* We now reinterpret Theorem 9.3.5 using the geometry of the corresponding Shimura varieties. When $r=1$, namely the case of Picard modular surfaces, we will reprove the theorem by an explicit computation; for the general case, we will reduce the theorem to a computation of the degree of a vector bundle on a Deligne–Lusztig variety.

If a coherent sheaf M over an $\mathcal{O}_{E,(p)}$-scheme is equipped with an $\mathcal{O}_E$-action, we write $M\cong M_1\oplus M_2$ such that $\mathcal{O}_E$ acts on M_1 by the structure map and on M_2 by the complex conjugate of the structure map. Let $\mathcal{A}$ denote the universal abelian scheme over $\overline{\mathscr{S}}_{G,K}$ (which is an $\mathcal{O}_E/(p)$-scheme). There is a natural $\mathcal{O}_E$-action on invariant differentials $\omega_{\mathcal{A}}$, and the Kottwitz condition implies that, under the decomposition by action of $\mathcal{O}_E$

$$\omega_{\mathcal{A}} \cong \omega_{\mathcal{A},1} \oplus \omega_{\mathcal{A},2},$$

where $\omega_{\mathcal{A},1}$ is locally free of rank 1 and $\omega_{\mathcal{A},2}$ is locally free of rank $2r$.

Remark 9.3.7. When $r=1$, one can describe the basic locus (or equivalently supersingular) $\mathcal{N}_b$ of the Picard modular surface as follows. The Verschiebung map $V : \mathcal{A}^{(p)} \to \mathcal{A}$ induces two maps

$$h_1 : \omega_{\mathcal{A},1} \to \omega_{\mathcal{A}^{(p)},1} \cong \omega_{\mathcal{A},2}^{(p)} \quad \text{and} \quad h_2 : \omega_{\mathcal{A},2} \to \omega_{\mathcal{A},1}^{(p)},$$

where for a coherent sheaf $\mathcal{F}$ on $\overline{\mathscr{S}}_{G,K}$, $\mathcal{F}^{(p)} = \mathcal{F} \otimes_{\overline{\mathscr{S}}_{G,K},\mathrm{Frob}} \overline{\mathscr{S}}_{G,K}$ is pullback along the relative Frobenius. The composition $h := h_2^{(p)} \circ h_1 : \omega_{\mathcal{A},1} \to (\omega_{\mathcal{A},1})^{p^2}$ can be regarded as the Hasse invariant for the Picard modular surface. Then $\mathcal{N}_b$ is (the reduced subscheme of) the vanishing locus of h. Indeed, we can work pointwise. So let $\bar{x}$ be an $\overline{\mathbb{F}}_p$-point, and let $\mathbb{D}(\mathcal{A}_{\bar{x}})$ be the contravariant Dieudonné module of $\mathcal{A}_{\bar{x}}$. In fact, for the given signature condition, there are only two possibilities of the Newton polygon of $\mathbb{D}(\mathcal{A}_{\bar{x}})$, namely, either with slopes $0, \frac{1}{2}, 1$ (the μ-ordinary case), or with slopes $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ (the supersingular case). In the first case, any power of the Verschiebung is nonzero on $\omega_{\mathcal{A}_{\bar{x}},1}$ and hence h is nonzero at $\bar{x}$. In the second case, the Frobenius is topologically nilpotent on $\mathbb{D}(\mathcal{A}_{\bar{x}})$, so h has to be zero at $\bar{x}$. This proves the claim.

Since we are working with the PEL type Shimura varieties, one can also make the cohomological correspondence (9.3.8) explicit as follows. Let $\overline{\mathscr{S}}_{G',K}$ denote the special fiber of the Shimura variety associated to the unitary group G'. In particular, making use of the same auxiliary CM elliptic curve A_0 from § 9.3.1, $\overline{\mathscr{S}}_{G',K}$ classifies triples (A', λ', η') over a noetherian $\mathbb{F}_{p^2}$-scheme S, where

- A' is an abelian schemes over S with a faithful action by $\mathcal{O}_E$, satisfying the Kottwitz' condition: $\omega_{A',1} = H^1_{\mathrm{dR}}(A')_1$ and $\omega_{A',2} = 0$;
- $\lambda' : A' \to A'^{\vee}$ is a principal polarization in the category of abelian varieties up to prime-to-p isogenies such that the Rosati involution induces complex conjugation on $\mathcal{O}_E$, and
- η' is a level structure as defined in § 9.3.1 with A replaced by A'.

