INTRODUCTION TO p-ADIC HODGE THEORY

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Abstract. We first give an introduction to the (ϕ,Γ)-modules and basic rings and invariants for p-adic Hodge theory. Then we study the their relationships, in particular, we will sketch a proof for weakly admissible $\Rightarrow$ admissible. After that, we study the cohomology of (ϕ,Γ)-modules and triangulations. In the final lectures, we explain some recent work by Pottharst on application to Iwasawa theory, extending Iwasawa Main Conjecture to the nonordinary case.

Lecture I: p-adic Hodge theory.

The study of p-adic Hodge theory is motivated by the seek of appropriate period rings for the comparison theorems between p-adic étale cohomology and the algebraic de Rham cohomology. We first introduce Grothendieck's mysterious functor question, and then define Fontaine's p-adic period rings $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{cris}}$, and $\mathbb{B}_{\mathrm{st}}$. Finally, we wrap the lecture by stating the comparison theorems.

Lecture II: De Rham and crystalline representations.

Fontaine's rings apply to abstract continuous p-adic representations. We give the definition of de Rham, crystalline, and semistable representations and talk about their basic properties, in particular, the (semi)linear algebra objects associated to the representations. Some easy examples will be given.

Lecture III: The associated Weil-Deligne representation.

We say a representation is potentially semistable if it becomes semistable over a finite extension of K. To a potentially semistable representation, Fontaine can naturally associate a Weil-Deligne representation. We discuss this construction and its relation with the conjecture on independence of l.

Lecture IV: (ϕ,Γ)-modules origins.

Fontaine introduced an alternative way of looking at p-adic representations, by transforming them into (ϕ,Γ)-modules, which are some (semi)linear algebra objects. In this story, a ground-breaking observation is that the Galois group of $\mathbb{Q}(\mu_{p^\infty})$ is isomorphic to the Galois group of $\mathbb{F}_p((T))$.

Lecture V: (ϕ,Γ)-modules over annuli.

Motivated by the study of (over)convergent (F-)isocrystals, Chebonnier and Colmez proved that the (ϕ,Γ)-modules associated to Galois representations in fact live over a smaller ring, the bounded Robba ring, which consists of analytic functions on some annulus with outer radius 1 that take bounded values on the annulus. We also introduce the Robba ring, which consists of possibly unbounded functions on some annulus with outer radius 1. Kedlaya's slope filtration governs the behavior of (ϕ,Γ)-modules over this Robba ring.

Lecture VI: (ϕ, Γ)-modules v.s. p-adic Hodge theory invariants (1).
In this lecture, we answer the question of linking (ϕ, Γ)-modules with the p-adic Hodge theory invariants given by Fontaine. We start with Berger's discovery on recovering $\mathbb{D}_{\mathrm{cris}}(V)$ from the (ϕ, Γ)-module associated to V. Then we discuss the reverse process, i.e., starting from a filtered (ϕ, N)-module, we construct a (ϕ, Γ)-module over the Robba ring.

Lecture VII: (ϕ, Γ)-modules v.s. p-adic Hodge theory invariants (2).
We continue the discussion by introducing Berger's differential equation. This together with the construction from previous lectures enables him to prove (1) de Rham $\Rightarrow$ potentially semistable, using the p-adic local monodromy theorem proved by Kedlaya, Christol-Mekhbout, and André independently; and (2) weakly admissible filtered (ϕ, N)-modules actually come from Galois representation, as a simple corollary of Kedlaya's slope filtration theorem.

Lecture VIII: Galois cohomology via (ϕ, Γ)-modules.
Since we have established an equivalence of categories between the p-adic representations and the (ϕ, Γ)-modules, one naturally expects an interpretation of the Galois cohomology in terms of (ϕ, Γ)-modules. This is all established by Herr in her thesis. We will also talk about Ruochuan Liu's generalization of Tate local duality and the Euler Poincaré characteristic formula to overconvergent (ϕ, Γ)-modules.

Lecture IX: Eigencurves.
We take a digression to introduce overconvergent p-adic modular forms. Coleman-Mazur and Buzzard proved that all the overconvergent p-adic modular forms on modular curves can be parameterized by a rigid analytic space, called the eigencurves. We discuss its basic properties. This result is later gereneralized to higher dimensional case by Chenevier and Beillaïche.

Lecture X: Triangulation.
After Kisin's finite slope space construction, Colmez realized that it is more natural to interpret his result using triangulation of (ϕ, Γ)-modules. He observed that even if a Galois representation is irreducible, the associated (ϕ, Γ)-module may still be reducible and this is almost always the case for all p-adic representations we can get from overconvergent p-adic modular forms. We discuss this point of view. One of the open problem is to obtain a global triangulation of the (ϕ, Γ)-modules associated to the family of p-adic representations parametrized by eigencurves.

Lecture XI: Bloch-Kato's local condition and triangluordinary condition.
Bloch and Kato introduced a local condition on the H^1 for p-adic cohomology served as an analogue of the unramified condition of H^1 for l-adic representations. This allows them to construct global Selmer group with reasonable structure at p. Pottharst realized that, as in the ordinary case, the triangulation can help to rewrite this local condition. This breakthrough allows him to carry many important work in Iwasawa theory to the nonordinary case.

Lecture XII: Nonordinary Iwasawa theory for modular curves.
We discuss Iwasawa Main Conjecture, relating the p-adic L-function of an eigen new form and the characteristic ideal of the Selmer group of the associated representation. Pottharst used Kato's Euler system to prove (half of) Iwasawa Main Conjecture at a nonordinary prime; his interpretation of local Bloch-Kato local

condition is the new ingredient. If time permits, we briefly talk about Kato's construction of his famous Euler system, which is obtained by p-adically interpolating the Beilinson elements for modular curves.

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Lecture I p-adic Hodge theory

I. Mysterious Functor

may not be crucial   usually characteristic 0

- Let X be a proper smooth variety of dimension d over a field K

$\rightsquigarrow$ de Rham complex $\Omega_X^\bullet$: $\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \longrightarrow \cdots \longrightarrow \Omega_X^d$

Define the algebraic de Rham cohomology of X to be $H_{dR}^\bullet(X/K) := H^\bullet(X, \Omega_X^\bullet)$

We in fact have a spectral sequence $E_1^{pq} = H^p(X, \Omega_X^q) \Rightarrow H_{dR}^{p+q}(X/K)$

It degenerates at E_1-terms if char $K = 0$.

- If $K = \mathbb{R}$, i.e. X is a proper smooth variety over $\mathbb{R}$, we have an isomorphism

Betti

$$(*_{\mathbb{C}}) \quad \underset{\sigma}{\circlearrowleft}\, H_B^i(X(\mathbb{C})^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \underset{\sigma}{\circlearrowleft}\,\mathbb{C} \simeq H_{dR}^i(X/\mathbb{R}) \otimes_{\mathbb{R}} \underset{\sigma}{\circlearrowleft}\,\mathbb{C} \qquad \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$$

This isomorphism is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant.

- Now, if K is a CDVF of mixed char. (ring of integers $\mathcal{O}_K$, residue field k perfect)

we hope to obtain something like

$$(*_{dR}) \quad \underset{G_K = \mathrm{Gal}(K^{alg}/K)}{\circlearrowleft}\, H_{et}^i(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \underset{G_K}{\circlearrowleft}\, ? \simeq H_{dR}^i(X/K) \otimes_K \underset{G_K}{\circlearrowleft}\, ?$$

We require it to be a G_K-equivariant isomorphism.

- Moreover, if X has a proper smooth model $\mathfrak{X}$ over $\mathcal{O}_K$, that is $\exists\, \mathfrak{X}/\mathcal{O}_K$ proper smooth, such that $X \simeq \mathfrak{X} \otimes_{\mathcal{O}_K} K$, we have an isom $H_{dR}^\bullet(X/K) \simeq H_{cris}^\bullet(\mathfrak{X}_k/W(k)) \otimes K$ where k is the residue field of K, and $H_{cris}^i(\mathfrak{X}_k/W(k))$ is a finite $W(k)$-module with a semilinear action φ ($\varphi(av) = \varphi(a)\cdot\varphi(v)$ for $a \in W(k)$, φ is the Frobenius on $W(k)$)

In this case, we expect not only $(*_{dR})$, but also

$$(*_{cris}) \quad \underset{G_K}{\circlearrowleft}\, H_{et}^i(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \underset{G_K \,\&\, \varphi}{\circlearrowleft}\, ? \;\simeq\; \underset{\varphi}{\circlearrowleft}\, H_{cris}^i(\mathfrak{X}_k/W(k)) \otimes_{W(k)} \underset{G_K \,\&\, \varphi}{\circlearrowleft}\, ?$$

This isomorphism is both G_K-equiv. and φ-equiv.

Remark: There is a story for semi-stable reduction; we don't discuss here.

II. Fontaine's rings

- Let K be a CDVF, mixed char. $\mathcal{O}_K, k$ as before. k perfect.

Let $\mathbb{C}_p$ denote the completion of the algebraic closure of K.

Let $\tilde{\mathbb{E}}^+ = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ $\quad (x_n) + (y_n) := (x_n + y_n)$, $\quad (x_n)\cdot(y_n) = (x_n y_n)$

$\simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$ $\quad x^{(n)} = \lim_{m\to+\infty} (\tilde{x}_{n+m})^{p^m}$, hence $(x^{(n)}) + (y^{(n)}) = \left(\lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}\right)$

a lift of x_{n+m}

$v : \tilde{\mathbb{E}}^+ \longrightarrow \mathbb{Q}$

$(x^{(n)}) \longmapsto v(x^{(0)})$ or $(x_n) \longmapsto p^n \cdot v(x_n)$ for any n s.t. $x_n \neq 0$ in $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$

Fact: $\tilde{\mathbb{E}}^+$ is complete for the valuation v and has characteristic p.

Special element: $\epsilon = (1, \zeta_p, \zeta_{p^2}, \cdots)$, where ζ_p is a non-trivial p^{th} root of unity

$v(\epsilon - 1) = p/p-1$

G_K acts on ϵ via $\chi : G_K \longrightarrow \mathbb{Z}_p^\times$, the cyclotomic character determined by $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ $\forall n$

- $g(\epsilon) = \epsilon^{\chi(g)}$

We define $\tilde{\mathbb{E}} = \tilde{\mathbb{E}}^+[\frac{1}{\epsilon - 1}]$; it's an algebraically closed field of characteristic $p > 0$.

(One can show that $\tilde{\mathbb{E}} = (k((\epsilon-1)))^{alg,\wedge}$)

- Consider $\tilde{\mathbb{A}}^+ = W(\tilde{\mathbb{E}}^+)$, the Witt vectors of $\tilde{\mathbb{E}}^+$; it's equipped with a homomorphism

$$\theta : \tilde{\mathbb{B}}^+ = \tilde{\mathbb{A}}^+[\tfrac{1}{p}] \longrightarrow \mathbb{C}_p$$
$$\sum_{k \gg -\infty} p^k [x_k] \longmapsto \sum p^k x_k^{(0)}$$

Definition. $B_{dR}^+ = \varprojlim_n \tilde{\mathbb{B}}^+/(\mathrm{Ker}\,\theta)^n$,

$A_{cris}^+ =$ complete divided power envelop of $\tilde{\mathbb{A}}^+$, w.r.t. $\mathrm{Ker}\,\theta \cap \tilde{\mathbb{A}}^+ =: I$, $\quad B_{cris}^+ = A_{cris}^+[\frac{1}{p}]$

$= \left(\tilde{\mathbb{A}}^+ + \sum_{n\in\mathbb{N}} \frac{I^n}{n!}\right)^\wedge$ $\qquad B_{cris}^+ \subset B_{dR}^+$

Special element: $t = \log[\epsilon] = ([\epsilon]-1) - \frac{([\epsilon]-1)^2}{2} + \frac{([\epsilon]-1)^3}{3} - \cdots$

It is a p-adic analogue of $2\pi i$! $\quad gt = \chi(g)t$ $\quad$ (analogous to $\sigma(i) = -i$ for complex conj σ)

- t lies in both B_{dR}^+ and A_{cris}^+, but not in $\tilde{\mathbb{A}}^+$ or $\tilde{\mathbb{B}}^+$

Definition $B_{dR} = B_{dR}^+[\frac{1}{t}]$, $B_{cris} = B_{cris}^+[\frac{1}{t}]$

$\mathbb{B}_{st} := \mathbb{B}_{cris}[X]$, embedding into $\mathbb{B}_{dR}$ by sending X to $\log[\tilde{p}]$, $\varphi(X) = pX$

$\circlearrowleft N = \frac{d}{dX}$ is a nilpotent operator and $N\varphi = p\varphi N$

Here, $\tilde{p} = (p, p^{1/p}, p^{1/p^2}, \dots) \in \tilde{\mathbb{E}}$. $\log[\tilde{p}]$ in $\mathbb{B}_{dR}$ is expressed as

$$\log[\tilde{p}] = \log p + \log([\tilde{p}]/p) = \log p + ([\tilde{p}]/p - 1) - ([\tilde{p}]/p - 1)^2/2 + \cdots$$

↑ choice of a p-adic monodromy

<u>Facts</u>: ① $\mathbb{B}_{dR}$ is a CDVF, with uniformizer t and residue field $\mathbb{C}_p$.

Hence $\mathbb{B}_{dR}$ is abstractly isomorphic to $\mathbb{C}_p((t))$, but <u>not</u> topologically. (no cont. G_K-equiv. $\mathbb{C}_p \hookrightarrow \mathbb{B}_{dR}$)

② $\mathbb{B}_{cris}$ and $\mathbb{B}_{st}$ both admit actions of φ, and $\varphi t = pt$

③ $\mathbb{B}_{dR}$, $\mathbb{B}_{cris}$, and $\mathbb{B}_{st}$ all admit continuous actions by G_K

• We also have a descending filtration on $\mathbb{B}_{dR}$ defined by the t-valuation: $\mathrm{Fil}^i \mathbb{B}_{dR} = t^i \mathbb{B}_{dR}^+$

Back to comparison theorems, we have

<u>Theorem</u> (Faltings, Tsuji)

Let X be a proper smooth variety over K. We have a G_K-equiv & filtration preserving isom.

$$H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dR} \simeq H^i_{dR}(X/K) \otimes_K \mathbb{B}_{dR}$$

↑ filtration here is given by $\mathrm{Fil}^j(H^i_{dR}(X/K)) := \mathbb{H}^i(X, \Omega^j_X \to \cdots \to \Omega^d_X)$

Hence $gr^j(H^i_{dR}(X/K)) = H^{j-i}(X, \Omega^j_X)$

If moreover, X has a smooth model $\mathcal{X}$ over $\mathcal{O}_K$, we have a (G_K, φ)-equivariant isom.

$$H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{cris} \simeq H^i_{cris}(\mathcal{X}_k / W(k)) \otimes_{W(k)} \mathbb{B}_{cris}.$$

• There is a story for $\mathbb{B}_{st}$, which we omit here.

