A note on critical p-adic L-functions

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We study the adjunction property of the Jacquet-Emerton functor in certain neighborhoods of critical points in the eigencurve. As an application, we construct two-variable p-adic L-functions around critical points via Emerton's representation theoretic approach.

Contents

1 Introduction

Let $N \in \mathbb{Z}_{>1}$, $p \nmid N$, $k \in \mathbb{Z}_{\geq 0}$, and f be a classical newform of level $\Gamma_1(N)$ of weight $k+2$ over E (which is a finite extension of $\mathbb{Q}_p$ sufficiently large). Let a_p (resp. b_p) be the eigenvalue of the Hecke operator T_p (resp. S_p) on f, and α be a root of the Hecke polynomial $X^2 - a_p X + p b_p$. To f and α, one can associate an eigenform f_α of level $\Gamma_1(N) \cap \Gamma_0(p)$ satisfying that f_α has the same prime-to-p Hecke eigenvalues as f, and that $U_p(f_\alpha) = \alpha f_\alpha$. The eigenform f_α is called a *refinement* (or a *p-stabilization*) of f. We have $\mathrm{val}_p(a_p) \geq 0$ and $\mathrm{val}_p(b_p) = k$ (where val_p the additive p-adic valuation on $\overline{\mathbb{Q}_p}$ normalized with $\mathrm{val}_p(p) = 1$). Since $\alpha^2 - a_p\alpha + p b_p = 0$, we easily deduce $\mathrm{val}_p(\alpha) \leq k+1$. The refinement f_α is called

- *of non-critical slope* if $\mathrm{val}_p(\alpha) < k+1$,
- *of critical slope* if $\mathrm{val}_p(\alpha) = k+1$,
- *critical* if $\rho_{f,p}$ is split (which implies $\mathrm{val}_p(\alpha) = k+1$),

where $\rho_{f,p}$ is the 2-dimensional $\mathrm{Gal}_{\mathbb{Q}_p}$-representation associated to f.

To the form f_α, we can associate a p-adic L-function $L(f_\alpha, -)$, which is a distribution on $\mathbb{Z}_p^\times$ (for example see [27] [28] [1] [25]... see also Proposition 4.5 of this note). When f_α is non-critical, $L(f_\alpha, -)$ interpolates the critical values of the classical L-function attached to f_α (e.g. see (4.18)). If f_α is moreover of non-critical slope, we know $L(f_\alpha, -)$ can be determined (up to non-zero scalars) by the interpolation property. When f_α is of critical slope, the interpolation property is however not enough to determine $L(f_\alpha, -)$. And in this case, the problem of proving that the p-adic L-functions constructed by different methods coincide is not completely settled yet (to the author's knowledge). We remark that when f_α is critical, then $L(f_\alpha, \phi x^j)=0$ for all smooth characters ϕ on $\mathbb{Z}_p^\times$ and $j\in\{0,\cdots,k\}$ where x^j denotes the algebraic character $a\mapsto a^j$ on $\mathbb{Z}_p^\times$ (e.g. see Proposition 4.7). In all cases, $L(f_\alpha,-)$ fits into the so-called two-variable p-adic L-functions $L(-,-)$ where the first variable runs around points on the eigencurve (we recall that such points correspond to overconvergent eigenforms g), and $L(g,-)$ is, up to non-zero scalars, equal to the p-adic L-function of g when g is classical.

In [15], Emerton provided a representation theoretic approach of the construction of p-adic L-functions, using results in the p-adic Langlands program. Via this approach, in [17, § 4.5], Emerton also constructed two-variable p-adic L-functions in certain neighborhoods of non-critical points in the eigencurve (we call a point critical if its associated eigenform is critical)). A key ingredient in this construction is an adjunction property of the Jacquet-Emerton functor around non-critical points. In this note, we study the adjunction property around critical points in the eigencurve. We show that by adding "poles", the adjunction can extend to critical points (cf. Theorem 3.2 and Theorem 4.2). Using this result, together with the smoothness of the eigencurve at critical points (due to Bellaïche [1]), we construct two-variable p-adic L-functions (see (4.16)) in some neighborhoods of critical points via Emerton's approach. Evaluating the two-variable p-adic L-functions at the critical points then allows us to associate p-adic L-functions to the corresponding critical eigenforms.

The note is organised as follows. In § 2.1, §2.2, we recall some basic facts on the completed H^1 of modular curves, and recall Emerton's construction of the eigencurves. Nothing is new in these twos sections. In § 3, we show an adjunction property of the Jacquet-Emerton functor in neighborhoods of a critical point on the eigencurve. In § 4.1, we use this adjunction property to construct two-variable p-adic L-functions around critical points. Finally, we study some properties of our p-adic L-functions in § 4.2.

1.1 Notations

In this note, E will be a finite extension of $\mathbb{Q}_p$ with $\mathcal{O}_E$ its ring of integers, $\varpi_E\in\mathcal{O}_E$ a uniformiser and $k_E:=\mathcal{O}_E/\varpi_E$ its residue field. We let B denote the Borel subgroup of GL_2 of upper triangular matrices, $T\subseteq B$ the subgroup of diagonal matrices, and $N\subseteq B$ the unipotent radical. We use $\mathfrak{t}$ to denote the Lie algebra of $T(\mathbb{Q}_p)$ over E, $\mathfrak{g}$ the Lie algebra of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E. For a continuous character $\chi: T(\mathbb{Q}_p)\to E^\times$, we denote by $d\chi:\mathfrak{t}\to E$ the induced E-linear map with

$$d\chi(X)=\lim_{t\to 0}\frac{\chi(\exp tX)-1}{t}|_{t=0},$$

for $X\in\mathfrak{t}$. Similarly, for a continuous character $\chi:\mathbb{Q}_p^\times\to E^\times$, we denote by $\mathrm{wt}(\chi):\mathbb{Q}_p\to E$ the induced E-linear map, called the weight of χ. For $a\in E^\times$, we denote by $\mathrm{unr}(a):\mathbb{Q}_p^\times\to E^\times$ the unramified character (i.e. $\mathrm{unr}(a)|_{\mathbb{Z}_p^\times}=1$) with $\mathrm{unr}(a)(p)=a$.

For an E-vector space V that is equipped with an E-linear action of A (with A a set of operators), and for a system of eigenvalues χ of A, we denote by $V[A=\chi]$ the χ-eigenspace for A, and $V\{A=\chi\}$ the generalized χ-eigenspace for A.

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2 Eigencurves

2.1 Completed cohomology of modular curves

Let $\mathbb{A}^\infty$ denote the finite adeles of $\mathbb{Q}$. For a compact open subgroup K of $\mathrm{GL}_2(\mathbb{A}^\infty)$, we denote by Y_K the affine modular curve over $\mathbb{Q}$ such that the $\mathbb{C}$-points of Y_K are given by

$$Y_K(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\big(\mathbb{R}_+^\times \mathrm{SO}_2(\mathbb{R})K\big).$$

Let $K^p = \prod_{\ell\neq p} K_\ell$ be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$, and

$$\Sigma(K^p) := \{p\} \cup \{\ell \neq p,\ K_\ell \text{ is not maximal}\}.$$

Let $\mathcal{H}(K^p)$ denote the Hecke $\mathcal{O}_E$-algebra of K^p double cosets in $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$ (which is non-commutative in general). Following Emerton (cf. [17, (2.1.1)]), we put

$$\widetilde{H}^1_{\text{ét},c}(K^p, \mathcal{O}_E) := \varprojlim_n \varinjlim_{K_p} H^1_{\text{ét},c}\big(Y_{K_pK^p,\overline{\mathbb{Q}_p}}, \mathcal{O}_E/\varpi_E^n\big)$$

which is a complete $\mathcal{O}_E$-module equipped with a continuous action of $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathcal{H}(K^p) \times \mathrm{Gal}_{\mathbb{Q}} \times \pi_0$, where π_0 is the 2-element group generated by the archimedean Hecke operator

$$T_\infty := (\mathbb{R}_+^\times \mathrm{SO}_2(\mathbb{R}))\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathrm{SO}_2(\mathbb{R}).$$

Let $\mathcal{H}^p$ be a commutative $\mathcal{O}_E$-subalgebra of $\mathcal{H}(K^p)$ containing $\mathcal{H}(K^p)^{\mathrm{sph}} := \otimes'_{\ell\notin\Sigma(K^p)}\mathcal{O}_E[T_\ell, S_\ell^{\pm}]$ with $T_\ell := K^p\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}K^p$, $S_\ell := K^p\begin{pmatrix}\ell & 0\\ 0 & \ell\end{pmatrix}K^p$. One has the Eichler-Shimura relations on $\widetilde{H}^1_{\text{ét},c}(K^p, \mathcal{O}_E)$ (e.g. see [6, Prop. 3.2.3] and the proof):

$$\mathrm{Frob}_\ell^{-2} - T_\ell \mathrm{Frob}_\ell^{-1} + \ell S_\ell = 0$$

for $\ell \notin \Sigma(K^p)$, where Frob_ℓ denotes an arithmetic Frobenius at ℓ. Put

$$\widetilde{H}^1_{\text{ét},c}(K^p, E) := \widetilde{H}^1_{\text{ét},c}(K^p, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E,$$

which is a unitary admissible E-Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ equipped with an action of $\mathcal{H}^p \times \mathrm{Gal}_{\mathbb{Q}} \times \pi_0$, commuting with $\mathrm{GL}_2(\mathbb{Q}_p)$.

Notation 2.1. *(1) For an $\mathcal{O}_E$-module M equipped with a π_0-action, denote by $M^\pm$ the ±1-eigenspace for T_∞, one has thus $M = M^+ \oplus M^-$.*

(2) Let $\overline{\rho}$ be a 2-dimensional continuous representation of $\mathrm{Gal}_{\mathbb{Q}}$ over k_E, unramified outside $\Sigma(K^p)$, we denote by $\mathfrak{m}(\overline{\rho})$ the maximal ideal of $\mathcal{H}(K^p)^{\mathrm{sph}}$ attached to $\overline{\rho}$ satisfying that the quotient map $\mathcal{H}(K^p)^{\mathrm{sph}} \twoheadrightarrow \mathcal{H}(K^p)^{\mathrm{sph}}/\mathfrak{m}(\overline{\rho}) \cong k_E$ sends T_ℓ to $\mathrm{Tr}(\overline{\rho}(\mathrm{Frob}_\ell^{-1}))$ and S_ℓ to $\ell^{-1}\det(\overline{\rho}(\mathrm{Frob}_\ell^{-1}))$ for $\ell \notin \Sigma(K^p)$.

(3) For a finite ϖ_E-torsion $\mathcal{O}_E$-module M that is equipped with an $\mathcal{H}(K^p)^{\mathrm{sph}}$-action, denote by $M_{\overline{\rho}}$ the localisation of M at $\mathfrak{m}(\overline{\rho})$. We put

$$\widetilde{H}^1_{\text{ét},c}(K^p, \mathcal{O}_E)_{\overline{\rho}} := \varprojlim_n \varinjlim_{K_p} H^1_{\text{ét},c}\big(Y_{K_pK^p,\overline{\mathbb{Q}_p}}, \mathcal{O}_E/\varpi_E^n\big)_{\overline{\rho}},$$

and $\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}} := \widetilde{H}^1_{\text{ét},c}(K^p, \mathcal{O}_E)_{\overline{\rho}} \otimes_{\mathcal{O}_E} E$.