We will apply the general construction of correspondences between Shimura varieties, but over a subspace of $\mathrm{Sht}^{\mathrm{loc}}_{\tau|\mu}$, where the isogeny of the local G-shtukas is restricted a particular type corresponding to $X_{\mu^*}^{\mathbf{b},\min}$ in (9.3.7). More precisely, we define a moduli space $\widetilde{\mathcal{N}}_b$ that classifies tuples $(A, \lambda, \eta, A', \lambda', \eta', f)$ over noetherian $\mathbb{F}_{p^2}$-schemes S, where

- (A, λ, η) is an S-point of $\overline{\mathscr{S}}_{G,K}$ and (A', λ', η') is an S-point of $\overline{\mathscr{S}}_{G',K}$,
- $f : A' \dashrightarrow A$ is an $\mathcal{O}_E$-equivariant quasi-isogeny such that $\lambda' = f^{\vee} \circ \lambda \circ f$, and $\eta = f_* \circ \eta'$, and
- at each geometric point $\bar{x}$, f induces inclusions $(f^{-1})^* : \mathbb{D}(A'_{\bar{x}})_1 \subset \mathbb{D}(A_{\bar{x}})_1$ and $f^* : \mathbb{D}(A_{\bar{x}})_2 \subset \mathbb{D}(A'_{\bar{x}})_2$ whose cokernels are all isomorphic to $k(\bar{x})^{\oplus r}$.

The forgetful map $\pi : \widetilde{\mathcal{N}}_b \to \overline{\mathscr{S}}_{G,K}$ remembering only A is exactly the Rapoport–Zink uniformization map (9.3.8). The other forgetful map $\pi' : \widetilde{\mathcal{N}}_b \to \overline{\mathscr{S}}_{G',K}$ remembering only A' identifies $\widetilde{\mathcal{N}}_b$ with the left hand side of (9.3.8); more precisely, the fiber over each coset $G'(\mathbb{Q})gK \in G'(\mathbb{A}_f)$ corresponds to an $\overline{\mathbb{F}}_p$-point $g \in \overline{\mathscr{S}}_{G',K}$; $X_g := \pi'^{-1}(g)$ classifies r-dimensional subspaces $H \subset H_1^{\mathrm{dR}}(A'_g)_1$ such that $H \subset \mathrm{Frob}(H)^{\perp}$. In terms of the moduli problem above, the subspace H is the image $f_2^* \omega_{A,2} \subseteq H^1_{\mathrm{dR}}(A'_g)_2 \cong H_1^{\mathrm{dR}}(A'_g)_1$ (which has dimension r). The perfection of each X_g is isomorphic to $X_{\mu^*}^{\mathbf{b},\min}(\tau^*)$ in (9.3.7) over $\overline{\mathbb{F}}_p$.

We first assume that $r = 1$, i.e. $\overline{\mathscr{S}}_{G,K}$ is a Picard modular surface. The space X_g may be identified with certain degree $(p+1)$ Fermat curve inside $\mathbb{P}(H_1^{\mathrm{dR}}(A'_g)_1) \simeq \mathbb{P}^2$. We denote the restriction of $\mathcal{O}_{\mathbb{P}^2}(n)$ to this curve by $\mathcal{O}(n)$. In this case, our computation of the intersection number (Theorem 9.3.5) is essentially equivalent to the following result.

Proposition 9.3.8. *When $r = 1$, the restriction of π to each X_g is a regular embedding (whose image we still denote by X_g), and its normal bundle in $\overline{\mathscr{S}}_{G,K}$ is $\mathcal{O}(1-2p)$. In particular, the self intersection of X_g in $\overline{\mathscr{S}}_{G,K}$ is $(p+1)(1-2p) = 1 - p - 2p^2$.*

Proof. We only sketch the proof here; the readers are invited to fill in the details. Let $\mathcal{H}$ denote the universal line subbundle of $H_1^{\mathrm{dR}}(A'_g)_1 \otimes \mathcal{O}_{X_g} \cong H^1_{\mathrm{dR}}(A'_g)_2 \otimes \mathcal{O}_{X_g}$ over X_g, which is the same as $\mathcal{O}(-1)$. Then the tangent bundle of X_g is given by

$$T_{X_g} \cong \mathcal{H}om\big(\mathcal{H}, (\mathcal{H}^{(p)})^{\perp}/\mathcal{H}\big).$$

One can verify that over X_g, the universal quasi-isogeny $f : \mathcal{A}' \to \mathcal{A}$ induces a homomorphism $f_2^* : H^1_{\mathrm{dR}}(\mathcal{A})_2 \to H^1_{\mathrm{dR}}(\mathcal{A}')_2$. From this, we deduce a canonical isomorphism

$$(\mathcal{H}^{(p)})^{\perp}/\mathcal{H} \cong f_2^{*,-1}(H^1_{\mathrm{dR}}(\mathcal{A})_2)/f_2^{*,-1}(\omega_{\mathcal{A},2}) \xrightarrow[\cong]{f_2^*} H^1_{\mathrm{dR}}(\mathcal{A})_2/\omega_{\mathcal{A},2}.$$