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Lecture II De Rham and crystalline representations

Recall: K CDVF mixed characteristic, with perfect residue field k

G_K acts cont. $\circlearrowright$ B_{dR} is a CDVF, uniformizer $t=\log[\epsilon]$, residue field $\mathbb{C}_p$

has a G_K-equivariant filtration: $\mathrm{Fil}^i B_{dR} = t^i B_{dR}$

$\circlearrowright$ $B_{cris} \subset B_{dR}$, contains t, admits action of Frobenius φ ⟵ commuting with G_K-action

$B_{st} = B_{cris}[X] \hookrightarrow B_{dR}$ sending X to $\log[\tilde{p}]$, admits actions of φ and N, $N\varphi = p\varphi N$

I. p-adic Galois representations

Facts: $(B_{dR})^{G_K} = K$, $(B_{cris})^{G_K} = K_0 := \mathrm{Frac}(W(k)) \circlearrowleft \varphi$, $(B_{st})^{G_K} = K_0$

- Let V be a p-adic Galois representation of G_K.

  We always view V as a $\mathbb{Q}_p$-vector space and $d := \dim V$

① We define $D_{dR}(V) := (V \otimes B_{dR})^{G_K}$; it is a finite dimensional K-vector space.

It inherits a descending filtration by K-vector subspaces $\mathrm{Fil}^i D_{dR}(V) = (V \otimes \mathrm{Fil}^i B_{dR})^{G_K}$

We always have $\dim_K D_{dR}(V) \leq d$.

- We say that V is de Rham if $\dim_K D_{dR}(V) = d$.

  In this case, $\underset{G_K,\mathrm{Fil}}{\underset{\circlearrowleft}{D_{dR}(V)}} \otimes_K \underset{G_K,\mathrm{Fil}}{\underset{\circlearrowleft}{B_{dR}}} \xrightarrow{\sim} V \otimes \underset{G_K,\mathrm{Fil}}{\underset{\circlearrowleft}{B_{dR}}}$

  The numbers i, for which $\mathrm{Fil}^i D_{dR}(V)/\mathrm{Fil}^{i+1} D_{dR}(V) \neq 0$, are called Hodge-Tate weights of V, denoted by $HT(V)$ (with multiplicity).

  Denote $t_H(V) :=$ sum of the numbers in $HT(V)$.

② We define $D_{cris}(V) := (V \otimes B_{cris})^{G_K}$; it is a finite dimensional K_0-vector space

It has a semilinear action of φ.

We say that V is crystalline if $\dim_{K_0} D_{cris}(V) = d$

In this case, $\underset{\varphi}{\underset{\circlearrowleft}{D_{cris}(V)}} \otimes_{K_0} \underset{\varphi,G_K}{\underset{\circlearrowleft}{B_{cris}}} \simeq \underset{G_K}{\underset{\circlearrowleft}{V}} \otimes_{\mathbb{Q}_p} \underset{\varphi,G_K}{\underset{\circlearrowleft}{B_{cris}}}$

The Newton number $t_N(V) := t_N(\det V) =$ val of φ-action

↑1-dim'l $/K_0$ ↑independence of the basis

Example: $V = \mathbb{Q}_p(1) = \mathbb{Q}_p \cdot v$, where $g \cdot v = \chi(g) \cdot v$ for $g \in G_K$

$D_{cris}(V) = \mathbb{Q}_p(t^{-1}v)$ $\qquad$ $D_{dR}(V) = \mathbb{Q}_p \cdot (t^{-1}v)$

$\varphi(t^{-1}v) = p^{-1} \cdot t^{-1}v$ $\qquad$ $\mathrm{Fil}^i D_{dR}(V) = \begin{cases} \mathbb{Q}_p(t^{-1}v) & i \leq -1 \\ 0 & i \geq 0 \end{cases}$ $\qquad$ $HT(V) = -1$

Newton slope $= -1$

③ We have a same story for B_{st}.

$$\begin{aligned} D_{st} : \underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_K) &\longrightarrow (\varphi, N)\text{-}\underline{\mathrm{mod}}/K_0 \\ V &\longmapsto (V \otimes B_{st})^{G_K} \circlearrowleft \varphi, N \end{aligned}$$

- Since $B_{cris} \subset B_{st} \subset B_{dR}$, V is de Rham $\Rightarrow$ V is semistable $\Rightarrow$ V is crystalline.

II. Admissibility and weak admissibility.

If V is crystalline, $D_{dR}(V) = (V \otimes B_{dR})^{G_K} = (V \otimes B_{cris} \otimes_{B_{cris}} B_{dR})^{G_K}$

$= (D_{cris}(V) \otimes_{K_0} B_{cris} \otimes_{B_{cris}} B_{dR})^{G_K} = (D_{cris}(V) \otimes_{K_0} B_{dR})^{G_K} = D_{cris}(V) \otimes_{K_0} K$

Same for semistable representations.

Definition. A filtered (φ, N)-module is a K_0-vector space D, together with

- a semilinear action of φ
- an endomorphism N

$\Big\}$ $N\varphi = p\varphi N$ $\leftarrow$ This forces N to be nilpotent.

- a descending filtration on $D \otimes_{K_0} K$

For $D \in (\mathrm{Fil}, \varphi, N)\text{-}\underline{\mathrm{mod}}/K$, we may talk about $t_N(D)$ and $t_H(D)$ as before.

We say that D is <u>(weakly) admissible</u> if

- for any sub-$(\mathrm{Fil}, \varphi, N)$-module $D' \subset D$ (whose filtration is the induced one), $t_H(D') \leq t_N(D')$
- $t_N(D) = t_H(D)$

Remark: weakly admissibility $\Rightarrow$ "Newton above Hodge"

Theorem (Colmez-Fontaine, Berger) weakly admissible $\Rightarrow$ admissible.

We have an equivalence of tensor categories

$$\begin{aligned} \underline{\mathrm{Rep}}^{st}_{\mathbb{Q}_p}(G_K) &\longrightarrow (\mathrm{Fil}, \varphi, N)\text{-}\underline{\mathrm{mod}}^{wa}/K \\ V &\longmapsto D_{st}(V) \end{aligned}$$

$\leftarrow$ filtration is given by the identification $D_{cris}(V) \otimes_{K_0} K \simeq D_{dR}(V)$.

Proposition. The functor above is faithful, i.e. we may recover V from the associated filtered-(φ, N)-module.

$$V = \mathrm{Fil}^0\left((\mathbb{D}_{st}(V) \otimes \mathbb{B}_{st})^{\varphi=1, N=0} \right)$$

Proof: $\mathrm{Fil}^0\left((\mathbb{D}_{st}(V) \otimes_{K_0} \mathbb{B}_{st})^{\varphi=1,N=0}\right) = \mathrm{Fil}^0\left((V \otimes_{\mathbb{Q}_p} \mathbb{B}_{st})^{\varphi=1,N=0}\right) = V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0(\mathbb{B}_{cris}^{\varphi=1}) = V$

because we will see in later lectures that $\mathbb{B}_{cris}^{\varphi=1} \cap \mathrm{Fil}^0 \mathbb{B}_{dR} = \mathbb{Q}_p$

III. An example from elliptic curves.

E = elliptic curve $/ \mathbb{Q}_p$ with good reduction at p.

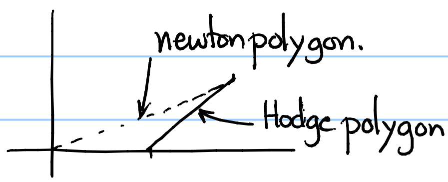
$V = H^1(E, \mathbb{Q}_p) \simeq V_p(E)(-1) = (\varprojlim_n E[p^n]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1)$; it is crystalline.

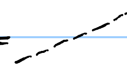
$\varphi \circlearrowright \mathbb{D}_{cris}(V)$ has characteristic polynomial $X^2 - a_p X + p = 0$

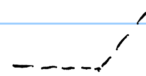
$\Rightarrow$ Newton slopes are either $\frac{1}{2}, \frac{1}{2}$, or $0, 1$

$\mathrm{Fil}^i(\mathbb{D}_{dR}(V)) = \mathrm{Fil}^i\left((H^1(E, \mathbb{Q}_p) \otimes \mathbb{B}_{dR})^{G_K}\right) \overset{\text{comparison}}{=\!=} \mathrm{Fil}^i\left((H^1_{dR}(E/\mathbb{Q}_p) \otimes \mathbb{B}_{dR})^{G_K}\right) = \mathrm{Fil}^i(H^1_{dR}(E/\mathbb{Q}_p))$

$\Rightarrow$ HT wts $= 0, 1$.



Newton polygon =  supersingular $v(a_p) \geq \frac{1}{2}$

 ordinary $v(a_p) = 0$

(When $p \geq 5$, $|a_p| < 2 \cdot |p|^{1/2}$, so we are really asking $a_p \overset{?}{=} 0$)

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Lecture III The associated Weil-Deligne representations

Recall: $D_{dR}: \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K) \longrightarrow \text{Fil-mod}/K$: $D_{dR}(V) = (V \otimes B_{dR})^{G_K}$

$D_{st}: \mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_K) \longrightarrow (\mathrm{Fil}, \varphi, N)\text{-mod}/K$: $D_{st}(V) = (V \otimes B_{st})^{G_K}$ with filtration given by $D_{dR}(V)$.

I. Potential theory

- If V is crystalline / semistable when viewed as a representation of G_L, for some L/K finite we say that V is potentially crystalline / semistable.
  In this case, we have $D_{cris,L}(V) := (V \otimes B_{cris})^{G_L}$, $D_{st,L} := (V \otimes B_{st})^{G_L}$ vector spaces over L_0

  $D_{cris,L}(V)$ ↺ φ, $D_{st,L}$ ↺ φ, N

  They both admit actions of $G_{L/K}$, commuting with the action of φ (and N).
- In the geometric picture, if the variety X has good/semistable reduction over L, then $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ is potentially crystalline/semistable for this L. ← not conversely.
- There is no "potentially de Rham", i.e. if V is de Rham over L, then it is automatically de Rham over K

Theorem (Berger, Colmez-Fontaine)
If V is de Rham, then V is potentially semistable.

Conjecture. Every proper smooth variety X over K has a potentially semistable reduction, that is for some finite extension L/K, X_L has a semistable model over $\mathcal{O}_L$

Remark: There is a general version of weakly admissible ⇒ admissible
If V is potentially semistable and becomes semistable over L,
then $D_{st,L}(V) \otimes_{L_0} L \simeq D_{dR,L}(V) = D_{dR}(V) \otimes_K L$
Together with de Rham ⇒ pst, we should have an equivalence of categories

$$\mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K) \longrightarrow (\mathrm{Fil}, G_K, \varphi, N)\text{-mod}^{wa}/K$$

$$V \longmapsto D_{pst}(V) = \bigcup_{L/K\ \mathrm{fin.}} (V \otimes B_{st})^{G_L}, \text{ filtration is given by } D_{dR}(V).$$

↑ vector space over $K_0^{ur} = \mathrm{Frac}\, W(k^{alg})$, G_K-action "locally finite"

II. Fontaine-Mazur Conjecture

Analogy $\begin{cases} p\text{-adic rep'n} & \text{crystalline} & , \text{semistable} & , & \text{de Rham} \\ & \updownarrow & \updownarrow & & \updownarrow \\ \ell\text{-adic rep'n} & \text{unramified} & , \text{tame part has image} = \mathbb{Z}_\ell & , & \text{all cont. rep'n} \end{cases}$ ← In the sense of coming from geometry.

Conjecture (Fontaine-Mazur)
Let $F/\mathbb{Q}$ be a number field and let $\rho: G_F \to GL_n(\mathbb{Q}_p)$ be a p-adic rep'n, s.t.
- $\rho|_{G_{F_v}}$ is unramified for all but finite places v of F.
- For any $v|p$, $\rho|_{G_{F_v}}$ is de Rham (and hence potentially semi-stable)

Then ρ comes from geometry, i.e. it is a subquotient of the étale cohomology of some proper variety$/F$.

Rmk: When $n=2$ and $F=\mathbb{Q}$, Kisin proved this conjecture by showing that in fact ρ comes from some modular form.

III. Weil-Deligne representations

- K = a finite extension of $\mathbb{Q}_p$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_K & \longrightarrow & G_K & \xrightarrow{\nu} & Gal(\bar{k}/k)=\hat{\mathbb{Z}} & \longrightarrow & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \longrightarrow & I_K & \longrightarrow & W_K & \xrightarrow{\nu} & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

W_K: Weil group of K $\qquad$ $1 \longleftrightarrow \varphi$ arithmetic Frobenius

Definition. A Weil-Deligne representation is a rep'n $\rho': W_K \to GL(V)=GL_n(\mathbb{C})$ together with a (nilpotent) endomorphism $N\in End(V)$, s.t. $\quad N\rho'(g) = |\#k|^{\nu(g)}\rho'(g)N$

- l-adic story: l-adic local monodromy theorem (Grothendieck) $\quad l\neq p$

We have an equivalence of tensor categories when identifying $\mathbb{Q}_l < \bar{\mathbb{Q}}_l = \mathbb{C}$

$$WD: \underline{Rep}_{\bar{\mathbb{Q}}_l}(W_K) \longleftrightarrow \underline{Rep}_{\mathbb{C}}(WD_K)$$

$$(V,\rho) \longmapsto N \text{ given by } \log(\rho(t^m))/\log(\chi_l(t^m)) \qquad \rho'(g)=\rho(g)\cdot\exp(-\log\chi_l(\phi^{-\nu(g)}g)N)$$

depends the choice of the lift of Frobenius ϕ

χ_l cyclotomic character

for $t\in I_K$, m suff. divisible

a lift of Frobenius

- p-adic story: We have a natural tensor functor

$$WD: \underline{Rep}^{pst}_{\mathbb{Q}_p}(G_K) \longrightarrow \underline{Rep}_{\mathbb{C}}(WD_K)$$

no filtration is considered on this side.

$V \longmapsto D_{pst}(V)$, let $\tilde{\rho}$ be the action of $G_{L/K}$ on $D_{pst}(V)$

$\rightsquigarrow \rho'(g) := \rho(g)\cdot\varphi^{\nu(g)}$ for $g\in W_K$

In some sense, we should view $dR \Rightarrow pst$ as the p-adic local monodromy theorem.

- On a separate note, somehow p-adic rep'ns blend the information from both the WD rep'ns and the Hodge structures. The "relative position" of the blending is related to the $\mathcal{L}$-invariants.

info at ∞-places (of motives)

IV Independence of ℓ

Let X be a proper smooth variety over K.