In the sequel, we fix a 2-dimensional continuous representation $\overline{\rho}$ of $\mathrm{Gal}_{\mathbb{Q}}$ over k_E, which is absolutely irreducible, unramified outside $\Sigma(K^p)$ and satisfies $\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}} \neq 0$. We summarize some properties of $\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}$:

Theorem 2.2. *(1) $\widetilde{H}^1_{ét,c}(K^p,E)_{\overline{\rho}}$ is a direct summand of $\widetilde{H}^1_{ét,c}(K^p,E)$ stable under $\mathrm{GL}_2(\mathbb{Q}_p)\times \mathrm{Gal}_{\mathbb{Q}}\times \mathcal{H}^p\times \pi_0$. For any compact open pro-p-subgroup K_p of $\mathrm{GL}_2(\mathbb{Z}_p)$, there exists $r\in\mathbb{Z}_{>0}$ such that we have an isomorphism of K_p-representations:*

$$\widetilde{H}^1_{ét,c}(K^p,E)_{\overline{\rho}}|_{K_p}\cong \mathcal{C}(K_p,E)^{\oplus r}, \tag{2.1}$$

where $\mathcal{C}(K_p,E)$ denotes the space of continuous functions on K_p with values in E, equipped with the right regular K_p-action.

(2) We have:

$$\widetilde{H}^1_{ét,c}(K^p,E)^{\mathrm{lalg}}_{\overline{\rho}}\xrightarrow{\sim}\oplus_{k\in\mathbb{Z}_{\geq 2},w\in\mathbb{Z}}H^1_{ét,c}(K^p,\mathcal{F}_{\mathrm{alg}(k,w)})_{\overline{\rho}}\otimes_E \mathrm{alg}(k,w)^\vee. \tag{2.2}$$

where $\widetilde{H}^1_{ét,c}(K^p,E)^{\mathrm{lalg}}_{\overline{\rho}}$ denotes the locally algebraic vectors for $\mathrm{GL}_2(\mathbb{Q}_p)$ inside $\widetilde{H}^1_{ét,c}(K^p,E)_{\overline{\rho}}$, $\mathrm{alg}(k,w)$ denotes the algebraic representation $\mathrm{Sym}^{k-2}E^2\otimes_E \det^w$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathcal{F}_{\mathrm{alg}(k,w)}$ denotes the local system associated to $\mathrm{alg}(k,w)$ on $Y_{K_pK^p}$ for compact open subgroups K_pK^p of $\mathrm{GL}_2(\mathbb{A}^\infty)$, and where

$$H^1_{ét,c}(K^p,\mathcal{F}_{\mathrm{alg}(k,w)})_{\overline{\rho}}:=\varinjlim_{K_p}H^1_{ét,c}(Y_{K^pK_p,\overline{\mathbb{Q}}},\mathcal{F}_{\mathrm{alg}(k,w)})_{\overline{\rho}}$$

denotes the (classical) étale cohomology with compact support of modular curves (localized at $\mathfrak{m}(\overline{\rho})$) with tame level K^p and coefficients in $\mathcal{F}_{\mathrm{alg}(k,w)}$ (with K_p running through compact open subgroups of $\mathrm{GL}_2(\mathbb{Z}_p)$).

Proof. For (1), see the discussion in [20, § 5.3] and [20, Cor. 5.3.19]. For (2), see [5, Thm. 4.1] (see also [17, Cor. 2.2.18, (4.3.4)]). □

2.2 Eigencurves (eigensurfaces)

We first briefly recall Emerton's construction of the eigencurves (eigensurfaces), and we refer to [17, § 2.3] for details. We let $\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}$ be the locally analytic subrepresentation of $\widetilde{H}^1_{\text{ét},c}(K^p,E)_{\overline{\rho}}$. By [16, Thm. 0.5], applying the Jacquet-Emerton functor to $\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}$, one gets an essentially admissible locally analytic representation $J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)$ of $T(\mathbb{Q}_p)$. Here $J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)$ being essentially admissible means that there exists a coherent sheaf $\mathcal{M}$ over $\widehat{T}$ such that (cf. [21, §6.4], [17, Prop. 2.3.2])

$$\mathcal{M}(\widehat{T})\cong J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)^\vee_b, \tag{2.3}$$

where "$-^\vee_b$" denotes the continuous dual equipped with the strong topology, and where $\widehat{T}$ denotes the rigid space parametrizing locally analytic characters of $T(\mathbb{Q}_p)$ (e.g. see [21, Prop. 6.4.5]). Moreover, $J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)$ inherits from $\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}$ a continuous action of $\mathcal{H}^p\times\pi_0$ commuting with $T(\mathbb{Q}_p)$. Hence $\mathcal{M}$ is equipped with a natural $\mathcal{O}(\widehat{T})$-linear action of $\mathcal{H}^p\times\pi_0$ such that the isomorphism in (2.3) is $\mathcal{H}^p\times\pi_0$-equivariant.

From $\{\mathcal{M},\widehat{T},\mathcal{H}^p\}$, one can construct as in [17, § 2.3] a rigid analytic space $\mathcal{S}$ over E equipped with a natural finite morphism $\kappa_1:\mathcal{S}\to\widehat{T}$ such that for any admissible affinoid open $U=\mathrm{Spm}\,A\subset\widehat{T}$, we have $\kappa_1^{-1}(\mathcal{U})\cong\mathrm{Spm}\,B$ where B is the finite A-subalgebra of $\mathrm{End}_A(\mathcal{M}(\mathcal{U}))$ generated by $\mathcal{H}^p$ (noting that $\mathcal{M}(\mathcal{U})$ is equipped with an A-linear $\mathcal{H}^p$-action). The rigid space $\mathcal{S}$ is referred to as the eigensurface of tame level K^p. An E-point z of $\mathcal{S}$ can be parametrized as (χ_z,λ_z) where χ_z is a locally analytic character of $T(\mathbb{Q}_p)$ over E, and $\lambda_z:\mathcal{H}^p\to E$ is a system of eigenvalues of $\mathcal{H}^p$. Moreover, such a point (χ_z,λ_z) lies in $\mathcal{S}$ if and only if the corresponding eigenspace

$$J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)[T(\mathbb{Q}_p)=\chi_z,\mathcal{H}^p=\lambda_z]\neq 0.$$

The $\mathcal{O}(\widehat{T})$-module $\mathcal{M}$ has a natural $\mathcal{O}(\mathcal{S})$-action, which makes $\mathcal{M}$ to be a coherent $\mathcal{O}(\mathcal{S})$-module. For any $z=(\chi_z,\lambda_z)\in\mathcal{S}$ with k_z the residue field, we have a natural isomorphism of finite dimensional k_z-vector spaces (cf. [17, Prop. 2.3.3 (iii)])

$$(z^*\mathcal{M})^\vee\cong J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)[T(\mathbb{Q}_p)=\chi_z,\mathcal{H}^p=\lambda_z]. \tag{2.4}$$

Since the action of the center $\mathbb{Q}_p^\times$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widetilde{H}^1_{\text{ét},c}(K^p,E)_{\overline{\rho}}$ is unitary, we easily see

$$\mathrm{val}_p(\chi_z(p))=0 \tag{2.5}$$

if $(\chi_z,\lambda_z)\in\mathcal{S}$. The following definition is standard.

Definition 2.3. *(1) A point $z=(\chi_z,\lambda_z)\in\mathcal{S}$ is called classical if*

$$J_B\big(\widetilde{H}^1_{\acute{e}t,c}(K^p,E)^{\mathrm{lalg}}_{\overline{\rho}}\big)[T(\mathbb{Q}_p)=\chi_z,\mathcal{H}^p=\lambda_z]\neq 0.$$

A point z is called very classical if z is classical and the natural injection

$$J_B\big(\widetilde{H}^1_{\acute{e}t,c}(K^p,E)^{\mathrm{lalg}}_{\overline{\rho}}\big)[T(\mathbb{Q}_p)=\chi_z,\mathcal{H}^p=\lambda_z]\lhook\joinrel\longrightarrow J_B\big(\widetilde{H}^1_{\acute{e}t,c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)[T(\mathbb{Q}_p)=\chi_z,\mathcal{H}^p=\lambda_z]$$

is bijective.

(2) Let $z=(\chi_z,\lambda_z)$ be a point in $\mathcal{S}$ with $\chi_z=(\psi_{z,1}x^{k_1})\otimes(\psi_{z,2}x^{k_2})$ where $\psi_{z,i}$ are smooth characters of $\mathbb{Q}_p^\times$, k_1, $k_2\in\mathbb{Z}$ and $k_2\geq k_2$ (we call such character locally algebraic of dominant weight). We call z of non-critical slope if $\mathrm{val}_p(p\psi_{z,1}(p))<1-k_2$.

We have by [17, Prop. 2.3.6]:

Proposition 2.4. *Let $z=(\chi_z,\lambda_z)\in\mathcal{S}$ with χ_z locally algebraic of dominant weight. If z is of non-critical slope, then z is very classical.*

Using Theorem 2.2 (1) and [16, Prop. 4.2.36], one can actually reformulate the construction of $\mathcal{S}$ using the spectral theory of compact operators (e.g. see [8, Lem. 3.10]). Let $\widehat{T}_0$ be the rigid space over E parametrizing locally analytic characters of $T(\mathbb{Z}_p)$, and we denote by κ the composition

$$\kappa:\mathcal{S}\xrightarrow{\kappa_1}\widehat{T}\longrightarrow\widehat{T}_0.$$

Let $\varpi_1:=\begin{pmatrix}p&0\\0&1\end{pmatrix}$, $\varpi_2:=\begin{pmatrix}p&0\\0&p\end{pmatrix}\in T(\mathbb{Q}_p)$. By the same argument as in the proof of [8, Prop. 3.11], we have:

Proposition 2.5. *There exists an admissible covering $\{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{S}$ by affinoids $\mathcal{U}_i$ such that for all i there exits an open affinoid W_i of $\widehat{T}_0$ such that*

- *the morphism κ induces a finite surjective morphism from each irreducible component of $\mathcal{U}_i$ onto W_i,*
- *$\mathcal{M}(\mathcal{U}_i)$ is a finite projective $\mathcal{O}(W_i)$-module equipped with $\mathcal{O}(W_i)$-linear operators ϖ_1, ϖ_2 and $T\in\mathcal{H}^p$,*
- *$\mathcal{O}(\mathcal{U}_i)$ is isomorphic to a $\mathcal{O}(W_i)$-subalgebra of $\mathrm{End}_{\mathcal{O}(W_i)}(\mathcal{M}(\mathcal{U}_i))$ generated by ϖ_1, ϖ_2 and the operators in $\mathcal{H}^p$.*

We assume that the $\mathcal{H}^p$-action on $\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{lalg}}_{\overline{\rho}}$ is semi-simple (e.g. this holds when $\mathcal{H}^p=\mathcal{H}(K^p)^{\mathrm{sph}}$). We summarize some (well-known) properties of $\mathcal{M}$ and $\mathcal{S}$ in the following theorem.

Theorem 2.6. *(1) The coherent sheaf $\mathcal{M}$ is Cohen-Macauly over $\mathcal{S}$.*

(2) The rigid space $\mathcal{S}$ is equidimensional of dimension 2, and the points of non-critical slope are Zariski-dense in $\mathcal{S}$, and accumulate at points (χ,λ) with χ locally algebraic.

(3) The rigid space $\mathcal{S}$ is reduced.