Under this isomorphism, we have an exact sequence over X_g:

$$0 \to T_{X_g} \cong \mathcal{H}om\big(\mathcal{H}, (\mathcal{H}^{(p)})^{\perp}/\mathcal{H}\big) \to T_{\overline{\mathscr{S}}_{G,K}|X_g} \cong \mathcal{H}om\big(\omega_{\mathcal{A},2}, H^1_{\mathrm{dR}}(\mathcal{A})_2/\omega_{\mathcal{A},2}\big) \to N_{X_g}(\overline{\mathscr{S}}_{G,K}) \to 0.$$

In particular, this implies that

$$\begin{aligned} N_{X_g}(\overline{\mathscr{S}}_{G,K}) &\cong \mathcal{H}om(\mathrm{Ker}(f_2^*), (\mathcal{H}^{(p)})^{\perp}/\mathcal{H}) \cong \mathcal{H}om\big(H^1_{\mathrm{dR}}(\mathcal{A}')_2/(\mathcal{H}^{(p)})^{\perp}, (\mathcal{H}^{(p)})^{\perp}/\mathcal{H}\big) \\ &\cong \mathcal{H}^{(p)} \otimes \big((\mathcal{H}^{(p)})^{\perp}/\mathcal{H}\big) \cong \mathcal{H}om(\mathcal{O}(p), \mathcal{O}(-p+1)) \cong \mathcal{O}(1-2p). \end{aligned}$$

□

On the other hand, X_g and $X_{g'}$ (for $g \neq g'$) intersects if and only if g and g' are related by the Hecke correspondence $T_{p,1}$, i.e. $\mathrm{Inv}(gK_p, g'K_p) = \lambda_{1,0}$, in which case X_g and $X_{g'}$ intersects properly at one point. This gives a direct proof of Theorem 9.3.5 when $r = 1$.

9.3.9. *Computation of $X_g \cdot X_{g'}$ in the general case.* We now describe the intersection of these cycles for general r. The computation is similar to [HTX17, §5–6] (which uses covariant Dieudonné modules). The cycles X_g and $X_{g'}$ intersect if and only if g and g' are related by the Hecke correspondence $T_{p,i}$ for some $0 \leq i \leq r$, after replacing coset representatives, g and g' have the same prime-to-p part and $\mathrm{Inv}(g_pK_p, g'_pK_p) = \lambda_i = \lambda_{1^i,0^{r-i},0}$. In this case, one can compute the intersection of X_g with $X_{g'}$ following the steps below.

(1) Let A'_g and $A'_{g'}$ be the universal abelian varieties at $G'(\mathbb{Q})gK$ and at $G'(\mathbb{Q})g'K$, respectively. Since $\mathrm{Inv}(gK_p, g'K_p) = \lambda_{1^i,0^{r-i},0}$, there is a quasi-isogeny $f' : A'_g \to A'_{g'}$ preserving the tame level structures and the polarizations on both abelian varieties such that under the identification $f' : \mathbb{D}(A'_g)[\frac{1}{p}] \cong \mathbb{D}(A'_{g'})[\frac{1}{p}]$, we have $(\mathbb{D}(A'_g)_j + \mathbb{D}(A'_{g'})_j)/\mathbb{D}(A'_g)_j \cong \overline{\mathbb{F}}_p^{\oplus i}$ for $j = 1, 2$. Moreover, we can show that the perfect pairing induced by polarization between $\mathbb{D}(A'_g)_2$ and $\mathbb{D}(A'_g)_1 = \mathrm{Frob}(\mathbb{D}(A'_g)_2)$ induces a perfect pairing between

$$\mathbb{D}_{g,g',2} := \big(\mathbb{D}(A'_g)_2 \cap \mathbb{D}(A'_{g'})_2\big)/\big(p\mathbb{D}(A'_g)_2 + p\mathbb{D}(A'_{g'})_2\big) \quad \text{and} \quad \mathrm{Frob}(\mathbb{D}_{g,g',2}), \tag{9.3.10}$$

where both spaces are of dimension $2r - 2i + 1$ over $\overline{\mathbb{F}}_p$.

(2) The intersection $X_g \cap X_{g'}$ parametrizes quasi-$\mathcal{O}_E$-isogenies $A'_g \xrightarrow{f_1} A \xleftarrow{f_2} A'_{g'}$ compatible with polarizations and tame level structures such that the composition $f_2^{-1} \circ f_1$ is just f', and at each geometric point $\bar{x}$, f_1 and f_2 induce includes $\mathbb{D}(A'_g)_1 \subset \mathbb{D}(A_{\bar{x}})_1 \supset \mathbb{D}(A'_{g'})_1$ and $\mathbb{D}(A'_g)_2 \supset \mathbb{D}(A_{\bar{x}})_2 \subset \mathbb{D}(A'_{g'})_2$ so that the cokernel of each inclusion is isomorphic to $k(\bar{x})^{\oplus r}$.