- If X has good reduction over K, one expects that

$$H^i_{et}(X, \mathbb{Q}_\ell) \text{ is unramified if } \ell \neq p, \quad \text{is crystalline if } \ell = p$$

Thm. The characteristic polynomial of φ on $\begin{cases} H^i_{et}(X, \mathbb{Q}_\ell) & \text{if } \ell \neq p \\ D_{cris}(H^i_{et}(X, \mathbb{Q}_\ell)) & \text{if } \ell = p \end{cases}$ is independent of ℓ.

- How about the general case?

Upshot of Weil-Deligne representations:

It allows one to compare ℓ-adic or p-adic rep'ns for different ℓ in a reasonable sense.

Conjecture. The Weil-Deligne representation $WD(H^i_{et}(X, \mathbb{Q}_\ell))$ is independent of ℓ

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Lecture IV (φ,Γ)-modules origin

I. Galois representation for a characteristic p field

- E field of char $p>0$, not necessarily perfect.
- $G_E = \mathrm{Gal}(E^{sep}/E)$
- $\varphi: E \to E$, $x \mapsto x^p$ the (arithmetic) Frobenius

<u>Definition</u>. A <u>φ-module</u> M over E is a finite dim'l vector space over E, together with a non-degenerate E-semilinear homomorphism $\varphi: M \to M$, i.e. $\Phi: \varphi^* M \xrightarrow{\sim} M$.

↳ means $\varphi(av) = \varphi(a)\varphi(v)$ for $a \in E$.

<u>Theorem</u> We have an equivalence of categories $\mathrm{Rep}_{\mathbb{F}_p}(G_E) \longleftrightarrow \varphi\text{-}\mathrm{mod}/E$

$$V \longmapsto \mathbb{D}(V) = (V \otimes_{\mathbb{F}_p} E^{sep})^{G_E}$$

$$\mathbb{V}(D) = (D \otimes_E E^{sep})^{\varphi=1} \longleftarrow D$$

Moreover, it preserves tensors and duals on both sides.

<u>Proof</u>: ① Let $d = \dim V$. We claim that $V \otimes_{\mathbb{F}_p} E^{sep}$ is isomorphic to $(E^{sep})^d$ as G_E-modules

Pick a basis of $V \otimes_{\mathbb{F}_p} E^{sep}$; let $a_g \in GL_d(E^{sep})$ denote the action of $g \in G_E$

Then $a_{gh} = a_g\, g(a_h)$ ← This is a cocycle in $Z^1(G_E, GL_d(E^{sep}))$

By Hilbert 90, $H^1(G_E, GL_d(E^{sep}))$ is trivial. choice of the basis changes this by a 1-coboundary.

$\Rightarrow V \otimes_{\mathbb{F}_p} E^{sep}$ has a basis of G_E-invariant vectors

Hence, $\dim_E (V \otimes_{\mathbb{F}_p} E^{sep})^{G_E} = d$; it has a φ-action coming from the φ-action on E^{sep}.

② Let $d = \dim D$. We need to show that $\dim_{\mathbb{F}_p} (D \otimes_E E^{sep})^{\varphi=1} = d$

Take a basis of D and denote the matrix of φ on D to be $P \in GL_n(E)$

Finding $(D \otimes_E E^{sep})^{\varphi=1}$ is equivalent to finding vectors $\underline{v} \in (E^{sep})^n$, s.t.

$$P\varphi(\underline{v}) = \underline{v}$$

This is equivalent to finding E^{sep}-points of $E[v_1, \dots, v_n] \Big/ \begin{pmatrix} v_1^p \\ \vdots \\ v_n^p \end{pmatrix} = P^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

This is an étale E-algebra

We have p^d solutions $\Rightarrow D \otimes_E E^{sep} \simeq (E^{sep})^d$ as a φ-module

$\Rightarrow \dim_{\mathbb{F}_p} (D \otimes_E E^{sep})^{\varphi=1} = d$.

<u>Wrap up</u>: $\mathbb{D}(\mathbb{V}(D)) = ((D \otimes_E E^{sep})^{\varphi=1} \otimes_{\mathbb{F}_p} E^{sep})^{G_E} \simeq (D \otimes_E E^{sep})^{G_E} \simeq D$

$$\mathbb{V}(\mathbb{D}(V)) = \left((V\otimes_{\mathbb{F}_p} E^{sep})^{G_E} \otimes_E E^{sep}\right)^{\varphi=1} \simeq \left(V\otimes_{\mathbb{F}_p} E^{sep}\right)^{\varphi=1} \simeq V \qquad \square$$

<u>Corollary</u>. Let C be a Cohen ring of E and pick a Frobenius lift φ on C.
Then we have an equivalence of tensor categories

$$\begin{aligned} \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_E) &\xrightarrow{\ \sim\ } \varphi\text{-}\underline{\mathrm{mod}}^{\acute{e}t}/C \\ V &\longmapsto \mathbb{D}(V) = (V\otimes_{\mathbb{Z}_p} \widehat{C^{ur}})^{G_E} \\ \mathbb{V}(D) = (D\otimes_C \widehat{C^{ur}})^{\varphi=1} &\longleftarrow\!\shortmid\ D \end{aligned}$$

- Here, we need the notion of étale φ-module:

An <u>étale φ-module</u> over C is a finite free C-module D together with an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$.

An <u>étale φ-module</u> D over $C[\frac{1}{p}]$ is a finite dim'l $C[\frac{1}{p}]$-vector space with a semilinear φ-action such that there is a φ-stable lattice D° which is étale as a φ-module.

(In particular, we <u>cannot</u> take φ to be mult. by p.)

<u>Corollary</u> We have an equivalence of tensor categories

$$\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_E) \xrightarrow{\ \sim\ } \varphi\text{-}\underline{\mathrm{mod}}^{\acute{e}t}/C[\tfrac{1}{p}]$$

II. <u>Big leap to the mixed characteristic case</u>

<u>Key Fact</u>: $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty})) \simeq \mathrm{Gal}_{\mathbb{F}_p((t))}$

Hence, we can turn a representation of $G_{\mathbb{Q}_p}$ into a φ-module and then remember the action of

$$\Gamma \simeq \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \xrightarrow[\simeq]{\chi} \mathbb{Z}_p^\times$$

- K CDVF of mixed characteristic, $\mathcal{O}_K$ ring integers, k residue field of char $p>0$ <u>perfect</u>

* For simplicity, we assume that K is <u>unramified</u>, i.e. $\mathcal{O}_K = W(k)$

- $K_n = K(\mu_{p^n})$, $K_\infty = \bigcup_n K_n$, $\mathrm{Gal}(K_\infty/K) = \Gamma_K \xrightarrow[\sim]{\chi} \mathbb{Z}_p^\times$, $H_K = \mathrm{Gal}(K^{alg}/K_\infty)$

$$\begin{array}{c} K^{alg} \\ \big| \ \big) H_K \\ K_\infty \\ \big| \ \big) \Gamma_K \\ K \end{array}$$

<u>Theorem</u> (Fontaine-Wintenberger)

$\varprojlim_{\text{norm}} K(\mu_{p^n})$ has a structure of field, and is isomorphic to $k((\epsilon-1)) =: \mathbb{E}_K$

where $\epsilon = (1, \zeta_p, \zeta_{p^2}, \cdots)$ ζ_p is a non-trivial p^{th} roots of unity and $\zeta_{p^n} = (\zeta_{p^{n+1}})^p$

Moreover, G_K acts on $\mathbb{E}_K^{sep}$ and $H_K \simeq \mathrm{Gal}(K^{alg}/K_\infty) \simeq \mathrm{Gal}(\mathbb{E}_K^{sep}/\mathbb{E}_K)$

- We denote $T = X = \overline{\pi} = \epsilon - 1 \rightsquigarrow \mathbb{E}_K = k((T))$ or $k((X))$ or $k((\overline{\pi}))$

  The action of Γ is given by $T \longmapsto (T+1)^{\chi(\gamma)} - 1$ and φ acts as p^{th}-power Frobenius.
- $\mathcal{O}_{\mathcal{E}} = \mathbb{A}_K =$ a Cohen ring of $\mathbb{E}_K = \mathcal{O}_K((T))^{\wedge}$ ← p-adic completion

  $\mathcal{E} = \mathbb{B}_K = \mathbb{A}_K[\frac{1}{p}] = \{ \sum_{n \in \mathbb{Z}} a_n T^n \mid v(a_n)$ is bounded below and $v(a_n) \to +\infty$ as $n \to -\infty \}$

  We lift the actions of Γ_K and φ on $\mathbb{A}_K$ or $\mathbb{B}_K$ to be

$$\gamma(T) = (1+T)^{\chi(\gamma)} - 1, \qquad \varphi(T) = (1+T)^p - 1$$

<u>Theorem</u>. We have equivalences of categories

same meaning as before.

$$\underline{\mathrm{Rep}}_{\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p}(G_K) \longleftrightarrow (\varphi, \Gamma)\text{-}\underline{\mathrm{mod}}^{\acute{e}t} / \mathbb{E}_K, \mathbb{A}_K, \mathbb{B}_K$$

$$V \longmapsto \mathbb{D}(V) = (V \otimes_{\mathbb{F}_p} \mathbb{E}_K^{sep})^{H_K} \text{ or } (V \otimes_{\mathbb{Z}_p} \widehat{\mathbb{A}_K^{ur}})^{H_K}$$

$$\mathbb{V}(D) = (D \otimes_{\mathbb{E}_K} \mathbb{E}_K^{sep})^{\varphi=1} \text{ or } (D \otimes_{\mathbb{A}_K} \widehat{\mathbb{A}_K^{ur}})^{\varphi=1} \longleftarrow D$$

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Lecture V (φ,Γ)-modules over annuli

Recall: We have an equivalence of categories

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p}(G_K) & \longleftrightarrow & (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}_K \\ V & \longmapsto & (V\otimes_{\mathbb{Q}_p}\widehat{\mathbb{B}_K^{ur}})^{H_K} \\ (D\otimes_{\mathbb{B}_K}\widehat{\mathbb{B}_K^{ur}})^{\varphi=1} & \longleftarrow\!\shortmid & D \end{array}$$

For simplicity, we assume that K is unramified

I Overconvergent story

Definition. We define a subring of $\mathbb{B}_K$ as follows

For $s>0$, $\mathbb{B}_K^{\dagger,s}=\left\{f(T)=\sum_{n\in\mathbb{Z}}a_nT^n \;\middle|\; |a_n| \text{ is bounded}, f(T) \text{ converges on the annulus } 0<v(T)\leq\frac{1}{s}\right\}$

$$=\left\{f(T)=\sum_{n\in\mathbb{Z}}a_nT^n \;\middle|\; \begin{array}{l}\text{When } n\to-\infty,\ v(a_n)+n/s\to+\infty \\ \text{When } n\to+\infty,\ v(a_n) \text{ is bounded below}\end{array}\right\}$$

$\mathbb{B}_K^{\dagger,s}$ is a subring of $\mathbb{B}_K$, stable under Γ, but $\underline{\underline{\text{not}}}$ φ, $\varphi:\mathbb{B}_K^{\dagger,s}\to\mathbb{B}_K^{\dagger,ps}$

Define $\mathcal{R}^{bd}=\mathcal{E}^\dagger=\mathbb{B}_K^\dagger=\bigcup_{s\to\infty}\mathbb{B}_K^{\dagger,s}$; it is stable under φ.

Fact: $\mathbb{B}_K^\dagger$ is a discrete valuation field with valuation $v(f(T))=\min_n\{v(a_n)\}$

Let $\mathcal{R}^{int}=\mathcal{O}_\mathcal{E}^\dagger=\mathbb{A}_K^\dagger$ denote the ring of integers of $\mathbb{B}_K^\dagger$ ← This notation might not be compatible with some literature but the difference is neglectable.

It is not complete but is henselian. ⇝ can talk about max'l unramified ext'n $\mathbb{A}_K^{\dagger,ur}$

The residue field of $\mathbb{A}_K^\dagger$ is again $k((T))$.

Theorem (Cherbonnier-Colmez) There exist equivalences of categories.

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Z}_p,\mathbb{Q}_p}(G_K) \arrow[rr, leftrightarrow] \arrow[dr, shift left, "\mathbb{D}"] & & (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{A}_K^\dagger,\mathbb{B}_K^\dagger \arrow[dl, "-\otimes_{\mathbb{A}_K^\dagger}\mathbb{A}_K"] \\
& (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{A}_K,\mathbb{B}_K \arrow[ul, shift left, "\mathbb{V}"] &
\end{tikzcd}$$

$V \longmapsto \mathbb{D}^\dagger(V)=(V\otimes_{\mathbb{Z}_p}\mathbb{A}_K^{\dagger,ur})^{H_K}$

defined similarly

In other words, the (φ,Γ)-modules we obtained from Galois representations can be descented to a smaller ring $\mathbb{A}_K^\dagger$ or $\mathbb{B}_K^\dagger$

Rmk: For (φ,Γ)-modules over $\mathbb{E}_K$, this overconvergency is "automatic" as we are dealing with Laurent series with finite tails. But it's not interesting on the other hand.

I (φ,Γ)-modules over Robba rings

<u>Definition</u> For $s>0$, $B^{\dagger,s}_{rig,K} = \left\{ f(T) = \sum_{n\in\mathbb{Z}} a_n T^n \,\middle|\, f(T) \text{ converges on the annulus } 0 < v(T) \leq \frac{1}{s} \right\}$

$$= \left\{ f(T) = \sum_{n\in\mathbb{Z}} a_n T^n \,\middle|\, \begin{array}{l} \text{When } n\to-\infty,\ v(a_n) + \frac{n}{s} \to +\infty \\ \text{When } n\to+\infty,\ v(a_n) + \frac{n}{r} \to +\infty,\ \forall r\to 1^- \end{array} \right\}$$

There is no boundedness condition here.

Similarly, $B^{\dagger,s}_{rig,K}$ is stable under Γ but <u>not</u> φ, $\varphi: B^{\dagger,s}_{rig,K} \to B^{\dagger,ps}_{rig,K}$ when $s>>0$

Denote $\mathcal{R} = B^{\dagger}_{rig,K} = \bigcup_{s\to\infty} B^{\dagger,s}_{rig,K}$; it is stable under φ

- A <u>(φ,Γ)-module</u> over $\mathcal{R}$ is a finite <u>free</u> $\mathcal{R}$-module D with semi-linear commutative actions of φ and Γ.

<u>Key</u>: There is no "ring of integers" for $\mathcal{R}$; we need something else to talk about étaleness

II. Slope filtration theorem

Let's forget about the Γ-actions for a moment.

<u>Definition</u>. We say a φ-module M over $\mathcal{R}$ is <u>étale</u> if there exists an étale φ-module M_0 on $\mathcal{R}^{bd}$ (or $\mathcal{R}^{int}$) s.t. $M = M_0 \otimes_{\mathcal{R}^{bd}} \mathcal{R}$.