Proof. (1) follows from Proposition 2.5 and the argument in the proof of [7, Lem. 3.8]. Since $\widehat{T}_0$ is equidimensional of dimension 2, by [9, Prop. 6.4.2] and Proposition 2.5, $\mathcal{S}$ is also equidimensional of dimension 2. The density and the accumulation property of the points of non-critical slope follow from standard arguments as in [9, § 6.4.5]. Finally, since the action of $\mathcal{H}^p$ on $\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{lalg}}_{\overline{\rho}}$ is semi-simple, (3) follows by the same argument as in [10, Prop. 6.4]. □

Remark 2.7. *By Theorem 2.6 (2) and Proposition 2.4, the very classical points are Zariski-dense in $\mathcal{S}$.*

Let $z=(\chi_z=\chi_{z,1}\otimes\chi_{z,2},\lambda_z)\in\mathcal{S}$. Recall that one can associate (by the theory of pseudo-characters) to z a semi-simple continuous representation $\rho_z:\mathrm{Gal}_{\mathbb{Q}}\to\mathrm{GL}_2(k_z)$ (where k_z denotes the residue field of z, which is a finite extension of E) satisfying that

1. the mod p reduction of ρ_z is isomorphic to $\overline{\rho}$,
2. the restriction $\rho_{z,\ell}:=\rho_z|_{\mathrm{Gal}_{\mathbb{Q}_\ell}}$ is unramified for all $\ell\notin\Sigma(K^p)$, and

$$\rho_z(\mathrm{Frob}_\ell^{-2})-\lambda_z(T_\ell)\rho_z(\mathrm{Frob}_\ell^{-1})+\ell\lambda_z(S_\ell)=0.$$

Note that the first property together with our assumption on $\overline{\rho}$ actually imply that ρ_z is absolutely irreducible. Note also that ρ_z is determined by the second property by Chebotarev's density theorem. Let $z=(\chi_z,\lambda_z)\in\mathcal{S}$ be such that $\mathrm{wt}(z):=\mathrm{wt}(\chi_{z,1})-\mathrm{wt}(\chi_{z,2})\in\mathbb{Z}_{\geq 0}$, we put

$$\chi_z^c:=\chi_z(x^{-\mathrm{wt}(z)-1}\otimes x^{\mathrm{wt}(z)+1}).$$

If $z^c:=(\chi_z^c,\lambda_z)\in\mathcal{S}$, we call z^c a *companion point* of z. Suppose z is classical, then we call z *critical* if z admits a companion point. By [17, Prop. 4.5.5] (and the proof), we have (noting that the bad points in *loc. cit.* are exactly the points admitting companion points in our terminology, see [17, Def. 4.5.4])

Proposition 2.8. *If z is of non-critical slope, then z is not critical.*

Theorem 2.9 ([6])**.** *The point z is critical if and only if the Galois representation $\rho_{z,p}:=\rho_z|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ splits.*

We recall the relation between Emerton's eigensurface $\mathcal{S}$ and Coleman-Mazur eigencurve. Let $\mathcal{W}$ denote the rigid space parameterizing locally analytic characters of $\mathbb{Z}_p^\times$, the decomposition

$$T(\mathbb{Q}_p)\cong\begin{pmatrix}\mathbb{Z}_p^\times & 0\\ 0 & 1\end{pmatrix}\times\begin{pmatrix}1 & 0\\ 0 & \mathbb{Z}_p^\times\end{pmatrix}\times\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}^{\mathbb{Z}}\times\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}^{\mathbb{Z}} \tag{2.6}$$

induces $\widehat{T}\cong\mathcal{W}_+\times\mathcal{W}_-\times\mathbb{G}_m\times\mathbb{G}_m$ with $\mathcal{W}_\pm\cong\mathcal{W}$. The trivial character on $\mathcal{W}_+$ induces an injection

$$\mathcal{W}_-\times\mathbb{G}_m\times\mathbb{G}_m\longhookrightarrow\widehat{T}.$$

Denote by $\mathcal{C}$ the pull-back of $\mathcal{S}$ over $\mathcal{W}_-\times\mathbb{G}_m\times\mathbb{G}_m$, by κ the induced map $\mathcal{C}\to\mathcal{W}_-\cong\mathcal{W}$. We still use $\mathcal{M}$ to denote the pull-back of the $\mathcal{O}(\mathcal{S})$-module $\mathcal{M}$ over $\mathcal{C}$. We have

$$\mathcal{M}(\mathcal{C})\cong\big(J_B(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}})^{T_1}\big)_b^\vee \tag{2.7}$$

where $T_1=\begin{pmatrix}\mathbb{Z}_p^\times & 0\\ 0 & 1\end{pmatrix}$. By [17, Prop. 4.4.6], we have $\mathcal{S}\cong\mathcal{C}\times\mathcal{W}$. Moreover, one can show that Proposition 2.5, Theorem 2.6 and Remark 2.7 also hold for $(\mathcal{C},\mathcal{M})$ (except that $\mathcal{C}$ is equidimensional of dimension 1). By [17, Prop. 4.4.2], we have an explicit relation between the classical points of $\mathcal{C}$ and those of Coleman-Mazur eigencurve (constructed from the finite slope overconvergent modular forms). Using the reducedness of $\mathcal{C}$, the density of (very) classical points and the same argument as in [2, Prop. 7.2.8], one can deduce that $\mathcal{C}$ is isomorphic to the corresponding Coleman-Mazur eigencurve. Finally we have by [17, Prop. 4.5.5]:

Proposition 2.10. *Let $z=(\chi_{z,1}\otimes\chi_{z,2},\lambda_z)\in\mathcal{C}$, then there exists an admissible open $U\subset\mathcal{S}$ containing z such that any point in $U\setminus\{z\}$ does not admit companion points in $\mathcal{S}$.*

3 Adjunctions

In this section, we study some adjunction properties of the Jacquet-Emerton functor. Let $z=(\chi_z=\chi_{z,1}\otimes\chi_{z,2},\lambda_z)$ be an E-point of $\mathcal{C}$. By Proposition 2.5 (with $\mathcal{S}$ replaced by $\mathcal{C}$ as discussed below Theorem 2.9) and Proposition 2.10, there exists an affinoid neighborhood $\mathcal{U}_0$ of z in $\mathcal{C}$ such that

- $\kappa:\mathcal{U}_0\to\kappa(\mathcal{U}_0)$ is a finite morphism of affinoids;

- $\mathcal{M}(\mathcal{U}_0)$ is finitely generated locally free over $\mathcal{O}(\kappa(\mathcal{U}_0))$;
- any $z' \in \mathcal{U}_0(\overline{E}) \setminus \{z\}$ does not admit companion points;
- $\kappa^{-1}(\kappa(z))^{\mathrm{red}} = \{z\}$.

Let $\mathcal{U} := \kappa^{-1}(\mathcal{V})$ with $\mathcal{V}$ an admissible strictly quasi-Stein neighborhood of $\kappa(z)$ in $\kappa(\mathcal{U}_0)$ (cf. [21, Def. 2.1.17 (iv)]. Thus $\mathcal{U}$ also satisfies the above listed properties. Since $\mathcal{M}(\mathcal{U})$ is a finitely generated locally free $\mathcal{O}(\mathcal{V})$-module and $\mathcal{V}$ is strictly quasi-Stein, we have that $\mathcal{M}(\mathcal{U})_b^\vee$ is an *allowable* locally analytic representation of $T(\mathbb{Q}_p)$ in the sense of [18, Def. 0.11] (e.g. using similar arguments as in [14, Ex. 6.3.15 (ii)]). Note also $\mathcal{M}(\mathcal{U})_b^\vee$ is naturally equipped with a continuous action of $\mathcal{H}^p$. The restriction maps $\mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{U})$ induce $\mathcal{H}^p$-invariant morphisms of locally analytic $T(\mathbb{Q}_p)$-representations:

$$\mathcal{M}(\mathcal{U})_b^\vee \longrightarrow \mathcal{M}(\mathcal{C})_b^\vee \longrightarrow \mathcal{M}(\mathcal{S})_b^\vee \cong J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big). \tag{3.1}$$

When z does not have companion points (hence all points in $\mathcal{U}$ do not admit companion points by assumption), one can prove as in [17, Lem. 4.5.12] that the composition in (3.1) is *balanced* in the sense of [18, Def. 0.8] and induces an $\mathcal{H}^p$-invariant morphism of locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations

$$\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}\big)^{\mathrm{an}} \longrightarrow \widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}, \tag{3.2}$$

where $\delta_B := \mathrm{unr}(p^{-1}) \otimes \mathrm{unr}(p)$ is the modulus character of $B(\mathbb{Q}_p)$ (which we also view as a character of $\overline{B}(\mathbb{Q}_p)$ via $\overline{B}(\mathbb{Q}_p) \twoheadrightarrow T(\mathbb{Q}_p)$). The morphism (3.2) plays a crucial role in Emerton's construction of two-variable p-adic L-functions (cf. [17, Thm. 4.5.7]). However, if z admits a companion point (e.g. if z is critical), then (3.1) does not (directly) induce a such morphism (e.g. see Proposition 3.3 below). The main result in this section is to establish a similar adjunction result in the locus of such z.

Suppose henceforward $k := \mathrm{wt}(\chi_{z,1}) - \mathrm{wt}(\chi_{z,2}) \in \mathbb{Z}_{\geq 0}$ (note $\mathrm{wt}(\chi_{z,1}) = 0$ by (2.7)), and let $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{t}$. We have a natural injection $\mathfrak{t} \hookrightarrow \mathcal{O}(\widehat{T}_0)$, and we view hence h as an element in $\mathcal{O}(\widehat{T}_0)$. By (2.7), the $\mathfrak{t}$-action on $\mathcal{M}(\mathcal{U})_b^\vee$ factors through the projection $\mathfrak{t} = \begin{pmatrix} \mathbb{Q}_p & 0 \\ 0 & \mathbb{Q}_p \end{pmatrix} \twoheadrightarrow \mathbb{Q}_p$ onto the bottom right factor. Shrinking $\mathcal{V}$ (and hence $\mathcal{U}$), we assume $\kappa(z)$ is the only point in $\mathcal{V}$ such that the associated character of $\mathbb{Z}_p^\times$ is of weight $-k$ (noting $\kappa(z) = \chi_{z,2}$). Consider $\mathcal{M}(\mathcal{U})_b^\vee[h = k]$ which is a finite dimensional E-vector space, and is a $T(\mathbb{Q}_p) \times \mathcal{H}^p$-stable subspace of $\mathcal{M}(\mathcal{U})_b^\vee$. By assumption, we have in fact:

$$\mathcal{M}(\mathcal{U})_b^\vee[h = k] = J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)[\mathfrak{t} = d\chi_z]\{T(\mathbb{Q}_p) = \chi_z, \mathcal{H}^p = \lambda_z\}. \tag{3.3}$$

We call a vector $v \in J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)$ *classical* if v lies in $J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{lalg}}_{\overline{\rho}}\big)$ (compare with Definition 2.3), and we call a vector $v \in \mathcal{M}(\mathcal{U})_b^\vee$ classical if it is sent to a classical vector in $J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)$ via the map (3.1).

Lemma 3.1. *Let $\gamma \in \mathrm{End}_{T(\mathbb{Q}_p)}\big(\mathcal{M}(\mathcal{U})_b^\vee\big)$, the composition*

$$\mathcal{M}(\mathcal{U})_b^\vee \xrightarrow{\gamma} \mathcal{M}(\mathcal{U})_b^\vee \xrightarrow{(3.1)} J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big) \tag{3.4}$$

is balanced if and only if any vector in $\gamma(\mathcal{M}(\mathcal{U})_b^\vee[h = k])$ is classical.