(3) One can then show that this moduli interpretation of $X_g \cap X_{g'}$ is equivalent to the moduli problem of an $(r-i)$-dimensional subspace

$$\overline{H} := (f_1)_2^*(\omega_{A,2}) \bmod \big(p\mathbb{D}(A'_g)_2 + p\mathbb{D}(A'_{g'})_2\big)$$

as a subspace of $\mathbb{D}_{g,g',2}$ such that $\overline{H} \in \mathrm{Frob}(\overline{H})^\perp$, where $\bullet^\perp$ means to take the annihilator under the perfect pairing between (9.3.10).

In particular, when $i = r$, X_g and $X_{g'}$ intersect properly.

(4) To compute the intersection number of X_g and $X_{g'}$, one needs to use the excessive intersection formula ([Fu98, §6.3]).

$$X_g \cdot X_{g'} := \int_{X_g \cap X_{g'}} c_{r-i}(\mathcal{E}_{g,g'}) \quad \text{for} \quad \mathcal{E}_{g,g'} := N_{X_g}(\overline{\mathscr{S}}_{G,K})|_{X_g \cap X_{g'}} \Big/ N_{X_g \cap X_{g'}}(X_g).$$

Using an argument similar to the discussion in Proposition 9.3.8, one can prove that, setting $\mathcal{H} := (f_1)_2^*(\omega_{\mathcal{A},2})$,

$$\begin{aligned} N_{X_g}(\overline{\mathscr{S}_K}) &\cong \mathcal{H}^{(p)} \otimes \big((\mathcal{H}^{(p)})^\perp/\mathcal{H}\big) \quad \text{and} \\ N_{X_g \cap X_{g'}}(X_g) &\cong \big((\mathcal{H}^{(p)})^\perp/\mathcal{H}\big) \otimes (p\mathbb{D}(A'_g)_2 + p\mathbb{D}(A'_{g'})_2)/p\mathbb{D}(A'_g)_2. \end{aligned}$$

Therefore, using the fact that $(\overline{\mathcal{H}}^{(p)})^\perp/\overline{\mathcal{H}} \cong (\mathcal{H}^{(p)})^\perp/\mathcal{H}$, we deduce that

$$\mathcal{E}_{g,g'} \cong \overline{\mathcal{H}}^{(p)} \otimes \big((\overline{\mathcal{H}}^{(p)})^\perp/\overline{\mathcal{H}}\big).$$

In other words, we essentially need to compute the top degree of a fixed vector bundle over a Deligne–Lusztig varieties (but for all different dimensional unitary spaces). This is the following lemma (which then justifies the expression of intersection of cycles in Proposition 9.3.5).

Now the above discussions imply that Theorem 9.3.5 is in fact equivalent to the following statement.

Proposition 9.3.10. *Consider the r-dimensional Deligne–Lusztig variety* (9.3.7)*, namely*

$$\mathrm{DL}_r := \big\{H \subset \Lambda/p\Lambda \text{ of dimension } r \mid H \subseteq \mathrm{Frob}(H)^\perp\big\}.$$

Let $\mathcal{H}$ denote the universal subbundle of rank r, then for $\mathcal{E} = \mathcal{H}^{(p)} \otimes \big((\mathcal{H}^{(p)})^\perp/\mathcal{H}\big)$, we have

$$\int_{\mathrm{DL}_r} c_r(\mathcal{E}) = \sum_{i=0}^{r} (-1)^i (2i+1) p^{i^2+i} \begin{bmatrix} 2r+1 \\ r-i \end{bmatrix}_{v=-p}$$

Proof. Indeed, we may deduce it from Theorem 9.3.5 by simply retrieving the coefficient of $T_{p,0}$ in (9.3.9). And vice versa, the proposition gives the coefficent of □

When $r = 1$, this is Proposition 9.3.8 above. In general, unfortunately, we do not know a direct proof to this lemma.

Remark 9.3.11. We hope to convey through the above discussion that formulating our main theorem in terms of geometric Satake theory allows us to overcome the difficult combinatorics questions like the above lemma.

9.4. **The $G = \mathrm{PSO}(2, n-2)$-Shimura varieties.** Consider a quadratic space $\mathcal{V}$ over $\mathbb{Q}$ of signature $(2, n-2)$ at infinity, and let $\mathrm{PSO}(\mathcal{V})$ denote the corresponding projective orthogonal group. Let p be an odd prime such that $\mathcal{V}_p := \mathcal{V} \otimes \mathbb{Q}_p$ is unramified, i.e. the Hasse invariant is 1 and its determinant is a p-adic unit modulo $(\mathbb{Q}_p^\times)^2$. We take K_p to be the stabilizer in $G(\mathbb{Q}_p)$ of a self-dual lattice $\Lambda \subset \mathcal{V}_p$. The corresponding Shimura variety is of abelian type (witnessed by the $\mathrm{GSpin}(\mathcal{V})$-Shimura variety), and its integral model $\mathscr{S}_{G,K}$ over $\mathbb{Z}_{(p)}$ is constructed by Kisin [Kis10]; its special fiber $\overline{\mathscr{S}}_{G,K}$ is of dimension n. We focus on its basic locus $\mathcal{N}_b \subset \overline{\mathscr{S}}_{G,K}$.