- We say that a φ-module M_0 over $\mathcal{R}^{bd}$ has <u>pure slope</u> μ if the valuations of all the eigenvalues of the matrix for φ w.r.t. some basis, are all equal to μ

This does not dependent on the choice of the basis

We say $M_0 \otimes_{\mathcal{R}^{bd}} \mathcal{R}$ has pure slope μ.

<u>Remark</u>: pure of slope $0 \Leftrightarrow$ étale

<u>Slope filtration theorem</u> (Kedlaya)

For a φ-module M over $\mathcal{R}$, there exists a canonical filtration

$$0 = \mathrm{fil}_0 M \subsetneq \mathrm{fil}_1 M \subsetneq \cdots \subsetneq \mathrm{fil}_n M = M$$

s.t. $\mathrm{gr}_i M$ has pure slopes μ_i, and moreover $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$

In particular, $\mathrm{gr}_i M$ comes from a φ-module over $\mathcal{R}^{bd}$

This also applies to the (φ,Γ)-module case.

- Let's record a "theorem by definition"

Theorem. We have equivalences of categories

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \xrightarrow[\sim]{\mathbb{D}} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}_K \xleftarrow[\sim]{-\otimes_{\mathbb{B}_K^\dagger}\mathbb{B}_K} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}_K^\dagger \xrightarrow[\sim]{-\otimes_{\mathbb{B}_K^\dagger}\mathbb{B}^\dagger_{\mathrm{rig},K}} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}^\dagger_{\mathrm{rig},K}$$

$$(\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}^\dagger_{\mathrm{rig},K} \xrightarrow{\text{fully faithful}} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}/\mathbb{B}^\dagger_{\mathrm{rig},K}$$

follows from slope filtration

IV Harder-Narasimhan filtration

We try to explain a bit of the proof of the slope filtration theorem.

- Fact: $\mathcal{R}^\times \subseteq \mathcal{R}^{bd}$
- For a φ-module M of rank 1, $\varphi v = a\cdot v$ for $a \in \mathcal{R}^\times \subset \mathcal{R}^{bd}$
  $\deg(M) := v(a)$ where v is the p-adic valuation on the DVF $\mathcal{R}^{bd}$.
- For a φ-module M of rank d, we define $\deg(M) := \deg(\Lambda^d M)$ and
  $\mu(M) = \deg(M)/d \leftarrow$ slope of M.

Definition A φ-module M is called semistable if for any φ-submodule N of M,
$$\mu(N) \geq \mu(M).$$

Theorem. Let M be a φ-module over $\mathcal{R}$. There exists a canonical filtration $\mathrm{fil}_i M$ of M s.t. $\mathrm{gr}_i M$ is semistable and $\mu(\mathrm{gr}_1 M) < \mu(\mathrm{gr}_2 M) < \cdots < \mu(\mathrm{gr}_n M)$

- The idea of the proof is to inductively taking the φ-submodule with minimal slope.
- This theorem is not entirely abstract nonsense; we need to show that the slopes of all φ-submodules are bounded below.

Key part of the slope filtration: Semistable $\Longrightarrow$ pure !

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Lecture VI (φ,Γ)-modules v.s. p-adic Hodge theory invariants (I)

<u>Recall</u>: $D_{cris}^{(+)}: \underline{Rep}_{\mathbb{Q}_p}(G_K) \longrightarrow \varphi\text{-mod}/K_0$, $D_{cris}^{(+)}(V) = (V\otimes B_{cris}^{(+)})^{G_K}$

We have equivalences of categories:

slope filtration tells you what happens here

$$Rep_{\mathbb{Q}_p}(G_K) \underset{\sim}{\xrightarrow{\mathbb{D}}} (\varphi,\Gamma)\text{-}\underline{mod}^{et}/\mathbb{B}_K \underset{\sim}{\xleftarrow{-\otimes_{\mathbb{B}_K^\dagger}\mathbb{B}_K}} (\varphi,\Gamma)\text{-}\underline{mod}^{et}/\mathbb{B}_K^\dagger \underset{\sim}{\xrightarrow{-\otimes_{\mathbb{B}_K^\dagger}\mathbb{B}_{rig,K}^\dagger}} (\varphi,\Gamma)\text{-}\underline{mod}^{et}/\mathbb{B}_{rig,K}^\dagger \subset (\varphi,\Gamma)\text{-}\underline{mod}/\mathbb{B}_{rig,K}^\dagger$$

call it $\mathbb{D}_{rig}^\dagger$

For simplicity, we assume that K is unramified, and we only deal with the crystalline case.

I. Recover $D_{cris}(V)$ from $\mathbb{D}_{rig}^\dagger(V)$

<u>Theorem</u>. For any $V \in \underline{Rep}_{\mathbb{Q}_p}(G_K)$, we have

$$\mathbb{D}_{cris}^+(V) := \left(V\otimes \mathbb{B}_{cris}^+\right)^{G_K} = \left(\mathbb{D}_{rig}^\dagger(V)\right)^{\Gamma_K} \quad \text{and} \quad \mathbb{D}_{cris}(V) = \left(\mathbb{D}_{rig}^\dagger(V)\left[\tfrac{1}{t}\right]\right)^{\Gamma_K}$$

<u>Remark</u>: This theorem does not require V to be crystalline!

<u>Idea of the proof</u>: The second one follows from the first one by twisting V.

Indeed, $\mathbb{D}_{cris}(V) = \bigcup_n \left(V\otimes t^{-n}\mathbb{B}_{cris}^+\right)^{G_K} \underset{\sim}{\xrightarrow{\text{as } \Gamma\text{-module}}} \bigcup_n \left(V(-n)\otimes \mathbb{B}_{cris}^+\right)^{G_K}$ but φ-actions are mult'd by p^n

$$\left(\mathbb{D}_{rig}(V)\left[\tfrac{1}{t}\right]\right)^{\Gamma_K} = \bigcup_n \left(t^{-n}\mathbb{D}_{rig}(V)\right)^{\Gamma_K} \underset{\varphi\text{-action mult'd by } p^n}{\simeq} \bigcup_n \left(\mathbb{D}_{rig}(V(-n))\right)^{\Gamma_K}$$

• To prove the first equality, one would like to write

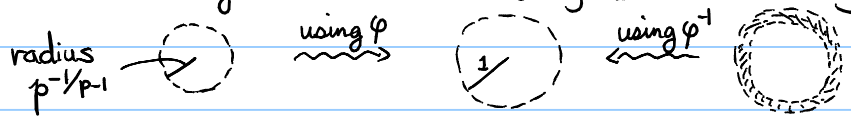
$$\left(V\otimes \mathbb{B}_{cris}^+\right)^{G_K} \overset{?}{=} \left(V\otimes \mathbb{B}_{rig,K}^{\dagger,ur}\right)^{G_K} = \left(\left(V\otimes \mathbb{B}_{rig,K}^{\dagger,ur}\right)^{H_K}\right)^{\Gamma_K}$$

we need some subtitude for this in the actual proof.

It is however not so straightforward to compare $\mathbb{B}_{cris}^+$ with $\mathbb{B}_{rig,K}^{\dagger,ur}$

<u>Point</u>: compare $\left(V\otimes \mathbb{B}_{cris}^+\right)^{G_K} = \left(V\otimes \widetilde{\mathbb{B}}_{max}^+\right)^{G_K} = \left(V\otimes \widetilde{\mathbb{B}}_{rig,K}^{\dagger,ur}\right)^{G_K} = \left(V\otimes \mathbb{B}_{rig,K}^{\dagger,ur}\right)^{G_K}$

something over | something over | something over | untilde process



• $\sim$ means residue field perfect

• no $\sim$ means not perfect.

When the HT weights of V are all positive, we have $D_{cris}(V) = \left(\mathbb{D}_{rig}^\dagger(V)\right)^{\Gamma_K}$

<u>Observation</u> If V is crystalline, we expect $D_{cris}(V)\otimes \mathbb{B}_{rig,K}^\dagger\left[\tfrac{1}{t}\right] \simeq \mathbb{D}_{rig}^\dagger(V)\left[\tfrac{1}{t}\right]$

II. From filtered φ-modules to (φ,Γ)-modules

Goal: construct a functor $(\mathrm{Fil},\varphi)\text{-}\underline{\mathrm{mod}}/K \longrightarrow (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}/\mathbb{B}^{\dagger}_{\mathrm{rig},K}$

not that we don't need weak admissibility here.

Idea: Given $D \in (\mathrm{Fil},\varphi)\text{-}\underline{\mathrm{mod}}/K$, we cut off the right piece from $D \otimes_K \mathbb{B}^{\dagger}_{\mathrm{rig},K}[\frac{1}{t}]$

· Definition. Let $s_n = (p-1)p^{n-1}$ for $n \gg 0$. Let $K_n = K(\mu_{p^n})$

We have a canonical injective homomorphism, "localization at $T = \zeta_{p^n}^j - 1$ for all $j \in (\mathbb{Z}/p^n\mathbb{Z})^{\times}$"

$$\iota_n : \mathbb{B}^{\dagger, s_n}_{\mathrm{rig},K} \longrightarrow K_n[\![t]\!]$$
$$T \longmapsto \zeta_{p^n} \exp(t/p^n) - 1 \quad \& \quad \varphi^{-n} \text{ on } K$$

$\rightsquigarrow$ commutative diagram

$$\begin{tikzcd} \mathbb{B}^{\dagger,s_n}_{\mathrm{rig},K} \arrow[r, "\iota_n"] \arrow[d, "\varphi"] & K_n[\![t]\!] \arrow[d] \arrow[loop right, "\varphi t = pt"] \\ \mathbb{B}^{\dagger,s_{n+1}}_{\mathrm{rig},K} \arrow[r, "\iota_{n+1}"] & K_{n+1}[\![t]\!] \end{tikzcd}$$

Now, fix a (φ,Γ)-module $D^{\dagger}_{\mathrm{rig}}$ over $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$. Then $D^{\dagger,s_n}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,s_n}_{\mathrm{rig},K},\iota_n} K_n[\![t]\!]$ is a finite $K_n[\![t]\!]$-module

From n to $n+1$, we have a natural map

$$\varphi_n : D^{\dagger,s_n}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,s_n}_{\mathrm{rig},K},\iota_n} K_n((t)) \otimes_{K_n((t))} K_{n+1}((t)) \longrightarrow D^{\dagger,s_{n+1}}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,s_{n+1}}_{\mathrm{rig},K},\iota_{n+1}} K_{n+1}((t))$$
$$v \otimes x \otimes y \longmapsto \varphi(v) \otimes xy$$

Consider the following category

$$\mathcal{C} = \left\{ (M_n)_{n\gg 0} \;\middle|\; \begin{array}{l} M_n \text{ is a finite free } K_n[\![t]\!]\text{-submodule of } D^{\dagger,s_n}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,s_n}_{\mathrm{rig},K},\iota_n} K_n((t)) \\ \text{stable under } \Gamma_K\text{-action}, \ \varphi_n(M_n) \otimes_{K_n[\![t]\!]} K_{n+1}[\![t]\!] = M_{n+1} \end{array} \right\} \Big/ \sim$$

$(M_n) \sim (M_n')$ if $M_n = M_n'$ for $n \gg 0$

Key proposition A sub-(φ,Γ)-module of $D^{\dagger}_{\mathrm{rig}}[\frac{1}{t}]$ over $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$ is in one-to-one correspondence to the objects (M_n) in $\mathcal{C}$

Proof: The correspondence is given by

$$\tilde{D}^{\dagger}_{\mathrm{rig}} \subset D^{\dagger}_{\mathrm{rig}}[\tfrac{1}{t}] \longleftrightarrow \tilde{D}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,s_n}_{\mathrm{rig},K}} K_n[\![t]\!] = M_n \subset D^{\dagger,s_n}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,s_n}_{\mathrm{rig},K}} K_n((t))$$

We need a technical lemma from Lazard to go back.

Now, it is a matter of fact how one define appropriate M_n's from the filtration

Construction of the (φ,Γ)-module.

We start with $D \in (\mathrm{Fil},\varphi)\text{-}\underline{\mathrm{mod}}/K$, and apply the key proposition to $D \otimes \mathbb{B}^{\dagger}_{\mathrm{rig},K}[\frac{1}{t}]$ and to

$M_n = \mathrm{Fil}^0(D \otimes_{K_0,\varphi^n} K_n((t)))$ where the filtration on $K_n((t))$ is given by t-valuation.

(On checks that M_n is a compatible system in the sense that

$\varphi_n(M_n) \otimes_{K_n[\![t]\!]} K_{n+1}[\![t]\!] = M_{n+1}$)

$\rightsquigarrow$ (φ,Γ)-module over $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$. :)

If the HT weights of D are all negative, $D^{\dagger}_{\mathrm{rig}} \subseteq D \otimes \mathbb{B}^{\dagger}_{\mathrm{rig},K}$

If the HT weights of D are all positive, $D \otimes \mathbb{B}^{\dagger}_{\mathrm{rig},K} \subseteq D^{\dagger}_{\mathrm{rig}}$

- Somehow, the shifts on t correspond to the change on Hodge-Tate weights.

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Lecture VII (φ,Γ)-modules v.s. p-adic Hodge theory invariants (2)

Recall: $D_{cris}(V) = \left(\mathbb{D}_{rig}(V)[\frac{1}{t}]\right)^{\Gamma_K}$

Localizing at $T=\zeta_{p^n}-1$ gives a homomorphism $B^{\dagger,s_n}_{rig,K} \xrightarrow{\iota_n} K_n[[t]]$

$T \longmapsto \zeta_{p^n}\exp(t/p^n)-1$ & φ^{-n} on K

We have a functor $(Fil,\varphi)\text{-}\underline{mod}/K \longrightarrow (\varphi,\Gamma)\text{-}\underline{mod}/B^{\dagger}_{rig,K}$

$D \longmapsto$ something cut off from $D\otimes B^{\dagger}_{rig,K}[\frac{1}{t}]$ according to the filtration on D.

I. From (φ,Γ)-modules to differential equations

- To determine $\left(\mathbb{D}_{rig}(V)[\frac{1}{t}]\right)^{\Gamma_K}$ & get the filtration, we use differential equations

We work with more general $D^{\dagger}_{rig} \in (\varphi,\Gamma)\text{-}\underline{mod}/B^{\dagger}_{rig,K}$.

Define $\nabla = \log\gamma / \log(\chi(\gamma))$ for $\gamma\in\Gamma_K$ with $\gamma\to 1$; it is a differential operator on $D^{\dagger}_{rig}$ and $B^{\dagger}_{rig,K}$.