Proof. By definition and the same argument as in the proof of [17, Lem. 4.5.12], (3.4) is balanced if and only if for any $k' \in \mathbb{Z}_{\geq 0}$, the image of the composition

$$\mathcal{M}(\mathcal{U})_b^\vee[h = k'] \xrightarrow{(3.4)} J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big) \longrightarrow (\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}})^{N_0} \hookrightarrow \widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}} \tag{3.5}$$

is annihilated by the operator $X_-^{k'+1}$ where $X_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{U}(\mathfrak{g})$ naturally acts on the locally analytic representation $\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}$, $N_0 = N(\mathbb{Z}_p) = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$, and where the second map in (3.5) is the "canonical lifting" of [18, (3.4.8)] (with respect to N_0), equivariant under the action of

$$T(\mathbb{Q}_p)^+ := \Big\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in T(\mathbb{Q}_p) \mid a/d \in \mathbb{Z}_p \setminus \{0\} \Big\}.$$

One can show as in the proof of [17, Lem. 4.5.12] that if $k' \neq k$, then any vector v in the image of the following composition (which is obtained in the same way as (3.5) replacing (3.4) by (3.1))

$$\mathcal{M}(\mathcal{U})_b^\vee[h=k'] \xrightarrow{(3.1)} J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\text{an}}_{\overline{\rho}}\big) \longrightarrow \widetilde{H}^1_{\text{ét},c}(K^p,E)^{\text{an}}_{\overline{\rho}} \tag{3.6}$$

is annihilated by $X_-^{k'+1}$. Indeed, using the fact that $\mathcal{M}(\mathcal{U})_b^\vee[h=k']$ is finite dimensional (and $T(\mathbb{Q}_p)\times\mathcal{H}^p$-stable), we can assume without loss of generality that v is a (χ',λ')-eigenvector for $T(\mathbb{Q}_p)^+\times\mathcal{H}^p$ with $(\chi',\lambda')\in\mathcal{U}$. In this case, by [14, Lem. 7.3.15] (which is an easy variation of [16, Prop. 4.4.4]), $X_-^{k'+1}\cdot v\in\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\text{an}}_{\overline{\rho}}\big)^{N_0}$ is a generalized $((\chi')^c,\lambda')$-eigenvector for $T(\mathbb{Q}_p)^+\times\mathcal{H}^p$. If $X_-^{k'+1}\cdot v\neq 0$, we deduce $\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\text{an}}_{\overline{\rho}}\big)^{N_0}\{T(\mathbb{Q}_p)^+=(\chi')^c,\mathcal{H}^p=\lambda'\}\neq 0$, and hence (see [16, Prop. 3.4.9] for the first equality):

$$J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\text{an}}_{\overline{\rho}}\big)[T(\mathbb{Q}_p)=(\chi')^c,\mathcal{H}^p=\lambda']=\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\text{an}}_{\overline{\rho}}\big)^{N_0}[T(\mathbb{Q}_p)^+=(\chi')^c,\mathcal{H}^p=\lambda']\neq 0.$$

But this implies $((\chi')^c,\lambda')\in\mathcal{S}$, contradicting the fact (χ',λ') does not admit any companion point. Since (3.5) factors through (3.6), we deduce the image of (3.5) for $k'\neq k$ is also annihilated by $X_-^{k'+1}$.

Now it suffices to show that for $k'=k$ and a vector v in the image of (3.5), $X_-^{k+1}\cdot v=0$ if and only v is classical. Using the fact v is fixed by N_0, $\mathfrak{t}v=d\chi_z v$ and the highest weight theory, it is not difficult to see that that followings are equivalent

- v is classical,
- $\mathrm{U}(\mathfrak{g})v\cong(\mathrm{Sym}^k E^2)^\vee$,
- $X_-^{k+1}\cdot v=0$.

The lemma then follows. ☐

Recall that in [18, (2.8)], Emerton introduced a closed $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation

$$I_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathcal{M}(\mathcal{U})_b^\vee\otimes_E\delta_B^{-1}\big)\longhookrightarrow\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathcal{M}(\mathcal{U})_b^\vee\otimes_E\delta_B^{-1}\big)^{\text{an}}, \tag{3.7}$$

which, by [18, Lem. 2.8.3], is the closed $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation of

$$\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathcal{M}(\mathcal{U})_b^\vee\otimes_E\delta_B^{-1}\big)^{\text{an}}$$

generated by $\mathcal{M}(\mathcal{U})_b^\vee$ via the following composition (see [18, Lem. 0.3] for the first map, and [16, (3.4.8)] for the second map)

$$\mathcal{M}(\mathcal{U})_b^\vee\longhookrightarrow J_B\Big(\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathcal{M}(\mathcal{U})_b^\vee\otimes_E\delta_B^{-1}\big)^{\text{an}}\Big)\longrightarrow\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathcal{M}(\mathcal{U})_b^\vee\otimes_E\delta_B^{-1}\big)^{\text{an}}.$$

However, by the assumption that $\mathcal{M}(\mathcal{U})$ is locally free over $\mathcal{O}(\kappa(\mathcal{U}))$, one can prove as in [17, Lem. 4.5.12 (ii)] that (3.7) is an isomorphism. The following theorem thus follows from Emerton's adjunction formula [18, Thm. 0.13] combined with Lemma 3.1.

Theorem 3.2. *Keep the notation of Lemma 3.1, and suppose γ commutes with $\mathcal{H}^p$ (so that(3.4) is $\mathcal{H}^p$-equivariant), then the followings are equivalent:*

(i) there exists an $\mathcal{H}^p$-equivariant morphism of locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations

$$\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathcal{M}(\mathcal{U})_b^\vee\otimes_E\delta_B^{-1}\big)^{\text{an}}\longrightarrow\widetilde{H}^1_{ét,c}(K^p,E)^{\text{an}}_{\overline{\rho}}$$

such that the induced morphism

$$\mathcal{M}(\mathcal{U})_b^\vee\longhookrightarrow J_B\Big(\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathcal{M}(\mathcal{U})_b^\vee\otimes_E\delta_B^{-1}\big)^{\text{an}}\Big)\longrightarrow J_B\big(\widetilde{H}^1_{ét,c}(K^p,E)^{\text{an}}_{\overline{\rho}}\big) \tag{3.8}$$

is equal to (3.4);

(ii) any vector in $\gamma(\mathcal{M}(\mathcal{U})_b^\vee[h=k])$ is classical.

Let $\mathcal{M}(\mathcal{U})_b^\vee[h=k]^{\mathrm{cl}} \subseteq \mathcal{M}(\mathcal{U})_b^\vee[h=k]$ be the subspace of classical vectors. Recall we have $\mathcal{M}(\mathcal{U})_b^\vee[h=k] \cong (\mathcal{M}(\mathcal{U})/(h-k))^\vee$ on which the $\mathcal{O}(\mathcal{U})$-action is determined by the action of $T(\mathbb{Q}_p) \times \mathcal{H}^p$. By definition (see also (3.3)), one easily verifies that $\mathcal{M}(\mathcal{U})_b^\vee[h=k]^{\mathrm{cl}}$ is stable under the action of $T(\mathbb{Q}_p)$ and $\mathcal{H}^p$. Hence $\mathcal{M}(\mathcal{U})_b^\vee[h=k]^{\mathrm{cl}}$ is stable under the $\mathcal{O}(\mathcal{U})$-action. Put

$$\mathcal{I}_z := \{\gamma \in \mathcal{O}(\mathcal{U}) \mid \gamma(\mathcal{M}(\mathcal{U})_b^\vee[h=k]) \subseteq \mathcal{M}(\mathcal{U})_b^\vee[h=k]^{\mathrm{cl}}\}. \tag{3.9}$$

We see $\mathcal{I}_z$ is either an ideal of $\mathcal{O}(\mathcal{U})$ of $\mathcal{I}_z = \mathcal{O}(\mathcal{U})$. It is also clear that $h-k \in \mathcal{I}_z$.

Proposition 3.3. *Suppose χ_z is a product of an algebraic character and an unramified character, then $\mathcal{I}_z = \mathcal{O}(\mathcal{U})$ if and only if z does not admit companion points.*

Proof. If z does not have companion points, by the same argument as in the proof of Lemma 3.1, we see any vector in $\mathcal{M}(\mathcal{U})_b^\vee[h=k]$ is classical, hence $\mathcal{I}_z = \mathcal{O}(\mathcal{U})$.

Suppose now $\mathcal{I}_z = \mathcal{O}(\mathcal{U})$. By Theorem 3.2, any vector in $\mathcal{M}(\mathcal{U})_b^\vee[h=k]$ is classical. By (3.3), any vector in $J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an}}_{\overline{\rho}}\big)[h=k]\{T(\mathbb{Q}_p)=\chi_z, \mathcal{H}^p=\lambda_z\}$ is classical (in particular, the point z is classical). Assume first $\chi_z = \chi_{z,1} \otimes \chi_{z,2}$ satisfies $\chi_{z,1}\chi_{z,2}^{-1} \neq \mathrm{unr}(p^{-1})x^k$. In this case, by similar (and easier) argument as in the proof of [13, Thm. 7] (using Chenevier's method [11, § 4.4]), we can deduce that $\mathcal{C}$ is étale over $\mathcal{W}$ at the point z. However, by [3, Thm. 1.1], if z is critical, then $\mathcal{C}$ will be ramified over $\mathcal{W}$ at z, that leads to a contradiction. Now assume $\chi_{z,1}\chi_{z,2}^{-1} = \mathrm{unr}(p^{-1})x^k$. Together with (2.5), we easily deduce $\mathrm{val}_p(\chi_{z,1}(p)) = (k-1)/2$. Since $z \in \mathcal{C}$, the character $\chi_{z,1}$ is actually smooth. By Definition 2.3 (2), the point z is of non-critical slope, and hence non-critical by Proposition 2.8. This concludes the proof. □

Under the π_0-action, the coherent sheaf $\mathcal{M}$ naturally decomposes into $\mathcal{M}^+ \oplus \mathcal{M}^-$. We deduce that $\mathcal{M}^\pm(\mathcal{U})$ are both locally free over $\mathcal{O}(\kappa(\mathcal{U}))$. Moreover by considering their fibers at classical points, we can see both of them are non-zero. The precedent results also hold for $\{\mathcal{M}^\pm(\mathcal{U}), \widetilde{H}^1_{\text{ét},c}(K^p,E)^{\mathrm{an},\pm}_{\overline{\rho}}\}$. In particular, we can define ideals $\mathcal{I}_z^\pm$ in a similar way, and we can prove that the same statement in Proposition 3.3 holds with $\mathcal{I}_z$ replaced by $\mathcal{I}_z^\pm$.

4 Two-variable p-adic L-functions

We use the results in § 3 to construct two-variable p-adic L-functions in a neighborhood of critical points in $\mathcal{C}$.