When n is odd, we do not expect any middle dimensional Tate classes for the dimension reason. This is related to the fact that the associated representation V_{μ^*} of the dual group $\hat{G} = \mathrm{Sp}_{n-1}$ is the vector representation, and is minuscule. So $V_{\mu^*}^{\mathrm{Tate}} = 0$. (More generally, if G splits over $\mathbb{Q}_p$, $V_{\mu^*}^{\mathrm{Tate}}$ is always trivial.)

For the rest of this subsection, we assume $n = 2m$ is even. The description of the basic locus was essentially contained in [HP17] following [GH15]. We explain how their findings fit in our philosophy and how [XZ17$^+$, Theorem 7.4.6] applies to this case.

The group G is of type D_m so $\hat{G} = \mathrm{Spin}_{2m}$. We need to separate two cases:

(a) when $\det \mathcal{V}_p \equiv (-1)^m \bmod (\mathbb{Q}_p^\times)^2$, $G_{\mathbb{Q}_p}$ splits over $\mathbb{Q}_p$, and

(b) when $\det \mathcal{V}_p \bmod (\mathbb{Q}_p^\times)^2$ is a p-adic unit by is not equal to $(-1)^m$, $G_{\mathbb{Q}_p}$ is unramified but non-split over $\mathbb{Q}_p$, and the Frobenius acts non-trivially on the Dynkin diagram of G.

We identify $\mathbb{G}_m^m$ with the diagonal maximal torus of SO_{2m} by

$$\underline{t} = (t_1, \dots, t_m) \mapsto \mathrm{Diag}(t_1, \dots, t_m, t_m^{-1}, \dots, t_1^{-1}),$$

and take the maximal torus $\hat{T}$ of the dual group $\hat{G} = \mathrm{Spin}_{2m}$ to be the preimage of $\mathbb{G}_m^m$ under the natural 2-to-1 map $\mathrm{Spin}_{2m} \to \mathrm{SO}_{2m}$. For $i = 1, \dots, m$, let ε_i denote the character of $\mathbb{G}_m^m$ that sends $\underline{t}$ to t_i; then we may identify the weight space of $\hat{G}$ with

$$\mathbb{X}^\bullet(\hat{T}) = \mathbb{Z} \cdot \frac{\varepsilon_1 + \cdots + \varepsilon_m}{2} \oplus \bigoplus_{i=1}^{m-1} \mathbb{Z}\varepsilon_i,$$

and the roots of $\hat{G}$ are $\pm(\varepsilon_i - \varepsilon_j)$ and $\pm(\varepsilon_i + \varepsilon_j)$ for $1 \leq i < j \leq m$. A weight $\sum_{i=1}^m a_i \varepsilon_i$ is dominant if $a_1 \geq \cdots \geq a_{m-1} \geq |a_m|$. The absolute Weyl group W of $\hat{G}$ is $R \rtimes S_m$, where $R \subset \{\pm 1\}^m$ is the subgroup of elements with total product 1 which acts on $\mathbb{X}^\bullet(\hat{T})$ by coordinate-wise multiplication, and S_m acts on $\mathbb{X}^\bullet(\hat{T})$ by permuting the factors. So for a dominant weight $\mu = \sum_{i=1}^m a_i \varepsilon_i$, $\mu^* = \sum_{i=1}^{m-1} a_i \varepsilon_i - a_m \varepsilon_m$.

The Hodge cocharacter of G induces the weight $\mu = \varepsilon_1$ of $\hat{G}$ and the associated highest weight representation V_{μ^*} is the vector representation $\mathrm{Spin}_{2m} \to \mathrm{SO}_{2m} \to \mathrm{GL}(Q)$ with $Q = \overline{\mathbb{Q}}_\ell^{\oplus 2m}$; its weights are $\pm\varepsilon_1, \dots, \pm\varepsilon_m$.

In case (a) above, i.e. when the projective orthogonal group $G_{\mathbb{Q}_p}$ splits over $\mathbb{Q}_p$, we have $V_{\mu^*}^{\mathrm{Tate}} = V_{\mu^*}(0) = 0$. In this case, the philosophy behind [XZ17$^+$, Theorem 1.1.4(1)] predicts that the dimension of the basic locus $\mathcal{N}_b$ should be strictly less than $\frac{1}{2} \dim \overline{\mathscr{S}}_{G,K} = \frac{n}{2}$. This agrees with the findings in [HP17, Theorem C].

We now assume that we are in case (b), i.e. $G_{\mathbb{Q}_p}$ is unramified and non-split over $\mathbb{Q}_p$.