∇ is $t\cdot\partial$ with $\partial = (1+T)\frac{d}{dT}$ $(=\frac{d}{dt})$ on $B^{\dagger}_{rig,K}$

Now, consider $M_n' = D^{\dagger,s_n}_{rig} \otimes_{B^{\dagger,s_n}_{rig,K},\iota_n} K_n((t)) \circlearrowleft \nabla$ for $n\gg 0$

<u>Definition</u> $D^{\dagger}_{rig}$ is called <u>locally trivial</u> if M_n' is a trivial ∇-module for $n\gg0$

<u>Fact</u>: When $D^{\dagger}_{rig}$ is locally trivial, $\exists!$ $K_n[[t]]$-submodule $M_n\subset M_n'$ s.t. $\nabla(M_n)\subset t\cdot M_n$

By Key proposition again, these M_n define N_{dR}, a sub-(φ,Γ)-module of $D^{\dagger}_{rig}[\frac{1}{t}]$ over $B^{\dagger}_{rig,K}$

<u>Upshot</u>: N_{dR} is not only a differential module for $\nabla = t\cdot\partial$, but also a differential module for ∂!!

<u>Theorem</u>. We have equivalence among categories.

$D_{cris}(V) = N_{dR}^{\Gamma_K} = (\mathbb{D}^{\dagger}_{rig}(V)[\frac{1}{t}])^{\Gamma_K}$ if V is crystalline

$(Fil, G_K, \varphi, N)\text{-}\underline{mod}/K$ $\underset{D\longmapsto (D\otimes B^{\dagger}_{log,L})^{H_L/H_K, N=0}}{\overset{\text{solving diff equation}}{\rightleftarrows}}$ p-adic diff. equations N_{dR} + infomation of the filtration & (φ,Γ_K)-action

Construction $\sim$: $D \mapsto$ cut off something from $(D\otimes B^{\dagger}_{log,L}[\frac{1}{t}])^{H_L/H_K, N=0}$ → locally trivial $(\varphi,\Gamma)\text{-}\underline{mod}/B^{\dagger}_{rig,K}$

locally trivial $(\varphi,\Gamma)\text{-}\underline{mod}/B^{\dagger}_{rig,K}$ → p-adic diff. equations: $D^{\dagger}_{rig} \longmapsto$ find the sub-$B^{\dagger}_{rig,K}$-module of $D^{\dagger}_{rig}[\frac{1}{t}]$, on which ∂ acts

- We explain how one retrieves the filtration from the (φ,Γ)-module $D^{\dagger}_{rig}$

(For simplicity, we assume that the differential equation N_{dR} is trivial, i.e. crystalline case)

Let $D = N_{dR}^{\Gamma_K}$ be the φ-module over K, given by solving N_{dR}.

Then
$$D \simeq D \otimes_{K,\varphi^n} K \hookrightarrow D \otimes \mathbb{B}^{\dagger,s_n}_{rig,K} \otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K},\iota_n} K_n((t))$$
$$\cong N_{dR} \otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K},\iota_n} K_n((t))$$
$$\cong D^{\dagger}_{rig} \otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K},\iota_n} K_n((t)) \supset D^{\dagger}_{rig} \otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K},\iota_n} K_n[\![t]\!]$$

Define $\mathrm{Fil}^i D = D \cap t^i \left(D^{\dagger}_{rig} \otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K},\iota_n} K_n[\![t]\!] \right)$

This is independent of the choice of n.

Remark: If the HT weights of D are all ≤ 0, $D^{\dagger}_{rig} \subseteq N_{dR}$

If the HT weights of D are all ≥ 0, $N_{dR} \subseteq D^{\dagger}_{rig}$ (this is consistent with $D \subset D^{\dagger}_{rig}$)

II. Application #1: weakly admissible $\Rightarrow$ admissible

If we start with a weakly admissible filtered-(G_K,φ,N)-module D, the machinery generates a (φ,Γ)-module $D^{\dagger}_{rig}$ over the Robba ring $\mathbb{B}^{\dagger}_{rig,K}$. All we need to show is that this $D^{\dagger}_{rig}$ comes from a representation, or equivalently, it is étale. By Kedlaya's slope filtration, this is further equivalent to check that $D^{\dagger}_{rig}$ is "semistable" as a (φ,Γ)-module.

Proposition. If we get $D^{\dagger}_{rig}$ out of D, then $\deg(D^{\dagger}_{rig}) = t_N(D) - t_H(D)$

Proof: It suffices to check this for the case $\mathrm{rank}\, D = \mathrm{rank}\, D^{\dagger}_{rig} = 1$

Say $D = K \cdot e_0$, $\varphi e_0 = \alpha e_0$ for $\alpha \in K$ and $\mathrm{Fil}^i D = K e_0 \supsetneq \mathrm{Fil}^{i+1} D = 0$

$\rightsquigarrow t_N(D) = v(\alpha)$, $t_H(D) = i$

By the construction of (φ,Γ)-modules, $D^{\dagger}_{rig} = t^{-i} \mathbb{B}^{\dagger}_{rig,K} \cdot e_0$ and $\varphi(t^{-i} e_0) = \alpha \cdot p^{-i} t^{-i} e_0$

$\Rightarrow \deg(D^{\dagger}_{rig}) = v(\alpha) - i$ $\square$

Corollary: D weakly admissible $\Rightarrow D^{\dagger}_{rig}$ étale $\Rightarrow D$ comes from a representation.

III. Application #2: de Rham $\Rightarrow$ potentially semi-stable.

We need to show that V de Rham $\Rightarrow \mathbb{D}_{rig}(V)$ is locally trivial

Input (Fontaine) For any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, there is a unique maximal finitely generated

$K_\infty[\![t]\!]$

$\updownarrow$

sub-$K_\infty((t))$-module of $(V\otimes \mathbb{B}_{dR})^{H_K}$, denoted by $\mathbb{D}_{dif}^{(+)}(V)$, that is stable under Γ_K-action

<u>Proposition</u>. (1) $K_\infty \otimes \mathbb{D}_{dR}(V)$ is the kernel of ∇ on $\mathbb{D}_{dif}(V)$

(2) V is de Rham $\Longleftrightarrow \mathbb{D}_{dif}(V)$ is a trivial differential module for ∇,

equivalently, $\exists$ a sub-$K_\infty[\![t]\!]$-module M_0 of $\mathbb{D}_{dif}(V)$ s.t. $\nabla(M_0) < tM_0$

Proof: (1) is true because $\mathbb{D}_{dR}(V)$ is the Γ_K-inv. of $\mathbb{D}_{dif}(V)$

(2) follows from (1) immediately.

<u>Key</u>: $\mathbb{D}_{rig}^{\dagger,s_n}(V)\otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K,2n}} K_\infty((t)) \xrightarrow{\sim} \mathbb{D}_{dif}(V)$ for $n>>0$ $\quad\Gamma_K$-equiv $\Rightarrow$ respect ∇-action.

Proof: We show this by proving $\mathbb{D}_{rig}^{\dagger,s_n}(V)\otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K,2n}} K_\infty[\![t]\!] \xrightarrow{\sim} \mathbb{D}^+_{dif}(V)$

Suffices to show $\mathbb{D}_{rig}^{\dagger,s_n}(V)\otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K,2n}} K_\infty[\![t]\!]/(t) \xrightarrow{\sim} \mathbb{D}^+_{dif}(V)/t$

$$\begin{array}{ccc} \mathbb{D}_{rig}^{\dagger,s_n}(V)\otimes_{\mathbb{B}^{\dagger,s_n}_{rig,K,2n}} K_\infty[\![t]\!]/(t) & \xrightarrow{\sim} & \mathbb{D}^+_{dif}(V)/t \\ \wr\wr & & \wr\wr \\ \mathbb{D}_{rig}^{\dagger,s_n}(V)/\varphi^{n-1}\left(\frac{\varphi(T)}{T}\right)\otimes_{K_n} K_\infty & & \mathbb{D}_{Sen}(V) \end{array}$$

$\uparrow$ unique f.g. K_∞-submodule of $(V\otimes \mathbb{C}_K)^{H_K}$

This follows from dimension count.

Hence, V de Rham $\Rightarrow \mathbb{D}^\dagger_{rig}(V)$ locally trivial $\Rightarrow V$ pst.

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Lecture VIII Galois cohomolgy via (φ,Γ)-modules

Recall: For a field E of char $p>0$, we have an equivalence of categories

$$\mathrm{Rep}_{\mathbb{F}_p,\mathbb{Z}_p,\mathbb{Q}_p}(G_E) \xrightarrow{\mathbb{D}} \varphi\text{-}\underline{\mathrm{mod}}^{\text{ét}}/E, C_E, C_E[\tfrac{1}{p}] \qquad \mathbb{D}(V) = \big(V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}}\big)^{G_E}$$

For K CDVF of mixed char & perfect res. field, we have equivalences of categories

$$\mathrm{Rep}_{\mathbb{F}_p,\mathbb{Z}_p,\mathbb{Q}_p}(G_K) \xrightarrow{\mathbb{D}} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/E_K, A_K, B_K \qquad \mathbb{D}(V) = \big(V \otimes_{\mathbb{Z}_p} \widehat{A_K^{ur}}\big)^{G_{H_K}}$$

$$(\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/B_K \underset{\sim}{\xleftarrow{-\otimes_{B_K^\dagger} B_K}} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/B_K^\dagger \underset{\sim}{\xrightarrow{-\otimes_{B_K^\dagger} B^\dagger_{\mathrm{rig},K}}} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/B^\dagger_{\mathrm{rig},K} \subset (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}/B^\dagger_{\mathrm{rig},K}$$

I. Interprete Galois cohomology in terms of (φ,Γ)-modules

- E a field of char. $p>0$

Let $V \in \mathrm{Rep}_{\mathbb{F}_p,\mathbb{Z}_p,\mathbb{Q}_p}(G_E)$, we may define cohomology groups $H^i(G_E,V)$ as in Serre's local fields.

$$0 \to \mathbb{Z}_p \to \widehat{C_E^{ur}} \xrightarrow{\varphi-1} \widehat{C_E^{ur}} \to 0$$

$$\Rightarrow 0 \to V \to V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}} \xrightarrow{\varphi-1} V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}} \to 0$$

Note that $V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}} \simeq \mathbb{D}(V) \otimes_{C_E} \widehat{C_E^{ur}}$ is a trivial G_E-semilinear representation on E^{sep}

$$\Rightarrow H^i\big(G_E, \mathbb{D}(V) \otimes_{C_E} \widehat{C_E^{ur}}\big) = \begin{cases} \mathbb{D}(V), & \text{if } i=0 \\ 0, & \text{if } i\geq 1 \end{cases} \quad \leftarrow G_E\text{-acyclic}$$

Thus, $0 \to H^0(G_E,V) \to \mathbb{D}(V) \xrightarrow{\varphi-1} \mathbb{D}(V) \to H^1(G_E,V) \to 0$ & $H^i(G_E,V)=0$ for $i>1$

*One should view $H^*(G_E,V)$ as the complex $R\Gamma(G_E,V) \simeq [\mathbb{D}(V) \xrightarrow{\varphi-1} \mathbb{D}(V)]$

- Somehow, this complex contains more information than cohomology. We may work with complexes throughout the argument, only taking cohomology when we are requested.

Now, Let K be a CDVF, mixed char, residue field k perfect

Serre spectral sequence: $H^j(\Gamma_K, H^i(H_K,V)) \Rightarrow H^{i+j}(G_K,V)$

- A better way to view it: $H^*(H_K,V)$ is given by $\mathbb{D}(V) \xrightarrow{\varphi-1} \mathbb{D}(V)$

  $H^*(\Gamma_K, M)$ for a Γ-module is given by $M \xrightarrow{\gamma-1} M$,

  where γ is a generator of Γ_K (assuming $p\neq 2$ for simplicity)

$\rightsquigarrow$ So, $H^*(G_K,V)$ is given by the cohomology of the double complex (canonically indep. of γ)

$$\begin{array}{ccc} \mathbb{D}(V) & \xrightarrow{\varphi-1} & \mathbb{D}(V) \\ \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} \\ \mathbb{D}(V) & \xrightarrow{\varphi-1} & \mathbb{D}(V) \end{array} \quad \xrightarrow{\text{or}} \quad C_{\varphi,\gamma}: \mathbb{D}(V) \xrightarrow{(\varphi-1,\gamma-1)} \mathbb{D}(V) \oplus \mathbb{D}(V) \xrightarrow{(1-\gamma,\varphi-1)} \mathbb{D}(V)$$

$$x \longmapsto ((\varphi-1)x, (\gamma-1)x)$$

$$(x,y) \longmapsto (1-\gamma)x + (\varphi-1)y$$

II. ψ-operator

Note $\varphi: E_K^{sep} \to E_K^{sep}$ is injective and E_K^{sep} is free of rank p over $\varphi(E_K^{sep})$

Define $\psi = \varphi^{-1} \circ \frac{1}{p} \mathrm{Tr}_{E_K^{sep}/\varphi(E_K^{sep})} : E_K^{sep} \longrightarrow E_K^{sep}$

It lifts to $\psi = \varphi^{-1} \circ \frac{1}{p} \mathrm{Tr}_{\widehat{A}_K^{ur}/\varphi(\widehat{A}_K^{ur})} : \widehat{A}_K^{ur} \longrightarrow \widehat{A}_K^{ur}$

· In precise terms, $\epsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \rightsquigarrow [\epsilon] \in \widehat{A}_K^{ur}$

Then $1, [\epsilon], \cdots, [\epsilon^{p-1}]$ form a basis of $\widehat{A}_K^{ur}$ over $\varphi(\widehat{A}_K^{ur})$

If we write $x \in \widehat{A}_K^{ur}$ as $\varphi(a_0) + [\epsilon]\varphi(a_1) + \cdots + [\epsilon^{p-1}]\varphi(a_{p-1})$, then $\psi(x) = a_0$

In particular, $\psi\varphi = \mathrm{Id}$.

· For any $V \in \underline{\mathrm{Rep}}_{\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p}(G_K)$, $D(V) = (V \otimes \widehat{A}_K^{ur})^{G_K}$ has a surjective action of ψ

Consider the following commutative diagram

$$\begin{array}{cccccc}
C_{\varphi,\gamma}: & D(V) & \xrightarrow{(\varphi-1,\,\gamma-1)} & D(V) \oplus D(V) & \xrightarrow{(1-\gamma,\,\varphi-1)} & D(V) \\
& \downarrow \mathrm{id} & & -\psi \downarrow \quad \downarrow \mathrm{id} & & -\psi \downarrow \\
C_{\psi,\gamma}: & D(V) & \xrightarrow{(\psi-1,\,\gamma-1)} & D(V) \oplus D(V) & \xrightarrow{(1-\gamma,\,\varphi-1)} & D(V)
\end{array}$$

<u>Key Fact</u>: $\gamma - 1$ is bijective on $D(V)^{\psi=0}$!