4.1 Constructions

Let $N > 1$, $p \nmid N$, and let

$$K^p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p) \;\middle|\; c \equiv 0 \pmod{N},\ d \equiv 1 \pmod{N} \right\}.$$

In this section, we let $\mathcal{H}^p$ denote the $\mathcal{O}_E$-algebra generated by T_ℓ, S_ℓ for $\ell \nmid N$, $\ell \neq p$, and the diamond operators $\langle a \rangle$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$. Let $k \in \mathbb{Z}_{\geq 0}$, f be a newform of weight $k+2$, level $\Gamma_1(N)$ over E, i.e. f is an eigenform for $\mathcal{H}^p$ and T_p, S_p, such that there does not exist an eigenform (for $\mathcal{H}^p$) of weight $k+2$, of level $\Gamma_1(N')$ with N' is a proper divisor of N, having the same eigenvalue as f for the operators in $\mathcal{H}^p$. Let ρ_f be the associated $\mathrm{Gal}_{\mathbb{Q}}$-representation over E (enlarge E if necessary), and we assume that the mod ϖ_E reduction of ρ_f is isomorphic to $\overline{\rho}$. Let $\mathcal{C}$, $\mathcal{M}$ be as in § 2.2. Let $\epsilon : (Z/NZ)^\times \to E^\times$ be the character with $\langle a \rangle f = \epsilon(a) f$. Let $a_p, b_p \in E$ (enlarge E if necessary) be the eigenvalues of T_p, S_p of f respectively. Let α be a root of $X^2 - a_p X + p b_p = 0$, and put

$$f_\alpha(z) = f(z) - \frac{p b_p}{\alpha} f(pz),$$

which is a modular form of weight $k+2$, of level $\Gamma_1(N) \cap \Gamma_0(p)$, and is an eigenform of the same eigenvalues of f for $\mathcal{H}^p$, an eigenform for U_p of eigenvalues α. The form f_α is referred to as a *refinement* (or a *p-stabilization*) of f. By [17, Prop. 4.4.2], f_α corresponds to a classical point $z_\alpha = (\chi_{z_\alpha}, \lambda_f) \in \mathcal{C} \subset \mathcal{S}$

where λ_f denotes the system of eigenvalues of f for operators in $\mathcal{H}^p$, and where

$$\chi_{z_\alpha} = \mathrm{unr}(\alpha/p) \otimes x^{-k}\, \mathrm{unr}(pb_p/\alpha).$$

In this section, we construct two-variable p-adic L-functions in a neighborhood of z_α (especially in the case where z_α is critical), via Emerton's method [15][17].

Proposition 4.1. *$\mathcal{M}^{\pm}$ are locally free of rank 1 in a neighborhood of z_α in $\mathcal{C}$.*

Proof. Suppose first $\alpha \neq pb_p/\alpha$ (i.e. $X^2 - a_pX + pb_p = 0$ has two distinct roots). By [1, § 2.3], the eigencurve $\mathcal{C}$ is smooth at the classical point z_α. Since $\mathcal{M}$ is Cohen-Macauly (see Theorem 2.6 (1) and the discussion below (2.7)), by [22, Cor. 17.3.5 (i)], $\mathcal{M}$ is locally free around z_α. Hence $\mathcal{M}^{\pm}$ are also locally free around z_α. For any classical point $z' = (\chi_{z'}, \lambda_{z'})$ of non-critical slope in $\mathcal{C}$, we have

$$\begin{aligned}\dim_{k_{z'}} (z')^* \mathcal{M}^{\pm} &= \dim_{k_{z'}} J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)^{\text{an},\pm}_{\overline{\rho}}\big)[T(\mathbb{Q}_p) = \chi_{z'}, \mathcal{H}^p = \lambda_{z'}] \\ &= \dim_{k_{z'}} J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)^{\text{lalg},\pm}_{\overline{\rho}}\big)[T(\mathbb{Q}_p) = \chi_{z'}, \mathcal{H}^p = \lambda_{z'}] = 1. \quad (4.1)\end{aligned}$$

where the first equality follows from (2.4), then second follows from Proposition 2.4, and the last equality follows from the multiplicity one result (see [17, Prop. 4.4.18]). Since such points accumulate at z_α (see Theorem 2.6 (2) and the discussion below (2.7)), we deduce that $\mathcal{M}^{\pm}$ are locally free of rank 1 in a neighborhood of z_α in $\mathcal{C}$. If $\alpha = pb_p/\alpha$, as in the proof of Proposition 3.3, we know z_α is of non-critical slope. The proposition in this case then follows from [17, Prop. 4.4.20] (and the proof). □

Let $\mathcal{U}$ be a neighborhood of z_α in $\mathcal{C}$ as in § 3. If $\alpha \neq pb_p/\alpha$, we shrink $\mathcal{U}$ such that the maximal ideal $\mathfrak{m}_{z_\alpha}$ associated to z_α is generated by one element $r_{z_\alpha} \in \mathcal{O}(\mathcal{U})$ (using the fact $\mathcal{U}$ is one-dimensional and smooth at the point z_α). We let e be the ramification degree of $\mathcal{C}$ over $\mathcal{W}$ at z_α in this case. If $\alpha = pb_p/\alpha$, we put $r_{z_\alpha} = 1$ and $e = 1$.

Theorem 4.2. *The composition (that is $\mathcal{H}^p \times T(\mathbb{Q}_p)$-equivariant)*

$$\mathcal{M}^{\pm}(\mathcal{U})^{\vee}_b \xrightarrow{r_{z_\alpha}^{e-1}} \mathcal{M}^{\pm}(\mathcal{U})^{\vee}_b \longrightarrow J_B\big(\widetilde{H}^1_{\acute{e}t,c}(K^p, E)^{\text{an}}_{\overline{\rho}}\big) \quad (4.2)$$

induces an $\mathcal{H}^p$-equivariant morphism of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

$$\iota^{\pm} : \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}^{\pm}(\mathcal{U})^{\vee}_b \otimes_E \delta_B^{-1}\big)^{\text{an}} \longrightarrow \widetilde{H}^1_{\acute{e}t,c}(K^p, E)^{\text{an},\pm}_{\overline{\rho}}. \quad (4.3)$$

Proof. If $\alpha = pb_p/\alpha$, then z_α is of non-critical slope (see the proof of Proposition 3.3), and hence non-critical by Proposition 2.8. By Proposition 3.3 (and the discussion that follows), we see $\mathcal{I}_z^{\pm} = \mathcal{O}(\mathcal{U})$. The theorem then follows from Theorem 3.2.

We suppose $\alpha \neq pb_p/\alpha$. We prove first

$$\mathcal{M}^{\pm}(\mathcal{U})^{\vee}_b[r_{z_\alpha}] = \mathcal{M}^{\pm}(\mathcal{U})^{\vee}_b[h = k]^{\text{cl}}. \quad (4.4)$$

We have

$$\begin{aligned}\mathcal{M}^{\pm}(\mathcal{U})^{\vee}_b[h = k]^{\text{cl}} &= J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)^{\text{lalg},\pm}_{\overline{\rho}}\big)[\mathfrak{t} = d\chi_z]\{T(\mathbb{Q}_p) = \chi_z, \mathcal{H}^p = \lambda_z\} \\ &= J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)^{\text{lalg},\pm}_{\overline{\rho}}\big)[T(\mathbb{Q}_p) = \chi_z, \mathcal{H}^p = \lambda_z] \quad (4.5)\end{aligned}$$

where the first equality follows from (3.3), and the second from the assumption $\alpha \neq pb_p/\alpha$ and the semi-simplicity of the $\mathcal{H}^p$-action on $\widetilde{H}^1_{\text{ét},c}(K^p, E)^{\text{lalg}}_{\overline{\rho}}$ (using the classical fact that the $\mathcal{H}^p$-action on the right hand side of (2.2) is semi-simple). By (the proof of) [17, Prop. 4.4.18], the E-vector space in (4.5) is one dimensional. It is clear that we have an injection

$$\begin{aligned}&J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)^{\text{lalg},\pm}_{\overline{\rho}}\big)[T(\mathbb{Q}_p) = \chi_z, \mathcal{H}^p = \lambda_z] \\ &\quad\hookrightarrow J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)^{\text{an},\pm}_{\overline{\rho}}\big)[T(\mathbb{Q}_p) = \chi_z, \mathcal{H}^p = \lambda_z] \cong (z_\alpha^* \mathcal{M}^{\pm})^{\vee} = \mathcal{M}^{\pm}(\mathcal{U})^{\vee}_b[r_{z_\alpha}]. \quad (4.6)\end{aligned}$$

By Proposition 4.1, $\dim_E \mathcal{M}^{\pm}(\mathcal{U})_b^\vee[r_{z_\alpha}] = 1$. We see (4.6) is actually bijective. Combined with (4.5), (4.4) follows.

We have $(h-k)\mathcal{O}(\mathcal{U}) = r_{z_\alpha}^e \mathcal{O}(\mathcal{U})$. By Proposition 4.1, we have $\mathcal{M}^{\pm}(\mathcal{U})_b^\vee[h=k] \cong (\mathcal{O}(\mathcal{U})/(h-k))^\vee$ and by (4.4) $\mathcal{M}^{\pm}(\mathcal{U})_b^\vee[h=k]^{\mathrm{cl}} \cong (\mathcal{O}(\mathcal{U})/r_{z_\alpha})^\vee$. We deduce (see (3.9) and the discussion at the end of § 3)

$$\mathcal{I}_z = \mathcal{I}_z^{\pm} = (\frac{h-k}{r_{z_\alpha}})\mathcal{O}(\mathcal{U}) = r_{z_\alpha}^{e-1}\mathcal{O}(\mathcal{U}). \tag{4.7}$$

The theorem follows then from Theorem 3.2. □

We have $\mathcal{M}^{\pm}(\mathcal{U})_b^\vee[r_{z_\alpha}] \cong \chi_{z_\alpha}$ as $T(\mathbb{Q}_p)$-representation, thus (4.3) induces a morphism

$$\big(\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}} \longrightarrow \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}[\mathcal{H}^p = \lambda_f]. \tag{4.8}$$

The locally analytic representation $(\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1})^{\mathrm{an}}$ sits in a non-splitting exact sequence (e.g. see [4, Thm. 4.1])

$$0 \longrightarrow (\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{sm}}\delta_B^{-1})^\infty \otimes_E (\operatorname{Sym}^k E^2)^\vee \longrightarrow (\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1})^{\mathrm{an}} \longrightarrow (\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{c}}\delta_B^{-1})^{\mathrm{an}} \to 0, \tag{4.9}$$

where $\chi_{z_\alpha}^{\mathrm{sm}} := \mathrm{unr}(\alpha/p)\otimes\mathrm{unr}(pb_p/\alpha)$ is the "smooth" part of χ_{z_α}, $(\operatorname{Ind} -)^\infty$ denotes the smooth induction, and recall $\chi_{z_\alpha}^{\mathrm{c}} = \chi_{z_\alpha}(x^{1-k}\otimes x^{k-1})$.

Proposition 4.3. *The map (4.8) is non-zero, and its restriction on the locally algebraic vectors is zero if and only if z_α is critical.*

Proof. We first prove the second part of the proposition. Put

$$I(\chi_{z_\alpha}\delta_B^{-1}) := (\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{sm}}\delta_B^{-1})^\infty \otimes_E (\operatorname{Sym}^k E^2)^\vee$$

which is in fact the locally algebraic subrepresentation of $\big(\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}}$ (cf. (4.9)). Applying the (left exact) Jacquet-Emerton functor to the composition

$$I(\chi_{z_\alpha}\delta_B^{-1}) \hookrightarrow \big(\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}} \xrightarrow{(4.8)} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}[\mathcal{H}^p = \lambda_f], \tag{4.10}$$

one gets an injection of locally analytic representations of $T(\mathbb{Q}_p)$ (see [18, Lem. 0.3]):

$$\chi_{z_\alpha} \hookrightarrow J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}\big)[\mathcal{H}^p = \lambda_f]. \tag{4.11}$$

We have actually a commutative diagram (see (3.8) for the bottom maps)

$$\begin{array}{ccccc}
\chi_{z_\alpha} & \longrightarrow & J_B\big(\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}} & \longrightarrow & J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}\big)[\mathcal{H}^p = \lambda_f] \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}(\mathcal{U})_b^\vee & \longrightarrow & J_B\big(\big(\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}\big)^{\mathrm{an}}\big) & \longrightarrow & J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an}}\big)
\end{array}$$

Using Theorem 3.2, we see (4.11) is equal to the composition

$$\chi_{z_\alpha} \cong \mathcal{M}^{\pm}(\mathcal{U})_b^\vee[r_{z_\alpha}] \hookrightarrow \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \xrightarrow{(4.2)} J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}\big).$$

It is straightforward to see the above map is non-zero if and only if $e = 1$, which is equivalent to that z_α is non-critical by Proposition 3.3. The second part follows.