Lemma 9.4.1. *The Frobenius element $\phi_p = \sigma_p$ fixes $\varepsilon_1, \dots, \varepsilon_{m-1}$ and sends ε_m to $-\varepsilon_m$. We have*

$$V_{\mu^*}^{\mathrm{Tate}} = V_{\mu^*}(\lambda_+) \oplus V_{\mu^*}(\lambda_-) \qquad \text{for} \quad \lambda_+ = \varepsilon_m \text{ and } \lambda_- = -\varepsilon_m,$$

which is of 2-dimensional. We can write $\lambda_\pm = \sigma(\nu_\pm^*) - \nu_\pm^*$ (so $\tau_\pm = 0$) with

$$\nu_+ = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m), \quad \text{and} \quad \nu_- = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m-1} - \varepsilon_m).$$

In addition, each $\mathbb{MV}_{\mu^*}(\lambda_\pm) = \{\mathbf{b}_\pm\}$ *is a singleton, and is represented by nonzero homomorphisms*

$$\mathbf{b}_{\pm,\mathrm{in}} : V_{\sigma(\nu_\pm)} \otimes V_{\nu_\pm^*} \to V_\mu \quad \textit{and} \quad \mathbf{b}_{\pm,\mathrm{out}} : V_{\sigma(\nu_\pm^*)} \otimes V_\mu \otimes V_{\nu_\pm} \to \mathbf{1}. \tag{9.4.1}$$

Proof. The dimension of the two hom spaces in (9.4.1) can be computed as

$$\dim \mathrm{Hom}(V_{\sigma(\nu_\pm)} \otimes V_{\nu_\pm^*}, V_\mu) = \dim \mathrm{Hom}(V_{\sigma(\nu_\pm)}, V_{\nu_\pm} \otimes V_\mu).$$

Since μ is minuscule, $V_{\nu_\pm} \otimes V_\mu = \bigoplus_\eta V_{\nu_\pm+\eta}$ with the sum taken over all weights $\eta \in W\mu$ in the Weyl group orbit such that $\nu_\pm + \eta$ is dominant. One checks directly, $\sigma(\nu_\pm)$ belongs to this list of weights. □

Set $\hat{S} := \hat{T}/(\sigma_p - 1)\hat{T}$ and then

$$\mathbb{X}^\bullet(\hat{S}) = \mathbb{X}^\bullet(\hat{T})^{\sigma_p=1} = \bigoplus_{i=1}^{m-1} \mathbb{Z}\varepsilon_i.$$

The relative Weyl group is $W_0 = W^{\sigma_p} \cong (\mathbb{Z}/2\mathbb{Z})^{m-1} \rtimes S_{m-1}$, where S_{m-1} permutes $\varepsilon_1, \dots, \varepsilon_{m-1}$ and $(\mathbb{Z}/2\mathbb{Z})^{m-1}$ acts on each of $\varepsilon_1, \dots, \varepsilon_{m-1}$ by multiplication by ±1. So the invariants of $\overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]$ under W_0 are

$$\boldsymbol{J} := \overline{\mathbb{Q}}_\ell[\mathbb{X}^\bullet(\hat{S})]^{W_0} = \overline{\mathbb{Q}}_\ell[\mathfrak{S}_1, \dots, \mathfrak{S}_{m-1}], \tag{9.4.2}$$

where $\mathfrak{S}_i$ for $i = 1, \dots, m-1$ is the ith symmetric polynomial in $e^{\varepsilon_1} + e^{-\varepsilon_1}, \dots, e^{\varepsilon_{m-1}} + e^{-\varepsilon_{m-1}}$.

The relative root system $\Phi_{\mathrm{rel}}^\vee \subset \mathbb{X}^\bullet(\hat{S})$ consists of roots $\pm(\varepsilon_i + \varepsilon_j)$ and $\pm\varepsilon_i$ for $1 \le i \le j \le m-1$. In particular, the discriminant of the W_0-orbits of long roots $\pm2\varepsilon_1, \dots, \pm2\varepsilon_{m-1}$ is

$$\mathrm{disc}_{\mathrm{long}} = \prod_{i=1}^{m-1} (e^{2\varepsilon_i} - 1)(e^{-2\varepsilon_i} - 1).$$

Lemma 9.4.2. *For* $i, j \in \{+, -\}$*, the elements*

$$\mathfrak{M}_{ij} := \Xi_{\nu_i^*}(\mathbf{b}_{i,\mathrm{out}}) \circ \Xi_{\nu_j}(\mathbf{b}_{j,\mathrm{out}}) \in \mathrm{Hom}\big(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}\big) \cong \boldsymbol{J}$$

can be computed explicitly as follows (up to a nonzero constant in $\overline{\mathbb{Q}}_\ell$*):*

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{M}_{++} & \mathfrak{M}_{+-} \\ \mathfrak{M}_{-+} & \mathfrak{M}_{--} \end{pmatrix} = \begin{pmatrix} \sum_{i\ge0 \text{ even}} 2^{m-1-i}\mathfrak{S}_i & \sum_{i>0 \text{ odd}} 2^{m-1-i}\mathfrak{S}_i \\ \sum_{i>0 \text{ odd}} 2^{m-1-i}\mathfrak{S}_i & \sum_{i\ge0 \text{ even}} 2^{m-1-i}\mathfrak{S}_i \end{pmatrix}.$$

Moreover, the determinant of $\mathfrak{M}$ *is* $\pm\mathrm{disc}_{\mathrm{long}}$.