So, the two complexes compute the same cohomology group.

III. Euler characteristic

Assume that $K = \mathbb{Q}_p$, otherwise we may use Frobenius reciprocity law to reduce to this case.

<u>Theorem</u>. (1) For $V \in \underline{\mathrm{Rep}}_{\mathbb{F}_p}(G_K)$, $\prod_{i=0}^{2} \left(\# H^i(G_K, V)\right)^{(-1)^i} = p^{-[K:\mathbb{Q}_p]\cdot \dim V}$

(2) For $V \in \underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_K)$, $\sum_{i=0}^{2} (-1)^i \dim H^i(G_K, V) = -[K:\mathbb{Q}_p] \cdot \dim V$

Proof: (2) follows from (1). To prove (1), one studies $C_{\psi,\gamma}$ $\square$

IV. Tate local duality

Continue to assume $K = \mathbb{Q}_p$.

A more natural way of viewing $D(\mathbb{Q}_p(1))$ is to identify it with

$B_{\mathbb{Q}_p} \cdot \frac{dT}{1+T} \subset \mathbb{Q}_p[\![T]\!] \frac{dT}{1+T}$, where $\gamma\left(\frac{dT}{1+T}\right) = \chi(\gamma) \cdot \frac{dT}{1+T}$ & $\varphi\left(\frac{dT}{1+T}\right) = \frac{dT}{1+T}$

One may expect to have a factor p, but it wouldn't be étale then

Pairing $\{\cdot,\cdot\}: D(V) \otimes \left(D(V)^\vee \cdot \frac{dT}{1+T}\right) \longrightarrow \mathbb{Q}_p$

$$f \otimes g \frac{dT}{1+T} \longmapsto \mathrm{Res}_{T=0} \langle f \otimes g \rangle \frac{dT}{1+T}$$

$\{\psi(x), \psi(y)\} = \{\gamma x, \gamma y\} = \{x, y\}, \quad \{\psi x, y\} = \{x, \varphi y\}$

$$\begin{tikzcd}
C_{\varphi,\gamma}(D(V)): D(V) \arrow[r, "{(\varphi-1,\, \gamma-1)}"] & D(V) \oplus D(V) \arrow[r, "{(1-\gamma,\, \varphi-1)}"] & D(V) \\
\times & \times & \times \\
D(V)^\vee \cdot \frac{dT}{1+T} \arrow[d] & D(V)^\vee \cdot \frac{dT}{1+T} \oplus D(V)^\vee \cdot \frac{dT}{1+T} \arrow[l, "{(1-\psi,\, \gamma-1)}"'] \arrow[d] & D(V)^\vee \frac{dT}{1+T} : C_{\gamma,\psi}\left(D(V)^\vee \frac{dT}{1+T}\right) \arrow[l, "{(1-\gamma,\, 1-\psi)}"'] \arrow[d] \\
\mathbb{Q}_p & \mathbb{Q}_p & \mathbb{Q}_p
\end{tikzcd}$$

$\Longrightarrow$ Tate duality $H^i(G_K, V) = \left(H^{2-i}(G_K, V^\vee(1))\right)^\vee$. $\forall i$

In other words, $H^i(G_K, V) \times H^{2-i}(G_K, V^\vee(1)) \longrightarrow H^2(G_K, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$ is perfect pairing.

V Overconvergent story

<u>Definition</u> For $D \in (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}_K^\dagger$ or $(\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}/\mathbb{B}^\dagger_{\mathrm{rig},K}$, we define $H^i(D)$ to be the cohomology of the Herr complex

$$C_{\varphi,\gamma}: D \xrightarrow{(\varphi-1,\, \gamma-1)} D \oplus D \xrightarrow{(1-\gamma,\, \varphi-1)} D$$

<u>Theorem</u> (Liu) (i) We have a perfect pairing $H^i(D) \times H^{2-i}(D^\vee(1)) \longrightarrow H^2(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$

(ii) $\chi(D) := \sum_{i=0}^{2} (-1)^i \dim H^i(D) = -[K:\mathbb{Q}_p] \cdot \mathrm{rank}\, D$.

<u>Idea of the proof</u>: For $D \in (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}_K^\dagger$ and $(\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\text{ét}}/\mathbb{B}^\dagger_{\mathrm{rig},K}$, we compare its cohomolgy with the cohomology of the corresponding $\tilde{D} \in (\varphi,\Gamma)\text{-}\mathrm{mod}^{\text{ét}}/\mathbb{B}_K$.

For general $D \in (\varphi,\Gamma)\text{-}\mathrm{mod}/\mathbb{B}^\dagger_{\mathrm{rig},K}$, we first use slope filtration to reduce to pure case. Then we show that both (i) and (ii) are preserved under twisting D by t.

(using exact sequence $0 \to tD \to D \to D/t \to 0$)

study this

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Lecture IX Eigencurves

I. Galois rep'n associated to a classical eigenform

- Let $f = q + a_2 q^2 + a_3 q^3 + \cdots \in S_k(N, \epsilon)$ be a normalized Hecke eigen new form of weight $k \geq 2$ level $\Gamma_1(N)$ and charater $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$

Deligne-Serre associates f a representation $\rho_f : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{Q}}_p)$ s.t.

(1) When $\ell \nmid Np$, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is unramified and the char. poly for Frob_ℓ is
$$X^2 - a_\ell X + \ell^{k-1}\epsilon(\ell) = 0$$
geom frobenius

(2) When $\ell = p$. $\to$ If $p \nmid N$, $\rho_f|_{G_{\mathbb{Q}_p}}$ is crystalline & the char. poly for φ^{-1} is $X^2 - a_p X + p^{k-1}\epsilon(p) = 0$

$\to$ If $p \| N$, $\rho_f|_{G_{\mathbb{Q}_p}}$ is semistable.

$\to$ If $p^2 | N$, we can only say that $\rho_f|_{G_{\mathbb{Q}_p}}$ is deRham or potentially semistable.

II. Overconvergent modular forms

Let $(N, p) = 1$ & $N > 5$.

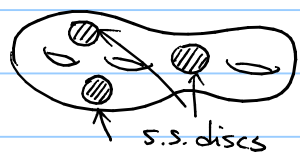
Consider the modular curve $X_0(N) \rightsquigarrow \mathfrak{X}_0(N)/\mathbb{Z}_p$ proper smooth model $/\mathbb{Z}_p$

$\mathfrak{X}_0(N)_{\mathbb{F}_p}$ parametrizes elliptic curves over $\mathbb{F}_p$ with level N structures

- Supersingular elliptic curves over $\bar{\mathbb{F}}_p$ corresponds to points on $\mathfrak{X}_0(N)_{\bar{\mathbb{F}}_p}$

They are the reductions of some discs of $X_0(N)^{an}$, viewed as a rigid analytic space.

To sum up, $X_0(N)^{an} = X_0(N)^{ord} \cup X_0(N)^{ss}$
a bunch of discs.



For $r \to 1^-$, let $X_0(N)(r) = X_0(N)^{an} \setminus$ the open discs of radius r

Def'n. An overconvergent modular form of wt k (≥ 2) level N is a section
$$H^0(X_0(N)(r), \underline{\omega}^{k-1}) \quad \text{for some } r \to 1^-$$
, where $\underline{\omega}$ = standard sheaf giving rise to modular forms.

We use $M_k^\dagger(\Gamma_0(N), r)$ to denote all ocvgt forms.

$\circlearrowleft$ Hecke algebra T_ℓ for $\ell \nmid N$

& U_p $\leftarrow$ We have U_p instead of T_p here

This is an important figure here

Coleman inequality.
Let $f \in M_k^\dagger(\Gamma_0(N), r)$ be an eigenform of U_p with eigenvalue λ.
If $v_p(\lambda) < k-1$, then f is a classical modular form of weight k and level possibly pN.

III Eigencurves

- $\chi = \chi_{cyc} : \Gamma = \Gamma_{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ $\quad(p>2)$

  $\Lambda := \mathcal{O}_L[[\Gamma]] = \varprojlim_n \mathcal{O}_L[[\Gamma/\Gamma^{p^n}]] \approx \mathcal{O}_L[[X]]^{(p-1)}$

- Weight space: space for $\mathrm{Hom}(\Gamma_{\mathbb{Q}_p}, \mathcal{O}_{\mathbb{C}_p}^\times) = (\mu_{p-1})^\vee \times (1 + \mathfrak{m}_{\mathbb{C}_p})$ ← the torsion part ← where a generator of Γ goes

  $\Gamma_{\mathbb{Q}_p} = \mathbb{Z}_p^\times$; $\quad$ = disjoint union of $(p-1)$ open discs

Theorem (Coleman–Mazur)
There exists a rigid space $\mathcal{C}$ parametrizing all overconvergent eigenforms in the following sense:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{U_p} & \mathbb{A}^1_{rig} \\ \downarrow \kappa & & \\ \mathcal{W} & & \end{array}$$

($\mathcal{C}$ is finite over a closed subspace of $\mathcal{W} \times \mathbb{A}^1_{rig}$)

in fact a disc suffices

If $c \in \mathcal{C}$ is a point such that
- $\kappa(c)$ is the character $\chi_{cyc}^k : \mathbb{Z}_p^\times \to \mathcal{O}_{\mathbb{C}_p}^\times \quad x \mapsto x^k$
- $U_p(c) = \lambda$

Then c corresponds to an overconvergent eigenform $f_c \in M_k^\dagger(\Gamma_0(Np), v)$.

Caution: $\mathcal{C}$ is not smooth. In general, it may not even be flat over $\mathcal{W}$.

Definition. We give a list of general properties expected for eigenvarieties X of unitary Shimura varieties, where the associated "representations" are d-dim'l. We have

(a) a family of pseudo representations $T : G_{\mathbb{Q}} \to \mathcal{O}(X)$ ← explained later. Think of actually rep'n for now.

(b) d analytic functions $\kappa_1, \dots, \kappa_d \in \mathcal{O}(X)^\times$ ← weight functions

(c) d analytic functions $F_1, \dots, F_d \in \mathcal{O}(X)^\times$ ← Frobenius eigenvalues, but not quite.

(d) a Zariski closed subsets $Z \subset X$ of points. ← classical modular forms with no level at p.

satisfying the following properties: ← makes sense for all p-adic rep'ns, not only the deRham ones.

(i) For every $x \in X$, the Hodge-Tate-Sen weights of ρ_x are $\kappa_1(x), \dots, \kappa_d(x)$

(ii) If $z \in Z$, ρ_z is crystalline (and hence, $\kappa_1(z), \dots, \kappa_d(z) \in \mathbb{Z}$)
Moreover, $\kappa_1(z) < \dots < \kappa_d(z)$

(iii) At $z \in Z$, the eigenvalues of φ on $D_{cris}(\rho_z)$ are $(p^{\kappa_1(z)} F_1(z), \dots, p^{\kappa_d(z)} F_d(z))$; they are distinct.

(iv) (Accumulation condition) For $C \in \mathbb{N}$, let Z_C denote the set

$\{z \in Z \mid |\kappa_I(z) - \kappa_J(z)| > C \ \forall I, J \in \{1, \cdots, d\}, |I| = |J| > 0, I \neq J\}$ Here $\kappa_I = \sum_{i \in I} \kappa_i$

Then Z_C accumulates at any $z \in Z$ for any C. ← essentially a corollary of Coleman inequality.

(*) For each n, $\exists$ continuous character $\mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$ whose derivative at 1 is κ_n and whose evaluation at each $z \in Z$ is the $\kappa_n(z)$-th power map.

will be important in the next chapter.

IV. Pseudo-representations

- From the Hecke eigenvalue, we only know the traces of the expected Galois representation. Over a field of char 0, one can recover the rep'n from the traces. However, given a family of traces, one may not be able to recover the rep'n.
- Let ρ be a family of rep'ns of a group G on a rank d free R-module

$$\rho \rightsquigarrow T : G \longrightarrow R$$
$$g \longmapsto \mathrm{tr}\,\rho(g)$$

Of course, $T(gh) = T(hg)$.

Given $T : G \to R$ satisfying $T(gh) = T(hg)$, we define $S_n(T) : \overbrace{G \times \cdots \times G}^{n} \longrightarrow R$ as

$$S_n(T)(x_1, \cdots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, T^\sigma(x_1, \cdots, x_n)$$

where if σ can be broken into circles $\sigma_1, \cdots, \sigma_r$, $T^\sigma(x_1, \cdots, x_n) = \prod_{j=1}^{r} T^{\sigma_j}(x_1, \cdots, x_n)$

Here if $\sigma_j = (x_{j_1}, \cdots, x_{j_s})$, $T^{\sigma_j}(x_1, \cdots, x_n) := T(x_{j_1} \cdots x_{j_s})$

We say that T is a pseudo-rep'n of dim d, if $S_{d+1}(T) \equiv 0$.

e.g. $d = 2$: $T(g)T(h)T(i) - T(g)T(hi) - T(h)T(ig) - T(i)T(gh) + T(ghi) + T(gih) = 0$

Theorem. Over a field of char $> d$, we can recover the rep'n from a pseudo-representation.

V. Hida family

Over the locus where U_p eigenvalues have norm 1, $\mathcal{C}$ is in fact finite & flat over $\mathcal{W}$

This is called the Hida family.

This works in a much more general context.

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Lecture X Triangulation

Recall: We have an equivalence of categories

Robba ring $= B^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}$

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \xrightarrow{D^{\dagger}_{\mathrm{rig}}} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}^{\acute{e}t}/\mathcal{R} \underset{\text{fully faithful}}{\longleftrightarrow} (\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}/\mathcal{R}$$

I. Triangulation

From now on, it is essential to assume that $K=\mathbb{Q}_p$

We allow $V\in\mathrm{Rep}_L(G_{\mathbb{Q}_p})$ to have coefficients in $L \rightsquigarrow \mathcal{R}=B^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}\otimes L$ with L-linear (φ,Γ)-actions

Upshot: Even if V is irreducible in $\mathrm{Rep}_L(G_{\mathbb{Q}_p})$, $D^{\dagger}_{\mathrm{rig}}(V)$ may still be reducible when considered as an object in $(\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}/\mathcal{R}$.