Now we prove (4.8) is non-zero. The non-critical case is an easy consequence of the second part proved above. Now assume z_α is critical (in particular $\alpha \neq pb_p/\alpha$). If (4.8) is zero, we see the morphism

$$(\operatorname{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1})^{\mathrm{an}} \longrightarrow \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}$$

factors as $\Big($noting the kernel of the first map of the following composition is

$$(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[r_{z_\alpha}] \otimes_E \delta_B^{-1})^{\mathrm{an}} \cong \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}}\Big):$$

$$(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee} \otimes_E \delta_B^{-1})^{\mathrm{an}} \xrightarrow{r_{z_\alpha}} (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee} \otimes_E \delta_B^{-1})^{\mathrm{an}} \longrightarrow \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm},$$

where the second morphism is also $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathcal{H}^p$-equivariant. Applying the Jacquet-Emerton functor, we deduce (4.2) can factor as

$$\mathcal{M}^{\pm}(\mathcal{U})_b^{\vee} \xrightarrow{r_{z_\alpha}} \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee} \xrightarrow{\iota'} J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}\big), \tag{4.12}$$

where ι' is $T(\mathbb{Q}_p)\times\mathcal{H}^p$-equivariant. By [18, Thm. 0.13], the map ι' is balanced. Using the same argument as in the proof of Lemma 3.1, we see that any vector in $\iota'(\mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[h=k])$ is classical. The composition

$$r_{z_\alpha} \circ \iota' : \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[h=k] \longrightarrow \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[h=k] \tag{4.13}$$

is equal to $r_{z_\alpha}^{e-1}$ (since the two maps (4.12) and (4.2) are equal, and their restriction on $\mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[h=k]$ is injective using (3.3)). However, $\iota'(\mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[h=k]) = \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[h=k]^{\mathrm{cl}} = \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[r_{z_\alpha}]$ and hence the morphism (4.13) is zero, which contradicts $r_{z_\alpha}^{e-1} \neq 0$ on $\mathcal{M}^{\pm}(\mathcal{U})_b^{\vee}[h=k] \cong (\mathcal{O}(\mathcal{U})/r_{z_\alpha}^{e})^{\vee}$. The proposition follows. □

Remark 4.4. *If z_α is not critical then (4.8) is injective, and its restriction on the locally algebraic subrepresentations is given by the local-global compatibility in classical local Langlands correspondence. If z_α is critical then (4.8) factors as*

$$\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}} \twoheadrightarrow (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{c}}\delta_B^{-1})^{\mathrm{an}} \hookrightarrow \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}[\mathcal{H}^p = \lambda_f]. \tag{4.14}$$

The existence of the second injection was proved in [6], which is an important fact on local-global compatibility in p-adic local Langlands program in critical case.

Let D^0 be the $\mathcal{O}_E$-module of degree zero divisors in $\mathbb{P}^1(\mathbb{Q})$. Recall that by [15, Prop. 4.2], one has the following pairing which interpolates the classical modular symbols

$$\widetilde{H}^1_{\text{ét},c}(K^p, \mathcal{O}_E)_{\overline{\rho}} \times D^0 \longrightarrow \mathcal{O}_E.$$

Evaluating at $\{\infty\} - \{0\}$ gives a continuous E-linear map (of norm less than 1)

$$\{\infty\} - \{0\} : \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}} \longrightarrow E. \tag{4.15}$$

We fix an isomorphism of $\mathcal{O}(\mathcal{U})$-modules $\mathcal{M}^{\pm}(\mathcal{U}) \cong \mathcal{O}(\mathcal{U})$. The following composition

$$\begin{aligned}\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, E)\widehat{\otimes}_E\mathcal{O}(\mathcal{U})_b^{\vee} \cong \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, \mathcal{O}(\mathcal{U})_b^{\vee}) &\hookrightarrow \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{O}(\mathcal{U})_b^{\vee} \otimes_E \delta_B^{-1}\big)^{\mathrm{an}} \\ &\xrightarrow{\sim} \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}^{\pm}(\mathcal{U})_b^{\vee} \otimes_E \delta_B^{-1}\big)^{\mathrm{an}} \xrightarrow{\iota^{\pm}} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\pm,\mathrm{an}} \xrightarrow{\{\infty\}-\{0\}} E,\end{aligned} \tag{4.16}$$

thus gives a global section $L^{\pm}$ of $\mathcal{U} \times \mathcal{W}$, where the second map sends F to $I(F)$ with $I(F)$ supported in $\overline{B}(\mathbb{Q}_p)N(\mathbb{Z}_p^\times)$ with $N(\mathbb{Z}_p^\times) := \begin{pmatrix} 1 & \mathbb{Z}_p^\times \\ 0 & 1 \end{pmatrix}$, and $I(F)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = F(x)$ for $x \in \mathbb{Z}_p^\times$. The function $L^{\pm}$ was constructed in [17, Thm. 4.5.7] when z_α is of non-critical slope. We also remark that $L^{\pm}$ depends on the (fixed) isomorphism $\mathcal{M}^{\pm}(\mathcal{U}) \cong \mathcal{O}(\mathcal{U})$, and hence is naturally defined up to units in $\mathcal{O}(\mathcal{U})$.

Denote by $L^{\pm}(z_\alpha, -) : \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, E) \to E$ the evaluation of $L^{\pm}$ at z_α which is a distribution of $\mathbb{Z}_p^\times$. Up to non-zero scalars, $L^{\pm}(z_\alpha, -)$ equals the distribution given by the following composition

$$\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, E) \hookrightarrow \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}} \xrightarrow{(4.8)} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \xrightarrow{(4.15)} E, \tag{4.17}$$

where the first map sends F to $I(F)$ with $I(F)$ supported in $\overline{B}(\mathbb{Q}_p)N(\mathbb{Z}_p^\times)$, and $I(F)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = F(x)$ for $x \in \mathbb{Z}_p^\times$. The following proposition is due to Emerton.

Proposition 4.5. *(1) The distribution $L^{\pm}(z_\alpha,-)$ is α-tempered (see [15, Def. 3.12] for the definition of α-tempered distributions).*

(2) Suppose z_α is not critical, then we have up to non-zero scalars (independent of ϕx^j),

$$L^{\pm}(z_\alpha,\phi x^j)=e_p(\alpha,\phi x^j)\frac{m^{j+1}}{(-2\pi i)^j}\frac{j!}{\tau(\phi^{-1})}\frac{L_\infty(f\phi^{-1},j+1)}{\Omega_f^{\pm}}, \tag{4.18}$$

where $\phi:\mathbb{Z}_p^\times\to\mathbb{Q}_p^\times$ is a smooth character of conductor p^v satisfying $\phi(-1)=\pm1$, $j\in\{0,\cdots,k\}$,

$$e_p(\alpha,\phi x^j):=\frac{1}{\alpha^v}\big(1-\frac{\phi^{-1}(p)\epsilon(p)p^{k-j}}{\beta}\big)\big(1-\frac{\phi(p)p^j}{\beta}\big),$$

$\tau(\phi^{-1})$ is the Gauss sum of ϕ, L_∞ is the archimedean L-function and $\Omega_f^{\pm}$ are the two archimedean periods of f.

Proof. (1) By [15, Lem. 3.22], the composition

$$\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times,E)\hookrightarrow\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}}\xrightarrow{(4.8)}\widetilde{H}^1_{\text{ét},c}(K^p,E)_{\overline{\rho}}^{\pm}$$

is α-tempered (in the sense of [15, Def. 3.12]). Since (4.15) sends $\widetilde{H}^1_{\text{ét},c}(K^p,\mathcal{O}_E)_{\overline{\rho}}$ to $\mathcal{O}_E$, we deduce by [15, Lem. 3.20] that $L^{\pm}(z_\alpha,-)$ is α-tempered.

(2) The interpolation result for the distribution (4.17) in non-critical slope case (i.e. $\mathrm{val}_p(\alpha)<k+1$) was proved in [15, Prop. 4.9], and the critical slope (i.e. $\mathrm{val}_p(\alpha)=k+1$) but non-critical case follows by the same argument. □

Remark 4.6. *By results of Amice-Vélu and Vishik (e.g. see [15, Lem. 3.14]), when z_α is not of critical slope, $L^{\pm}(z_\alpha,-)$ is determined (up to non-zero scalars) by the interpolation property.*

Proposition 4.7. *If z_α is critical, then $L^{\pm}(z_\alpha,\phi x^j)=0$ for all ϕx^j given as in (4.18).*

Proof. We denote by $\mathcal{C}^{\mathrm{lp},\leq k}(N(\mathbb{Z}_p^\times),E)$ the closed subspace of $\mathcal{C}^{\mathrm{la}}(N(\mathbb{Z}_p^\times),E)$ consisting of functions that are locally polynomial of degree $\leq k$. We have a natural commutative diagram (e.g. see [15, (3.15) (3.16)]):

$$\begin{array}{ccc}\mathcal{C}^{\mathrm{lp},\leq k}(N(\mathbb{Z}_p^\times),E) & \longrightarrow & (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{z_\alpha}^{\mathrm{sm}}\delta_B^{-1})^\infty\otimes_E(\mathrm{Sym}^k E^2)^\vee \\ \downarrow & & \downarrow \\ \mathcal{C}^{\mathrm{la}}(N(\mathbb{Z}_p^\times),E) & \longrightarrow & \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}}\end{array}$$

where the horizontal maps are injective and are given as in the first map in (4.17). The commutative diagram, together with Proposition 4.3, imply that $L^{\pm}(z_\alpha,\psi)=0$ for all $\psi\in\mathcal{C}^{\mathrm{lp},\leq k}(N(\mathbb{Z}_p^\times),E)$. It is easy to see that all the ϕx^j (given as in (4.18)) lie in $\mathcal{C}^{\mathrm{lp},\leq k}(N(\mathbb{Z}_p^\times),E)$. The proposition follows. □

Remark 4.8. *(1) It is not clear to the author whether $L^{\pm}(z_\alpha,-)$ is zero or not.*

(2) We can also directly construct the critical p-adic L-functions $L^{\pm}(z_\alpha,-)$ without using the two-variable p-adic L-functions $L^{\pm}(-,-)$. In fact, we have:

$$\begin{aligned}&\dim_E\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{z_\alpha}^c\delta_B^{-1})^{\mathrm{an}},\widetilde{H}^1_{\acute{e}t,c}(K^p,E)_{\overline{\rho}}^{\mathrm{an},\pm}[\mathcal{H}^p=\lambda_f]\big)\\&=\dim_E\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{z_\alpha}^{\mathrm{sm}}\delta_B^{-1})^\infty\otimes_E(\mathrm{Sym}^k E^2)^\vee,\widetilde{H}^1_{\acute{e}t,c}(K^p,E)_{\overline{\rho}}^{\mathrm{an},\pm}[\mathcal{H}^p=\lambda_f]\big)=1\end{aligned}\tag{4.19}$$

where the first equation is a consequence of the local-global compatibility of p-adic Langlands correspondence [20, Thm. 1.2.1], and the second equation follows from the multiplicity one result. Any non-zero element j in the 1-dimensional E-vector space on the left hand side of (4.19) induces (where the first map is the same as the first map in (4.17))

$$\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times,E)\hookrightarrow\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{z_\alpha}\delta_B^{-1}\big)^{\mathrm{an}}\twoheadrightarrow(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{z_\alpha}^c\delta_B^{-1})^{\mathrm{an}}\xrightarrow{j}\widetilde{H}^1_{\acute{e}t,c}(K^p,E)_{\overline{\rho}}^{\mathrm{an},\pm}\xrightarrow{(4.15)}E. \tag{4.20}$$

By (4.19), j is equal to the second morphism in (4.14) up to non-zero scalars. We deduce that (4.20) is equal to $L^{\pm}(z_\alpha,-)$ up to non-zero scalars. We also remark that this construction does not rely on the smoothness of the eigencurve.