Proof. See [XZ19, Lemma 6.4.5]. □

We remark that in the process of changing from σ_p to ϕ_p following [XZ17+, Remark 3.5.3], the formula in Lemma 9.4.2 is unchanged. We leave it to the interesting readers to make the Satake isomorphism explicit in this case.

Now we focus on the relevant affine Deligne–Lusztig varieties and the basic locus of $\overline{\mathscr{S}}_{G,K}$. These results are essentially contained in [HP17]. The $\hat{G}$-representation homomorphisms $\mathbf{b}_{\pm,\mathrm{in}}$ induce $\hat{G}$-homomorphisms

$$\mathbf{b}_\pm : V_{\nu_\pm^*} \otimes V_{\mu^*} \to V_{\sigma(\nu_\pm^*)},$$

which, by Geometric Satake theory, corresponds to a cohomological correspondence supported on $\mathrm{Gr}^{0,\mathbf{b}_\pm}_{(\nu_\pm^*,\mu^*)|\sigma(\nu_\pm^*)}$. Here Gr denotes the affine Grassmannian for the group $\mathrm{PSO}(\mathcal{V}_p)$. Recall that Λ is the self-dual $\mathbb{Z}_{p^2}$-lattice of the quadratic space $\mathcal{V}_p$.

Lemma 9.4.3. *Over $\mathbb{F}_{p^2}$, the correspondence*

$$\mathrm{Gr}^{0,\mathbf{b}_\pm}_{(\nu^*_\pm,\mu^*)|\sigma(\nu^*_\pm)} \subset \mathrm{Gr}_{\nu^*_\pm} \times \mathrm{Gr}_{\sigma(\nu^*_\pm)}$$

is L^+G-equivariantly isomorphic to

(9.4.3) $\quad H = \big\{(\mathscr{L}_1, \mathscr{L}_2)$ *m-dimensional Lagrangian subspaces of* $\Lambda/p\Lambda \mid \dim \mathscr{L}_1 + \mathscr{L}_2 \le m+1\big\}$.

The Frobenius map $\mathrm{Gr}_{\nu^*_\pm} \to \mathrm{Gr}_{\sigma(\nu_\pm)^*}$, $g \mapsto \sigma(g)$ is identified with the map sending $\mathscr{L}$ to $\sigma_p(\mathscr{L})$.

Proof. This is essentially contained in [HP14, § 3.2]; we only sketch the essence here. One may choose a $\mathbb{Z}_{p^2}$-basis $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ of $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$ such that

- $\langle e_1, \dots, e_m\rangle$ and $\langle f_1, \dots, f_m\rangle$ are (maximal) isotropic, and $\langle e_i, f_j\rangle = \delta_{ij}$,
- the Frobenius $1 \otimes \sigma$ fixes $e_1, \dots, e_{m-1}, f_1, \dots, f_{m-1}$, and interchanges e_m with f_m, and
- the parabolic subgroups $P_\pm$ generated by root subgroups U_α for those $\langle \alpha, \nu_\pm\rangle \le 0$, are precisely the stabilizer subgroups of the maximal isotropic subspaces $\mathscr{L}_+ = \langle f_1, \dots, f_m\rangle$ and $\mathscr{L}_- = \langle f_1, \dots, f_{m-1}, e_m\rangle$, respectively.

Then, we have a natural isomorphism

$$\begin{array}{ccccc} \mathrm{Gr}_{\nu^*_\pm} \cong \mathrm{Gr}_{\nu_\mp} & \xrightarrow{\cong} & G/P_\mp & \xrightarrow{\cong} & \{m\text{-dimensional Lagrangian subspaces of } \Lambda/p\Lambda\} \\ gp^{\nu_\mp}L^+G & \longmapsto & (g \bmod L^+G^{(1)}) \bmod P_\mp & \longmapsto & g\mathscr{L}_\mp. \end{array}$$

The lemma follows from the above isomorphism. $\square$

The relevant affine Deligne–Lusztig variety $X_{\mu^*}(1)$ is defined as

$$X_{\mu^*}(1) := \{g \in \mathrm{Gr} \mid g^{-1}\sigma(g) \in L^+G \cdot p^{\mu^*} L^+G\}.$$