Facts: (1) All 1-dim'l (φ,Γ)-modules over $\mathcal{R}$ are given by a character $\delta:\mathbb{Q}_p^{\times}\longrightarrow L^{\times}$

$$\delta \leftrightsquigarrow \mathcal{R}(\delta), \text{ generated by } v \text{ s.t. } \varphi(v)=\delta(p)v \text{ and } \gamma(v)=\delta(\chi(\gamma))v.$$

(2) Extensions of $\mathcal{R}(\delta_1)$ by $\mathcal{R}(\delta_2)$ is parametrized by $H^1(\mathcal{R}(\delta_2^{-1}\delta_1))$

$$* \quad H^0(\mathcal{R}(\delta))=\begin{cases} L\cdot t^i, & \text{if } \delta=x^{-i} \text{ for } i\in\mathbb{Z}_{\geq 0}\\ 0, & \text{otherwise}\end{cases}$$

$$\xRightarrow{\text{duality}} H^2(\mathcal{R}(\delta))=\begin{cases} 1\text{-dim'l}, & \text{if } \delta=|x|\cdot x^i \text{ for } i\in\mathbb{Z}_{\geq 0}\\ 0, & \text{otherwise}\end{cases}$$

$$\xRightarrow{\text{EP-char.}} H^1(\mathcal{R}(\delta))=\begin{cases} 2\text{-dimensional}, & \text{if } \delta=x^{-i} \text{ for } i\in\mathbb{Z}_{\geq 0} \text{ or } \delta=|x|\cdot x^i \text{ for } i\in\mathbb{N}\\ 1\text{-dimensional}, & \text{otherwise}\end{cases}$$

Definition A (φ,Γ)-module $D_{\mathrm{rig}}\in(\varphi,\Gamma)\text{-}\underline{\mathrm{mod}}/\mathcal{R}$ is called trianguline if we have a filtration

$$0=\mathrm{Fil}_0 D_{\mathrm{rig}}\subsetneq \mathrm{Fil}_1 D_{\mathrm{rig}}\subsetneq\cdots\subsetneq \mathrm{Fil}_d D_{\mathrm{rig}}=D_{\mathrm{rig}}$$

s.t. $\mathrm{Fil}_i D_{\mathrm{rig}}/\mathrm{Fil}_{i-1}D_{\mathrm{rig}}\simeq\mathcal{R}(\delta_i)$ is 1-dim'l.

We are interested in 2-dim'l trianguline (φ,Γ)-modules that come from irred. rep'ns

$$0\longrightarrow\mathcal{R}(\delta_1)\longrightarrow D(s)\longrightarrow\mathcal{R}(\delta_2)\longrightarrow 0$$

where $s=(\delta_1,\delta_2,\mathcal{L})$ with $\mathcal{L}\in\mathbb{P}(H^1(\mathcal{R}(\delta_2^{-1}\delta_1)))$

Let $u=v_p(\delta_1(p))=-v_p(\delta_2(p))>0$,

$w_i=$ weight of δ_i, i.e. $\delta_i(x)=x^{w_i}$ for $x\in\mathbb{Z}_p^{\times}$

$w=w_1-w_2$

$$\mathcal{S}_+ \begin{cases} \mathcal{S}_+^{ng} : \text{if } w \text{ is not a positive integer} \rightsquigarrow D(s) \text{ is not twist of HT-rep'ns} \\ \mathcal{S}_+^{cris} : \text{if } w \in \mathbb{N} \text{ and } w > u \ \& \ \mathcal{L} = \infty \rightsquigarrow D(s) \text{ comes from twists of crystabelian rep'ns} \\ \mathcal{S}_+^{st} : \text{if } w \in \mathbb{N} \text{ and } w > u \ \& \ \mathcal{L} \neq \infty \rightarrow D(s) \text{ comes from twists of semistabelian rep'ns} \\ \mathcal{S}_+^{ncl} : \text{if } w \in \mathbb{N} \text{ and } w < u \rightsquigarrow D(s) \text{ does not come from a Galois rep'n.} \\ \mathcal{S}_+^{ord} : \text{if } w = u \longrightarrow D(s) \text{ comes from an ordinary rep'n, up to a twist.} \end{cases}$$

Here, crystabelian/semistabelian mean crystalline/semistable over $\mathbb{Q}_p(\mu_{p^N})$ for some $N \in \mathbb{Z}_{\geq 0}$

<u>Remark</u>: (1) $s \in \mathcal{S}_+^{ncl}$, $D(s)$ does not come from a Galois rep'n because $\mathcal{R}(t^w \delta_2)$ is a sub.

(2) If $s \in \mathcal{S}_+^{cris}$, $D(s)$ admits another triangulation by

$$0 \to \mathcal{R}(t^w \delta_2) \to D(s) \longrightarrow \mathcal{R}(t^{-w}\delta_1) \longrightarrow 0$$

The triangulation for the other three types of (φ,Γ)-modules is unique.

II. Trianguline $\iff$ $D_{cris}(V) \neq 0$ up to twists

<u>Recall</u>: $D_{cris}(V) = \left(D^\dagger_{rig}(V)[\frac{1}{t}]\right)^{\Gamma_{\mathbb{Q}_p}}$

- $\dim V = 2$

  * If V is trianguline, may twist a unitary character of $\mathbb{Q}_p^\times$

    $\rightsquigarrow 0 \to \mathcal{R}(\delta_1) \longrightarrow D^\dagger_{rig}(V \otimes \chi) \longrightarrow \mathcal{R}(\delta_2) \longrightarrow 0$ such that $\delta_1(\mathbb{Z}_p^\times) = 1$

    $\Rightarrow D_{cris}(V)$ must contain $(\mathcal{R}(\delta_1))^{\Gamma_{\mathbb{Q}_p}} = \delta_1 \cdot e_1$

  * If $D_{cris}(V) \neq 0$, $\Rightarrow \exists$ eigenvector $e \in D_{cris}(V)^{\varphi = \alpha}$

    Let $\delta = \mathbb{Q}_p^\times \to L$ be given by $\delta(p) = \alpha$, $\delta(\mathbb{Z}_p^\times) = 1$

    $\Rightarrow \mathcal{R}(\delta) \longrightarrow D_{cris}(V) \otimes_L \mathcal{R}[\frac{1}{t}] \simeq D^\dagger_{rig}(V)[\frac{1}{t}]$

    $\Rightarrow t^? \mathcal{R}(\delta) \longrightarrow D^\dagger_{rig}(V)$ with quotient free over $\mathcal{R}$.

    $\Longrightarrow$ Trianguline (where the sub might be different from $\mathcal{R}(\delta)$ though)

- <u>A good dictionary</u>: V crystalline $\rightsquigarrow D_{cris}(V) \in \text{Fil-}\varphi\text{-}\underline{\text{mod}}/\mathbb{Q}_p$

  Assume that we have $0 \to D_1 \to D_{cris}(V) \to D_2 \to 0$ in $\text{Fil-}\varphi\text{-}\underline{\text{mod}}/\mathbb{Q}_p$

  This translates to an exact sequence $0 \to D_{rig,1} \to D^\dagger_{rig}(V) \to D_{rig,2} \to 0$ in $(\varphi,\Gamma)\text{-mod}/\mathcal{R}$

  by Berger's correspondence we discussed earlier.

  $\Longrightarrow$ If V is crystalline of arbitrary dimension, $D^\dagger_{rig}(V)$ is trianguline.

III Triangular Family of (φ,Γ)-modules

Recall: We have eigenvarieties parametrizing Galois representations coming from automorphic forms, being crystalline at a Zariski dense set of points Z.

$\rightsquigarrow$ trianguline over these points.

Conjecture. There exists a global triangulation, whose restriction to each point gives the triangulation at the point.

IV. Choice of Frobenius eigenvalues

Back to the classical modular form/eigencurves case.

Subtlety: Let f be a normalized classical new form of level N, $p \nmid N$.

$\rightsquigarrow \rho_f|_{G_{\mathbb{Q}_p}}$ is crystalline $\rightsquigarrow D_{cris}(\rho_f)$ usually has two eigenvalues α, β

- Eigenvalue of φ on $D_{cris}(\rho_f) \longleftrightarrow U_p$-eigenvalue.

$\Rightarrow \rho_f \longleftrightarrow$ two points on $\mathcal{C}$!!

$$f_\alpha(z) := f(z) - \beta f(pz) \quad \text{and} \quad f_\beta(z) := f(z) - \alpha f(pz)$$

They have U_p-eigenvalues α and β respectively.

Another point of view: $\mathcal{C} \longrightarrow$ deformation space of $\bar{\rho}$ some residue rep'n (mod p)

is not an embedding; it branches at classical points.

$\rightsquigarrow$ Image is dense (although the dimensions are different)

- p-adic L-function associated to f depends on the choice of α or β

$\rightsquigarrow \mathcal{L}_{p,\alpha}(f)$ and $\mathcal{L}_{p,\beta}(f)$

They vary in family on $\mathcal{C}$; note $\mathcal{L}_{p,\alpha}(f)$ shows up at the point f_α and $\mathcal{L}_{p,\beta}(f)$ shows up at the point f_β

When $|\alpha|=1$, we need to be a bit careful about $\mathcal{L}_{p,\beta}(f)$ (recent work by Beillaiche)

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Lecture XI Bloch-Kato local condition and triangluordinary condition

I. Bloch-Kato local condition

We assume that K is finite over $\mathbb{Q}_p$

Fact: A fundament exact sequence: $0 \to \mathbb{Q}_p \to B_{cris}^{\varphi=1} \to B_{dR}/\mathrm{Fil}^0 B_{dR} \to 0$

Variant: $0 \to \mathbb{Q}_p \to B_{cris} \xrightarrow{(\varphi-1,1)} B_{cris} \oplus B_{dR}/\mathrm{Fil}^0 B_{dR} \to 0$

Tensoring with $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we have long exact sequences

$$0 \to V^{G_K} \to D_{cris}(V)^{\varphi=1} \to D_{dR}(V)/\mathrm{Fil}^0 D_{dR}(V) \to H^1(K,V) \to H^1(K, V\otimes B_{cris}^{\varphi=1})$$

$$0 \to V^{G_K} \to D_{cris}(V) \xrightarrow{(\varphi-1,1)} D_{cris}(V) \oplus D_{dR}(V)/\mathrm{Fil}^0 D_{dR}(V) \to H^1(K,V) \to H^1(K, V\otimes B_{cris})$$

Define: $H^1_e(K,V) = \mathrm{Ker}\left(H^1(K,V) \to H^1(K, V\otimes B_{cris}^{\varphi=1})\right)$

$H^1_f(K,V) = \mathrm{Ker}\left(H^1(K,V) \to H^1(K, V\otimes B_{cris})\right)$

$H^1_g(K,V) = \mathrm{Ker}\left(H^1(K,V) \to H^1(K, V\otimes B_{dR})\right)$

Facts: (1) $H^1_e(K,V) \subseteq H^1_f(K,V) \subseteq H^1_g(K,V)$

(2) Under the Tate pairing $H^1(K,V) \times H^1(K,V^*(1)) \to H^2(K,\mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$

$H^1_e(K,V)^\perp = H^1_g(K,V^*(1))$, $H^1_f(K,V)^\perp = H^1_f(K,V^*(1))$, $H^1_g(K,V)^\perp = H^1_e(K,V^*(1))$

(3) $H^1_f(K,V)/H^1_e(K,V) = D_{cris}(V)/(\varphi-1)$, $H^1_g(K,V)/H^1_f(K,V) \simeq D_{cris}(V^*(1))^{\varphi=1}$

almost always zero.

Analogy with the ℓ-adic case:

denoted as $H^1_f(K,V)$

$V \in \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)$: $\quad 0 \subseteq H^1_{ur}(K,V) \subseteq H^1(K,V)$

$V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$: $\quad 0 \subseteq H^1_e(K,V) \subseteq H^1_f(K,V) \subseteq H^1_g(K,V) \subseteq H^1(K,V)$

- $H^1_f(K,V)$ is the "correct" local condition at p.

Now, let F be a number field and $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_F)$

Define $\mathrm{Sel}(F,V) = H^1_f(F,V) := \mathrm{Ker}\left(H^1(F,V) \to \prod_v \dfrac{H^1(F_v,V)}{H^1_f(F_v,V)}\right)$

(when $v|\infty$, $H^1(F_v,V)=0$)

Problem: Hard to get a hand on $H^1_f(F_v,V)$ when $v|p$.

II. Pachinski's condition

If V is crystalline and fits into an exact sequence

$$0 \to V_+ \to V \to V_- \to 0$$

where V_+ has all non-positive HT weights ← sorry for the confusing notation
V_- has all positive HT weights ← Note that this is opposite to the DR fil'n

Then (assuming $D_{cris}(V)^{\varphi=1} = D_{cris}(V^*(1))^{\varphi=1} = 0$) we have an exact sequence

$$0 \to H^0(K,V_-) \to H^1(K,V_+) \to H^1_f(K,V) \to 0$$

often zero too. same as e or g under the assumption.

<u>Proof</u>: It follows from the following commutative diagram

$$\begin{tikzcd}
 & & H^1_g(K,V_+) \arrow[r, dashed] \arrow[d] \arrow[d, phantom, shift left=6, ""] & H^1_g(K,V) \arrow[r, dashed] \arrow[d] & H^1_g(K,V_-) \arrow[d] \\
H^0(K,V) \arrow[r] & H^0(K,V_-) \arrow[r] \arrow[ur, dashed] & H^1(K,V_+) \arrow[r] \arrow[d] & H^1(K,V) \arrow[r] \arrow[d] & H^1(K,V_-) \arrow[d] \\
 & 0 \arrow[r] & H^1(K,V_+\otimes B_{dR}) \arrow[r] & H^1(K,V\otimes B_{dR}) \arrow[r] & H^1(K,V_-\otimes B_{dR})
\end{tikzcd}$$

$H^0(K,V) = 0$ b/c $D_{cris}(V)^{\varphi=1}$

$H^1_g(K,V_+) =$ all of $H^1(K,V_+)$ b/c V_+ has nonnegative & duality

$H^1_g(K,V_-) = 0$ because V_- has only positive HT-wts
$D^0_{dR}(V) \xrightarrow{\sim} D_{dR}(V_-)$
$\Rightarrow H^1_e(K,V_-) = 0$

b/c V is dR $\to 0$

<u>Upshot</u>: One can write $H^1_f(K,V)$ as a complex of usual cohomology.
This happens if V comes from an ordinary modular form. ($\rho_f|_{G_{\mathbb{Q}_p}}$ is reducible)
We can also talk about a family version of this via Hida family.

III. <u>Pottharst's triangluordinary condition</u>

Let $D \in$ Fil-(φ,N)-<u>mod</u>$/\mathbb{Q}_p$, not necessarily weakly admissible ← can talk about HT-wts.
$\rightsquigarrow D_{rig} \in (\varphi,\Gamma)$-<u>mod</u>$/\mathcal{R} = B^\dagger_{rig,\mathbb{Q}_p}$ not necessarily étale
Now, we start with a crystalline rep'n $\rho \rightsquigarrow D_{cris}(\rho) \in$ Fil-(φ,N)-<u>mod</u>$/\mathbb{Q}_p$
Assume that we have an exact sequence of filtered (φ,N)-modules

$$0 \to D_+ \to D_{cris}(\rho) \to D_- \to 0$$

s.t. D_+ has all non-positive HT weights ← But they are not weakly admissible, so are not
D_- has all positive HT weights ← actual rep'ns.

$$\rightsquigarrow 0 \to D_{rig,+} \to D^\dagger_{rig}(\rho) \to D_{rig,-} \to 0$$

<u>Theorem</u> (Pottharst)
Assume $H^0(\mathbb{Q}_p, D_{rig,-}) = 0$ and $(D_{rig,-}[\frac{1}{t}])^{\Gamma, \varphi=p^{-1}} = 0$.