4.2 Properties

Keep the notation in § 4.1, and assume z_α is critical. In this section, we study some properties of $L^\pm(z_\alpha, -)$ and the two-variable p-adic L-functions $L^\pm(-,-)$.

4.2.1 Relations with Bellaïche's critical p-adic L-functions

Recall in [1], Bellaiche constructed 2-variable $\pm$-p-adic L-functions $\mathscr{L}^\pm$ in a neighborhood of z_α. In this section, we compare $\mathscr{L}^\pm$ with our $L^\pm$ constructed in § 4.1.

Let $\mathcal{V} \subseteq \mathcal{U}$ be an affinoid open neighborhood of z_α in $\mathcal{C}$ such that both $L^\pm$ and $\mathscr{L}^\pm$ are defined, that the points of non-critical slope are Zariski-dense in $\mathcal{V}$, and that $\mathcal{O}(\mathcal{V})$ is a PID.

The isomorphism $\mathbb{Z}_p^\times \cong (\mathbb{Z}/q\mathbb{Z})^\times \times (1+q\mathbb{Z}_p)$ (with $q=p$ if $p \neq 2$, and $q = 2p$ if $p=2$) induces an isomorphism of rigid spaces $\mathcal{W} \cong \sqcup_{i \in (\mathbb{Z}/q\mathbb{Z})^\times} \mathcal{W}_i$ where all the $\mathcal{W}_i$ are isomorphic to the rigid space over E parametrizing continuous characters of $1+q\mathbb{Z}_p \cong \mathbb{Z}_p$. Note that the latter rigid space is isomorphic to the open unit disc: a point z of $\mathbb{C}_p$ with $|z|<1$ corresponds to the character $(1+z)^a$ for $a \in \mathbb{Z}_p$. For $\lambda \in \mathcal{O}(\mathcal{W})$ (resp. $\Lambda \in \mathcal{O}(\mathcal{V} \times \mathcal{W})$, we denote by $\lambda^i \in \mathcal{O}(\mathcal{W}_i)$ (resp. $\Lambda^i \in \mathcal{O}(\mathcal{V} \times \mathcal{W}_i)$) its restriction on $\mathcal{W}_i$ (resp. on $\mathcal{O}(\mathcal{V} \times \mathcal{W}_i)$).

Proposition 4.9. *Let $i \in (\mathbb{Z}/q\mathbb{Z})^\times$. Assume $\mathscr{L}^\pm(z_\alpha, -)^i \neq 0$ (resp. $L^\pm(z_\alpha,-)^i \neq 0$), then there exist an admissible affinoid $\mathcal{V}' \subset \mathcal{V}$ containing z_α and $a_\pm \in \mathcal{O}(\mathcal{V}')$ such that $L^\pm(-,-)^i = a_\pm \mathscr{L}^\pm(-,-)^i$ (resp. $\mathscr{L}^\pm(-,-)^i = a_\pm L^\pm(-,-)^i$).*

Proof. We only prove the case where $\mathscr{L}^\pm(z_\alpha,-)^i \neq 0$, the other case being symmetric. Let $w_1, w_2 \in \mathcal{W}_i$, put:

$$d^\pm_{w_1,w_2}(-) := L^\pm(-,w_1)\mathscr{L}^\pm(-,w_2) - L^\pm(-,w_2)\mathscr{L}^\pm(-,w_1) \in \mathcal{O}(\mathcal{V}).$$

We claim $d^\pm_{w_1,w_2} = 0$ (thus independent of w_1, w_2). Indeed for any point $z \in \mathcal{V}$ of non-critical slope, we know the distributions $L^\pm(z,-)$, $\mathscr{L}^\pm(z,-)$ equal up to non-zero scalars (by the interpolation property), thus $d^\pm_{w_1,w_2}(z) = 0$. Since such points are Zariski-dense in $\mathcal{V}$, the claim follows.

For $w \in \mathcal{W}_i$ such that $\mathscr{L}^\pm(z_\alpha, w) = \mathscr{L}^\pm(z_\alpha,w)^i \neq 0$ (by assumption, such w exists), we put $a'_\pm := \frac{L^\pm(-,w)}{\mathscr{L}^\pm(-,w)} \in \mathrm{Frac}(\mathcal{O}(\mathcal{V}))$. By the above claim, we see $a'_\pm$ is independent of the choice of w. We put $a_\pm := \frac{L^\pm(-,-)^i}{\mathscr{L}^\pm(-,-)^i} \in \mathrm{Frac}(\mathcal{O}(\mathcal{V} \times \mathcal{W}_i))$. We claim $a'_\pm = a_\pm$ (in other words $a_\pm \in \mathrm{Frac}(\mathcal{O}(\mathcal{V}))$). We view $a'_\pm$ as an element in $\mathrm{Frac}(\mathcal{O}(\mathcal{V} \times \mathcal{W}_i))$ by the natural inclusion. To prove $a'_\pm = a_\pm$, it is sufficient to prove

$$\mathfrak{d} := \mathscr{L}^\pm(-,w)L^\pm(-,-)^i - L^\pm(-,w)\mathscr{L}^\pm(-,-)^i = 0$$

where w satisfies $\mathscr{L}^\pm(z_\alpha, w) \neq 0$, $\mathscr{L}^\pm(-,w)$ and $L^\pm(-,w)$ are viewed as elements in $\mathcal{O}(\mathcal{V} \times \mathcal{W}_i)$ via the natural injection $\mathcal{O}(\mathcal{V}) \hookrightarrow \mathcal{O}(\mathcal{V} \times \mathcal{W}_i)$ (and so $\mathfrak{d} \in \mathcal{O}(\mathcal{V} \times \mathcal{W}_i)$). Let Z be the set of classical points of non-critical slope in $\mathcal{V}$. As discussed above, we know

$$\mathfrak{d}(z,w') = \mathscr{L}^\pm(z,w)L^\pm(z,w')^i - L^\pm(z,w)\mathscr{L}^\pm(z,w')^i = 0$$

for all $z \in Z$, and $w' \in \mathcal{W}_i$. Since $Z \times \mathcal{W}_i$ is Zariski-dense in $\mathcal{V} \times \mathcal{W}_i$, we deduce $\mathfrak{d} = 0$. Thus $a_\pm = a'_\pm \in \mathrm{Frac}(\mathcal{O}(\mathcal{V}))$. The proposition follows by shrinking $\mathcal{V}$ (and using the assumption $\mathscr{L}^\pm(z_\alpha,-)^i \neq 0$). □

Recall that the restriction $\mathscr{L}^\pm(z_\alpha,-)$ is equal, up to non-zero scalars, to the $\pm$-p-adic L-function for f_α constructed in [1], we can thus deduce from the proposition the following corollary.

Corollary 4.10. *Let $i \in (\mathbb{Z}/q\mathbb{Z})^\times$. Assume $\mathscr{L}^\pm(z_\alpha,-)^i \neq 0$ (resp. $L^\pm(z_\alpha,-)^i \neq 0$), then there exists $a^\pm \in E$ (enlarge E if necessary) such that $L^\pm(z_\alpha,-)^i = a^\pm \mathscr{L}^\pm(z_\alpha,-)^i$ (resp. $\mathscr{L}^\pm(z_\alpha,-)^i = a^\pm L^\pm(z_\alpha,-)^i$).*

4.2.2 Secondary critical p-adic L-functions

For $i = 1, \cdots, e-1$, we put (shrinking $\mathcal{U}$ if necessary) $L_i^\pm \in \mathcal{O}(\mathcal{U} \times \mathcal{W})$ such that (noting that r_{z_α} is a uniformiser of $\mathcal{O}(\mathcal{U})$ at z_α)

$$L_i^\pm(t,\sigma) := \frac{\partial^i L^\pm}{\partial r^i_{z_\alpha}}(t,\sigma), \forall\ (t,\sigma) \in \mathcal{U} \times \mathcal{W}.$$

The statement in the following proposition was proved in [1, § 4.4] for Bellaïche's two-variable p-adic L-functions. We show that it also holds for our $L^{\pm}(-,-)$.

Proposition 4.11. *(1) For $i = 1, \cdots, e-2$, $L_i^{\pm}(z_\alpha, \phi x^j) = 0$ for all ϕx^j given as in (4.18).*
(2) With the notation of (4.18), we have up to non-zero scalars (independent of ϕx^j):

$$L_{e-1}^{\pm}(z_\alpha, \phi x^j) = e_p(\alpha, \phi x^j) \frac{m^{j+1}}{(-2\pi i)^j} \frac{j!}{\tau(\phi^{-1})} \frac{L_\infty(f\phi^{-1}, j+1)}{\Omega_f^{\pm}}. \tag{4.21}$$

Proof. Step (a). We unwind a bit Emerton's adjunction formula ([18, Thm. 0.13]). By [16, Thm. 3.5.6], the composition in (4.2) first induces $B(\mathbb{Q}_p)$-equivariant morphisms

$$\mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \xrightarrow{r_{z_\alpha}^{e-1}} \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \xrightarrow{\iota_0} \widetilde{H}^1_{\text{ét},c}(K^p, E)^{\mathrm{an},\pm}_{\overline{\rho}}, \tag{4.22}$$

where $\mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1})$ denotes the space of locally constant, compactly supported functions on $N(\mathbb{Q}_p)$ with values in $\mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}$ (which is equipped with a natural $B(\mathbb{Q}_p)$-action as in [16, § 3.5]). The morphisms in (4.22) further induce $(\mathfrak{g}, B(\mathbb{Q}_p))$-equivariant morphisms (cf. [19, (5.11)])

$$\begin{aligned}\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) &\xrightarrow{r_{z_\alpha}^{e-1}} \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \\ &\xrightarrow{\iota_1^{\pm}} \widetilde{H}^1_{\text{ét},c}(K^p, E)^{\mathrm{an},\pm}_{\overline{\rho}}.\end{aligned} \tag{4.23}$$

On the other hand, we have a natural $(\mathfrak{g}, B(\mathbb{Q}_p))$-equivariant morphism (cf. [18, (2.8.7)])

$$\begin{aligned}\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) &\longrightarrow \mathcal{C}_c^{\mathrm{lp}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \\ &\cong \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p, \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \otimes_E E[z],\end{aligned} \tag{4.24}$$

where $\mathcal{C}_c^{\mathrm{lp}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1})$ denotes the space of locally polynomial functions on $N(\mathbb{Q}_p)$ with values in $\mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}$ and we refer to [18, Def. 2.5.21] for the precise definition and for the second isomorphism in (4.24). As in [17, (4.5.16)], the morphism in (4.24) is given by

$$X_-^l \otimes f \mapsto ((\prod_{i=0}^{l-1}(h-i)) \cdot f) z^l$$

for $l \in \mathbb{Z}_{\geq 1}$, and $f \in \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1})$ (and $X_-^0 \otimes f \mapsto f$). As in the last paragraph of the proof of [17, Lem. 4.5.12], the first morphism in (4.24) is surjective. Since the composition in (4.2) is balanced, the composition in (4.23) factors through (4.24) (noting that in contrary $\iota_1^{\pm}$ does not factor through (4.24)). In summary, we have a $(\mathfrak{g}, B(\mathbb{Q}_p))$-equivariant commutative diagram

$$\begin{array}{ccc}\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) & \xrightarrow{r_{z_\alpha}^{e-1}} & \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \\ \downarrow & & \downarrow \iota_1^{\pm} \\ \mathcal{C}_c^{\mathrm{lp}}(N(\mathbb{Q}_p), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) & \longrightarrow & \widetilde{H}^1_{\text{ét},c}(K^p, E)^{\mathrm{an},\pm}_{\overline{\rho}}\end{array}. \tag{4.25}$$