By [XZ17+, Theorem 4.4.5], two irreducible components of the affine Deligne–Lusztig are given by the following Cartesian diagram

$$\begin{tikzcd} X^{\mathbf{b}_\pm,\min}_{\mu^*}(1) \arrow[r] \arrow[d] & \mathrm{Gr}^{0,\mathbf{b}_i}_{(\nu^*_\pm,\mu^*)|\sigma(\nu^*_\pm)} \arrow[d, "\mathrm{pr}_1 \times m"] \\ \mathrm{Gr}_{\nu^*_\pm} \arrow[r, "1\times\sigma_p"] & \mathrm{Gr}_{\nu^*_\pm} \times \mathrm{Gr}_{\sigma(\nu^*_\pm)}. \end{tikzcd}$$

More precisely, we also needed here the explicit description in Lemma 9.4.3 to deduce that the fiber product is itself irreducible and is hence equal to $X^{\mathbf{b}_\pm,\min}_{\mu^*}(1)$. Explicitly,

(9.4.4)

$$X^{\mathbf{b}_\pm,\min}_{\mu^*}(1) = \big\{\mathscr{L}\ m\text{-dimensional Lagrangian subspace of } \Lambda/p\Lambda \mid \dim(\mathscr{L} + \sigma_p(\mathscr{L})) = m+1\big\}.$$

It is the Deligne–Lusztig variety over $\mathbb{F}_{p^2}$ for the unitary group $\mathrm{PSO}(\Lambda/p\Lambda)$ associated to the Coxeter elements.

In general we have a $G(\mathbb{Q}_p)$-equivariant surjective map

$$\coprod_{i\in\{\pm\}} G(\mathbb{Q}_p) \times^{K_p} X^{\mathbf{b}_i,\min}_{\mu^*}(1) \to X^{\mathbf{b}_+}_{\mu^*}(1) \cup X^{\mathbf{b}_-}_{\mu^*}(1) = X_{\mu^*}(1)$$

that induces a bijection on irreducible components. Let $\mathcal{V}'$ denote the $2m$-dimensional quadratic space over $\mathbb{Q}$ of signature $(0, n)$ at infinity and is isomorphic to $\mathcal{V}$ at all non-archimedean places. Set $G' := \mathrm{PSO}(\mathcal{V}')$ denote the associated projective orthogonal group. Then the Rapoport–Zink uniformization [XZ17+, Corollary 7.2.16] induces a natural surjective map

$$\coprod_{i\in\{\pm\}} G'(\mathbb{Q})\big\backslash\big(G(\mathbb{Q}_p) \times^{K_p} X^{\mathbf{b}_i,\min}_{\mu^*}(\tau^*) \times G'(\mathbb{A}^p_f)\big/K^p\big) \to \mathcal{N}_b \tag{9.4.5}$$

onto the basic locus $\mathcal{N}_b \subseteq \overline{\mathscr{S}}^{\mathrm{perf}}_{G,K}$, which induces a bijection on the set of irreducible components. This then defines the cycle class map (the Gysin map) and its dual:

$$\mathrm{cl}(\mathbf{b}_\pm) : C(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell) \to \mathrm{H}^{2m}_{\mathrm{et},c}(\overline{\mathscr{S}}_{G,K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(m)).$$
$$\mathrm{cl}^\vee(\mathbf{b}_\pm) : \mathrm{H}^{2m}_{\mathrm{et},c}(\overline{\mathscr{S}}_{G,K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(m)) \to C(G'(\mathbb{Q})\backslash G'(\mathbb{A})/K, \overline{\mathbb{Q}}_\ell).$$

Then [XZ17$^+$, Theorem 7.4.6] specialized to this case as follows.

Theorem 9.4.4. *For $i, j \in \{+, -\}$, the composition*

$$\mathrm{cl}^\vee(\mathbf{b}_i) \circ \mathrm{cl}(\mathbf{b}_j) : C(G'(\mathbb{Q})\backslash G'(\mathbb{A})/K, \overline{\mathbb{Q}}_\ell) \to C(G'(\mathbb{Q})\backslash G'(\mathbb{A})/K, \overline{\mathbb{Q}}_\ell)$$

is the multiplication by $\mathfrak{M}_{ij}$ in Lemma 9.4.2 under the Satake isomorphism $\boldsymbol{J} \cong C_c(K_p\backslash G'(\mathbb{Q}_p)/K_p, \overline{\mathbb{Q}}_\ell)$. Moreover, for π_f an irreducible module of the Hecke algebra $\mathcal{H}_K$, if the Satake parameter for $\pi_{f,p}$ avoids the zeros of $e^{2\varepsilon_i} = 1$ for all i, then the following map is injective:

$$C(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell)[\pi_f]^{\oplus 2} \xrightarrow{\mathrm{cl}(\mathbf{b}_+)+\mathrm{cl}(\mathbf{b}_-)} \mathrm{H}^{2m}_{\mathrm{et},c}\big(\overline{\mathscr{S}}_{G,K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(m)\big)[\pi_f].$$

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