Then $\mathrm{Ker}\left(H^1(G,V)\cong H^1(G,D^{\dagger}_{rig}(V))\to H^1(G,D^{\dagger}_{rig}(V)/D_{rig,+})\right)=H^1_g(G,V)$

IV Case of modular forms.

f normalized eigen new form of level N $(p\nmid N)$, character ϵ, weight k.

$\rightsquigarrow$ $\rho_f|_{\mathbb{Q}_p}$ is crystalline with $D_{cris}(\rho_f|_{\mathbb{Q}_p})=E\cdot e_\alpha\oplus E\cdot e_\beta$. E some coefficient field of f

$\varphi e_\alpha=\alpha\cdot e_\alpha$, $\varphi e_\beta=\beta\cdot e_\beta$, where α,β are roots of $X^2-a_pX+p^{k-1}\epsilon(p)=0$

$$\mathrm{Fil}^i D_{cris}(\rho_f|_{\mathbb{Q}_p})=\begin{cases}0 & i\geq k\\ E\cdot(e_\alpha+?e_\beta) & 1\leq i\leq k-1 \qquad ?\neq 0\\ E\cdot e_\alpha\oplus E\cdot e_\beta & i\leq 0\end{cases}$$

Each $E\cdot e_\alpha$ or $E\cdot e_\beta$ is a subobject of $D_{cris}(\rho_f|_{\mathbb{Q}_p})$ in Fil-φ-mod$/E$

$\rightsquigarrow$ Two triangulations $0\to\mathcal{R}_E\cdot e_\alpha\to D_{rig}(\rho_f)\to t^{-k+1}\mathcal{R}_E\cdot e_\beta\to 0$

$0\to\mathcal{R}_E\cdot e_\beta\to D_{rig}(\rho_f)\to t^{-k+1}\mathcal{R}_E\cdot e_\alpha\to 0$

They both satisfy Pottharst condition.

$\Rightarrow$ $H^1_f(G_{\mathbb{Q}_p},V_f)=H^1(\mathcal{R}_E\cdot e_\alpha)$ or $H^1(\mathcal{R}_E\cdot e_\beta)$ for most of the case.

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Lecture XII: Application to non-ordinary IMC.

<u>Recall</u>: $\mathrm{Sel}(F,V)=H^1_f(F,V):=\mathrm{Ker}\left(H^1(F,V)\longrightarrow\prod_v \frac{H^1(F_v,V)}{H^1_f(F_v,V)}\right)$

I. Iwasawa theory v.s. Galois deformation

Consider the cyclotomic tower $K_n=\mathbb{Q}_p(\mu_{p^n})$, $K_\infty=\bigcup_n K_n$, $\Gamma=\mathrm{Gal}(K_\infty/\mathbb{Q}_p)$

For $V\in\mathrm{Rep}_{\mathbb{Z}_p,\mathbb{Q}_p}(G_{\mathbb{Q}_p})$, define $H^i_{Iw}(\mathbb{Q}_p,V):=\varprojlim H^i(K_n,V)$, where the maps are corestrictions.

Let $\Lambda:=\mathbb{Z}_p[\![\Gamma]\!]$, equipped with an action by Γ (and hence G) via multiplication.

<u>Lemma</u>. $H^i_{Iw}(\mathbb{Q}_p,V)\simeq H^i(\mathbb{Q}_p,V\otimes\Lambda)$

Proof: This follows from Shapiro's lemma.

$$\begin{array}{ccc} H^i(K_n,V) & \simeq & H^i(\mathbb{Q}_p,V\otimes\mathbb{Z}_p[\mathrm{Gal}(K_n/\mathbb{Q}_p)]) \\ \uparrow \text{cores} & & \uparrow \text{natural} \\ H^i(K_{n+1},V) & \simeq & H^i(\mathbb{Q}_p,V\otimes\mathbb{Z}_p[\mathrm{Gal}(K_{n+1}/\mathbb{Q}_p)]) \end{array} \quad \square$$

<u>Conclusion</u>: Iwasawa theory = deformation via Galois representation Λ.

· Since $\Gamma=\Delta\times\mathbb{Z}_p$, $\Lambda=\mathbb{Z}_p[\Delta]\otimes\mathbb{Z}_p[\![\mathbb{Z}_p]\!]\simeq\mathbb{Z}_p[\Delta]\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\![T]\!]$

$$1+p \longleftarrow\!\!| \; T$$

$\rightsquigarrow$ <u>analytic version</u>: Let Λ_{an} = "analytification of $\Lambda[\frac{1}{p}]$"

$$=\mathbb{Q}_p[\Delta]\otimes_{\mathbb{Q}_p}\mathcal{R}^+(T)$$

$\uparrow$ ring of analytic functions on open unit disc.

II. Selmer complexes.

In order to study Selmer group in family, it is better to go to the derived category.

We restrict ourself to the case $F=\mathbb{Q}$. $V\in\mathrm{Rep}_{\mathbb{Q}_p}(G_\mathbb{Q})$ (I think I need p splits in F.)

Let S be the set of places of $\mathbb{Q}$ containing p,∞ and where V ramifies

Then $\mathrm{Sel}(\mathbb{Q},V)=H^1_f(\mathbb{Q},V)=\mathrm{Ker}\left(H^1(G_{\mathbb{Q},S},V)\longrightarrow\bigoplus_{v\in S}H^1(\mathbb{Q}_v,V)/H^1_f(\mathbb{Q}_v,V)\right)$

$G_{\mathbb{Q},S}$ = $\mathrm{Gal}(\mathbb{Q}_S/F)$, where $\mathbb{Q}_S$ = max'l ext'n of $\mathbb{Q}$ unram. outside of S.

Define $\tilde{C}^\bullet_f(G_{\mathbb{Q},S},V):=\mathrm{Cone}\left[R\Gamma(G_{\mathbb{Q},S},V)\oplus\bigoplus_{v\in S}U_v^{+,\bullet}\longrightarrow\bigoplus_{v\in S}R\Gamma(\mathbb{Q}_v,V)\right][-1]$

$\uparrow$ local conditions at $v\in S$

To recover $Sel(\mathbb{Q},V)$, we take $U_v^{+,\bullet} = R\Gamma(G_{\mathbb{Q}_v}/I_v, V^{I_v})$ when $p\nmid v$

$U_p^{+,\bullet} = \mathbb{H}^\bullet(D_{rig,+}(V))$ assuming V satisfies Pottharst's condition.

The upshot is that $D_{rig,+}(V)$ and V^{I_v} interpolate well.

$\rightarrow$ If V & $D_{rig,+}(V)$ come in family, we have a Selmer complex in family.

III. The Iwasawa Main Conjecture

f weight k normalized cuspidal new eigenform of level N $(p\nmid N)$ and character ϵ.

- Let α,β be two roots of $X^2 - a_pX + p^{k-1}\epsilon = 0$

$$\begin{array}{ccc} f & \longrightarrow & V_f : \text{Galois rep'n of } G_{\mathbb{Q}} \\ \wr\downarrow & & \wr\downarrow \\ \text{p-adic L-functions } \mathcal{L}_{p,\alpha}(f),\ \mathcal{L}_{p,\beta}(f) \in \Lambda^{an} & & \widetilde{Sel}_{\alpha}(G_{\mathbb{Q},S}, V_f\otimes\Lambda^{an}) \\ & & \widetilde{H}^2_{f,\alpha}(G_{\mathbb{Q},S}, V_f\otimes\Lambda^{an}) \end{array}$$

β

a torsion Λ^{an}-module

s.t. $\forall \chi = \eta\chi_{cyc}^n$ with $1\le n\le k-1$, $\eta:\Gamma\to\mathcal{O}_E^\times$ finite

$\rightarrow \chi : \Lambda^{an}\to E$ alg. homo.

we have $\chi_*\left(\mathcal{L}_{p,\alpha}(f)\right) = (\cdots)\,\alpha^{-cond(\eta)}\,\dfrac{L_\infty^{\pm}(f_\alpha,\eta,k)}{\Omega_f^{\pm}}$

$\beta \quad \beta \quad f_\beta$

Here at p, we can choose either $D_\alpha\otimes\Lambda^{an}$ or $D_\beta\otimes\Lambda^{an}$

$\downarrow$

$\mathrm{Char}\left(\widetilde{H}^2_{f,\alpha}(G_{\mathbb{Q},S},V_f\otimes\Lambda^{an})\right)$ is an ideal of Λ^{an}

Iwasawa Main Conjecture:

Let f^c = complex conjugation of f and assume that $\mathcal{L}_p(f^c,\alpha)$ does not have an exceptional zero.
Then $\mathrm{char}\,\widetilde{H}^2_f(G_{\mathbb{Q},S},V_f\otimes\Lambda^{an})$ contains the ideal generated by $\mathcal{L}_p(f^c,\beta)$.

local condition at p uses the α-eigenvector

Idea of the proof:

By the construction of our Selmer complex

Thing we care about

$$H^1(G_{\mathbb{Q},S},V_f\otimes\Lambda^{an}) \longrightarrow \bigoplus_{v\in S} H^1\left(\frac{R\Gamma(\mathbb{Q}_v, V_f\otimes\Lambda^{an})}{U_v^{+,\bullet}}\right) \longrightarrow \widetilde{H}^2_{f,\alpha}(G_{\mathbb{Q}},V_f\otimes\Lambda^{an})$$

$$= \begin{cases} H^1(I_v, V_f\otimes_E\Lambda^{an})^{G_v/I_v} = ((V_f)_{I_v}\otimes_E\Lambda^{an})^{G_v/I_v} = 0 & \text{if } v\neq p \\ H^1(t^{-(k-1)}\mathcal{R}_E e_\beta\otimes\Lambda^{an}) \end{cases}$$

$$\hookrightarrow H^2(G_{\mathbb{Q},S},V_f\otimes\Lambda^{an}) \longrightarrow \bigoplus_{v\in S} H^2\left(\frac{R\Gamma(\mathbb{Q}_v,V\otimes\Lambda^{an})}{U_v^{+,\bullet}}\right) \longrightarrow \widetilde{H}^3_{f,\alpha}(G_{\mathbb{Q}},V_f\otimes\Lambda^{an})$$

Kernel denoted by $H^2_{p,an}(G_{\mathbb{Q},S},V_f\otimes\Lambda^{an})$

$$= \begin{cases} H^2(\mathbb{Q}_v, V\otimes\Lambda^{an}) \\ H^2(t^{-(k-1)}\mathcal{R}_E e_\beta) & v=p \end{cases}$$ almost always zero

$$\rightsquigarrow H^1(G_{\mathbb{Q},S}, V_f\otimes\Lambda^{an}) \longrightarrow H^1(t^{-(k-1)}\mathcal{R}_E\otimes\Lambda^{an} e_\beta) \longrightarrow \widetilde{H}^2_{f,\Lambda}(G_{\mathbb{Q}}, V_f\otimes\Lambda^{an}) \longrightarrow H^2_{p,an}(G_{\mathbb{Q},S}, V_f\otimes\Lambda^{an}) \to 0$$

$$\kappa_f \in H^1(G_{\mathbb{Q},S}, V_f\otimes\Lambda^{an}) \qquad H^1(t^{-(k-1)}\mathcal{R}_E\otimes\Lambda^{an} e_\beta) \cong \left(t^{-(k-1)}\mathcal{R}_E^+\right)^{\psi=1}$$

Kato's element

$$\left(t^{-(k-1)}\mathcal{R}_E^+\right)^{\psi=1} \xrightarrow{\;t^{k-1}\circ(1-\varphi)\;} \mathcal{L}_p(f^c,\beta)\in\left(\mathcal{R}_E^+\right)^{\psi=0}\simeq\Lambda^{an}, \qquad \kappa_f \longmapsto \mathcal{L}_p(f^c,\beta)$$

Another input: char $H^2_{p,Iw}(G_{\mathbb{Q},S}, V_f)$ divides char $H^1_{Iw}(G_{\mathbb{Q},S}, V_f\otimes\Lambda)/\Lambda\cdot\kappa_f$

IV About the construction of κ_f (very rough story)

Beilinson: X proper curve $/\mathbb{Q}$

$$K_2(X)\otimes_{\mathbb{Z}}\mathbb{Q} \rightsquigarrow H^1_f(\mathbb{Q}, H^1(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p)(2))$$

coming from $\mathcal{O}_X^\times\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow H^1(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p(1))$

$$\begin{tikzcd}
\rightsquigarrow \mathcal{O}_X^\times\otimes\mathcal{O}_X^\times\otimes_{\mathbb{Z}}\mathbb{Q} \arrow[r, "\text{cup product}"] \arrow[d, dashed] & H^2(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p(2)) \arrow[d, "\text{only a subspace}"] \\
K_2(X)\otimes_{\mathbb{Z}}\mathbb{Q} \arrow[r] & H^1(\mathbb{Q}, H^1(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p)(2))
\end{tikzcd}$$

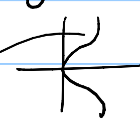
$\mathcal{O}_X^\times$ doesn't quite make sense because X is proper. Actually, we start with an affine subspace and then fill in the gap.

can put in f by a big theorem (weight monodromy)

Analogously, $K_2(E)\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow H^1(E_{\mathbb{C}},\mathbb{R}(1))$

$$\exists\, \kappa_E \longmapsto (*)\cdot L(E,0)\cdot\delta^-$$

Beilinson's element



Now, look at the tower $X(p^N)\to X(p^{N-1})\to\cdots\to X=X_0(N)$

$$\kappa_N \in X(p^N),\quad \kappa_{N-1}\in X(p^{N-1}),\quad \kappa_0 \in X$$

$$\rightsquigarrow \left(\varprojlim_N K_2(X(p^N))\right)\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow \varprojlim_N H^1\left(\mathbb{Q}(\mu_{p^N}), H^1(X(p^N)_{\bar{\mathbb{Q}}},\mathbb{Q}_p)(2)\right)$$

(κ_N) $\qquad\downarrow$ adjust weights.

$$(\kappa_N)\longmapsto \kappa_f \in H^1_{Iw}(\mathbb{Q}, V_f)$$

- It's completely non-trivial to see that κ_f encode any info of $\mathcal{L}(f,s)$ with $s\neq 0$!