We remark that the morphism (4.3) is actually induced by the bottom map in (4.25) (e.g. see [18, Cor. 4.3.3]). For $N(\mathbb{Z}_p^\times) = \begin{pmatrix} 1 & \mathbb{Z}_p^\times \\ 0 & 1 \end{pmatrix}$, we have a natural injection $\mathcal{C}^*(N(\mathbb{Z}_p^\times), -) \hookrightarrow \mathcal{C}_c^*(N(\mathbb{Q}_p), -)$, sending a function F to the function whose value at $x \in N(\mathbb{Z}_p^\times)$ is $F(x)$ and 0 outside $N(\mathbb{Z}_p^\times)$. We then easily deduce from (4.25) a commutative diagram (which is $\mathrm{U}(\mathfrak{g})$-equivariant)

$$\begin{array}{ccc}\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) & \xrightarrow{r_{z_\alpha}^{e-1}} & \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \\ \downarrow & & \downarrow \iota_1^{\pm} \\ \mathcal{C}^{\mathrm{lp}}(N(\mathbb{Z}_p^\times), \mathcal{M}^{\pm}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) & \longrightarrow & \widetilde{H}^1_{\text{ét},c}(K^p, E)^{\mathrm{an},\pm}_{\overline{\rho}}\end{array}. \tag{4.26}$$

The bottom map of (4.26) is in fact equal to the composition

$$\mathcal{C}^{\mathrm{lp}}(N(\mathbb{Z}_p^\times), \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \hookrightarrow \mathcal{C}^{\mathrm{la}}(N(\mathbb{Z}_p^\times), \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \longrightarrow \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}\big)^{\mathrm{an}} \xrightarrow{\iota^\pm} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm},$$

where the middle map is given in the same way as in the discussion below (4.16).

Step (b). Let $\mathcal{V}$ be a compact open subset of $\mathbb{Z}_p^\times$, and $1_{\mathcal{V}} \in \mathcal{C}^\infty(\mathbb{Z}_p^\times, E)$ be the function with $1_{\mathcal{V}}(x) = \begin{cases} 1 & x \in \mathcal{V} \\ 0 & \text{otherwise} \end{cases}$. For $j \in \mathbb{Z}_{\geq 0}$ and $i = 1, \cdots, e-1$, we have

$$L_i^\pm(-, 1_{\mathcal{V}} z^j) = \frac{dL^\pm(-, 1_{\mathcal{V}} z^j)}{d^i r_{z_\alpha}}. \tag{4.27}$$

Recall that we have fixed an isomorphism $\mathcal{O}(\mathcal{U}) \cong \mathcal{M}^\pm(\mathcal{U})$, which induces an isomorphism $\mathcal{O}(\mathcal{U})_b^\vee \cong \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}$, that we fix in the sequel. By definition, $L^\pm(-, 1_{\mathcal{V}} z^j) \in \mathcal{O}(\mathcal{U})$ is characterized by the following composition

$$\mathcal{O}(\mathcal{U})_b^\vee \cong \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1} \xrightarrow{h_{\mathcal{V},j}} \mathcal{C}^{\mathrm{lp}}(N(\mathbb{Z}_p^\times), \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \longrightarrow \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \xrightarrow{\{\infty\}-\{0\}} E, \tag{4.28}$$

where the map $h_{\mathcal{V},j}$ sends $m \in \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}$ to the element:

$$m \otimes 1_{\mathcal{V}} \otimes z^j \in \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), E) \otimes_E \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \otimes_E E[z] \cong \mathcal{C}^{\mathrm{lp}}(N(\mathbb{Z}_p^\times), \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}).$$

Denote by $F_{\mathcal{V},j} \in \mathcal{O}(\mathcal{U})$ the element given by the following composition

$$\mathcal{O}(\mathcal{U})_b^\vee \cong \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1} \xrightarrow{g_{\mathcal{V},j}} \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \xrightarrow{\iota_1^\pm \circ r_{z_\alpha}^{e-1}} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \xrightarrow{\{\infty\}-\{0\}} E, \tag{4.29}$$

where $g_{\mathcal{V},j}$ sends m to $X_-^j \otimes 1_{\mathcal{V}} \otimes m$. By the description of (4.24), we see

$$F_{\mathcal{V},j} = \Delta_j L^\pm(-, 1_{\mathcal{V}} z^j) \tag{4.30}$$

with $\Delta_j = \prod_{l=0}^{j-1}(h-l)$ for $j \in \mathbb{Z}_{>0}$, and $\Delta_0 = 1$. Similarly, the composition

$$\mathcal{O}(\mathcal{U})_b^\vee \cong \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1} \xrightarrow{g_{\mathcal{V},j}} \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \mathcal{M}^\pm(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \xrightarrow{\iota_1^\pm} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \xrightarrow{\{\infty\}-\{0\}} E \tag{4.31}$$

gives an element $G_{\mathcal{V},j} \in \mathcal{O}(\mathcal{U})$ as well. Comparing (4.29) with (4.31), we see

$$F_{\mathcal{V},j} = G_{\mathcal{V},j} r_{z_\alpha}^{e-1}. \tag{4.32}$$

Suppose $j \leq k$, then Δ_j and r_{z_α} are coprime (using $(r_{z_\alpha}^e) = (h-k)$). By (4.30) and (4.32), we deduce (shrinking $\mathcal{U}$ if necessary) that there exists $H_{\mathcal{V},j} \in \mathcal{O}(\mathcal{U})$ such that $L^\pm(-, 1_{\mathcal{V}} z^j) = r_{z_\alpha}^{e-1} H_{\mathcal{V},j}$ and $G_{\mathcal{V},j} = \Delta_j H_{\mathcal{V},j}$. Together with (4.27), the part (1) follows and we have $L_{e-1}^\pm(z_\alpha, 1_{\mathcal{V}} z^j) = (e-1)! H_{\mathcal{V},j}(z_\alpha)$.

Step (c). We prove the part (2). We fix an isomorphism of E-vector space $\chi_{z_\alpha} \cong E$. Denote by $i_{z_\alpha} : \mathcal{O}(\mathcal{U}) \twoheadrightarrow \mathcal{O}(\mathcal{U})/r_{z_\alpha} = E$ the natural projection, which induces $i_{z_\alpha}^* : \chi_{z_\alpha} \hookrightarrow \mathcal{O}(\mathcal{U})_b^\vee$. We remark that for $s \in \mathcal{O}(\mathcal{U})$ the evaluation $s(z_\alpha)$ is given by $i_{z_\alpha}(s)$. Since $i_{z_\alpha}^*$ has image in $\mathcal{M}^\pm(\mathcal{U})_b^\vee[h=k]^{\mathrm{cl}}$ (by (4.4)), as in [19, Ex. 5.22], the composition

$$\chi_{z_\alpha} \hookrightarrow \mathcal{O}(\mathcal{U})_b^\vee \cong \mathcal{M}^\pm(\mathcal{U})_b^\vee \longrightarrow J_B\big(\widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}\big)$$

is balanced and induces an injection

$$\iota_{z_\alpha}^{\pm} : (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{sm}} \delta_B^{-1})^\infty \otimes_E (\mathrm{Sym}^k E^2)^\vee \hookrightarrow \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm}.$$

We have hence a commutative diagram

$$\begin{array}{ccc} \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \chi_{z_\alpha} \otimes_E \delta_B^{-1}) & \xrightarrow{i_{z_\alpha}^*} & \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \mathcal{O}(\mathcal{U})_b^\vee \otimes_E \delta_B^{-1}) \\ \downarrow & & \downarrow \iota_1^{\pm} \\ (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{sm}})^\infty \otimes_E (\mathrm{Sym}^k E^2)^\vee & \xrightarrow{\iota_{z_\alpha}^{\pm}} & \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \end{array} \tag{4.33}$$

where the top morphism is induced by $i_{z_\alpha}^*$, and the left hand side map factors as

$$\begin{aligned} \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \chi_{z_\alpha} \otimes_E \delta_B^{-1}) &\longrightarrow \mathcal{C}^{\mathrm{lp},\leq k}(N(\mathbb{Z}_p^\times), \chi_{z_\alpha} \otimes_E \delta_B^{-1}) \\ &\hookrightarrow (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{sm}} \delta_B^{-1})^\infty \otimes_E (\mathrm{Sym}^k E^2)^\vee \Big(\hookrightarrow (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha} \delta_B^{-1})^{\mathrm{an}} \Big), \end{aligned}$$

with the first map given in the same way as in (4.24), and the second map given in the same way as in the discussion below (4.16). By (4.33), $G_{\mathcal{V},j}(z_\alpha)$ is equal to the evaluation at $1 \in E$ of the following composition

$$E \cong \chi_{z_\alpha} \otimes_E \delta_B^{-1} \xrightarrow{g_{\mathcal{V},j}} \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathcal{C}^{\mathrm{sm}}(N(\mathbb{Z}_p^\times), \chi_{z_\alpha} \otimes_E \delta_B^{-1}) \xrightarrow{\iota_1^{\pm} \circ i_{z_\alpha}^*} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \xrightarrow{\{\infty\}-\{0\}} E.$$

Consider the following composition (recall $j \leq k$, and the map $h_{\mathcal{V},j}$ is given in the same way as in (4.28))

$$\begin{aligned} \mu_{\mathcal{V},j} : E \cong \chi_{z_\alpha} \otimes_E \delta_B^{-1} &\xrightarrow{h_{\mathcal{V},j}} \mathcal{C}^{\mathrm{lp},\leq k}(N(\mathbb{Z}_p^\times), \chi_{z_\alpha} \otimes_E \delta_B^{-1}) \\ &\hookrightarrow (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{sm}} \delta_B^{-1})^\infty \otimes_E (\mathrm{Sym}^k E^2)^\vee \xrightarrow{\iota_{z_\alpha}^{\pm}} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \xrightarrow{\{\infty\}-\{0\}} E. \end{aligned}$$

By the commutative diagram (4.33) (and using the description of (4.24)), we see

$$G_{\mathcal{V},j}(z_\alpha) = \Delta_j(z_\alpha) \mu_{\mathcal{V},j}(1)$$

and hence (by step (b)) $L_{e-1}^{\pm}(z_\alpha, 1_{\mathcal{V}} z^j) = (e-1)! \mu_{\mathcal{V},j}(1)$. In summary, the values of $L_{e-1}^{\pm}(z_\alpha, -)$ at functions in $\mathcal{C}^{\mathrm{lp},\leq k}(\mathbb{Z}_p^\times, E)$ (in particular, at ϕx^j as in (4.18)) are characterized by the composition

$$\begin{aligned} \mathcal{C}^{\mathrm{lp},\leq k}(N(\mathbb{Z}_p^\times), \chi_{z_\alpha} \otimes_E \delta_B^{-1}) \hookrightarrow (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{z_\alpha}^{\mathrm{sm}} \delta_B^{-1})^\infty \otimes_E (\mathrm{Sym}^k E^2)^\vee& \\ \xrightarrow{\iota_{z_\alpha}^{\pm}} \widetilde{H}^1_{\text{ét},c}(K^p, E)_{\overline{\rho}}^{\mathrm{an},\pm} \xrightarrow{\{\infty\}-\{0\}} E.& \end{aligned}$$

Now we are in the same situation as in [15, (4.14)(4.15)]. The proof of [15, Prop. 4.9] carries over verbatim to this setting, and the part (2) follows. □

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