p-adic Hodge parameters in the crystabelline representations of $\mathrm{GL}_3(\mathbb{Q}_p)$

Yiwen Ding

Abstract

We build a one-to-one correspondence between 3-dimensional (generic) crystabelline representations of the absolute Galois group of $\mathbb{Q}_p$ and certain locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$. We show that the correspondence can be realized in subspaces of p-adic automorphic representations.

Contents

1 Introduction

The locally analytic p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ builds a one-to-one correspondence betwen two dimensional p-adic representations of the absolute Galois group $\mathrm{Gal}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, and certain locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (e.g. see [13, Thm. 0.1]). In particular, the correspondence provides a dictionary of parameters on the two sides. In the non-split crystalline case, the correspondence is rather straightforward, since a 2-dimensional crystalline representation ρ of $\mathrm{Gal}_{\mathbb{Q}_p}$ is simply determined by its associated Weil-Deligne representation and Hodge-Tate weights. Correspondingly, the locally analytic representation $\pi(\rho)$ associated to ρ is determined by its locally algebraic subrepresentation, as characterized by the classical local Langlands correspondence. This phenomenon collapses for other groups (say $\mathrm{GL}_n(\mathbb{Q}_p)$, $n > 2$, or $\mathrm{GL}_2(K)$, $K \neq \mathbb{Q}_p$), where additional parameters of the Hodge filtration appear in the crystalline case. An immediate question in the p-adic Langlands program for other groups is how to see such new parameters on the automorphic side. In the note, we address this question for $\mathrm{GL}_3(\mathbb{Q}_p)$.

We introduce some notation. Let ρ be a 3-dimensional crystalline representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ over a sufficiently large p-adic field E, of regular Hodge-Tate weights $h := h_1 > h_2 > h_3$. Let $\alpha_1, \alpha_2, \alpha_3$ be the eigenvalues of the crystalline Frobenius on $D_{\mathrm{cris}}(\rho)$. We assume $\alpha_i\alpha_j^{-1} \neq 1, p$ for $i \neq j$ (this is what we mean ρ is generic). Let e_i be an α_i-eigenvector for φ. We furthermore assume ρ is non-critical for all refinements, which means the Hodge filtration of $D_{\mathrm{cris}}(\rho)$ is in a relative general position with all the φ-stable flags: $Ee_i \subsetneq Ee_i \oplus Ee_j \subsetneq D_{\mathrm{cris}}(\rho)$. Multiplying e_i by certain non-zero scalars, the Hodge filtration of $D_{\mathrm{cris}}(\rho)$ can be explicitly described as follows:

$$\mathrm{Fil}^j D_{\mathrm{cris}}(\rho) = \begin{cases} D_{\mathrm{cris}}(\rho) & j \leq -h_1 \\ E(e_1+e_2) \oplus E(e_1 + a_\rho e_2 + e_3) & -h_1 < j \leq -h_2 \\ E(e_1 + a_\rho e_2 + e_3) & -h_2 < j \leq -h_3 \\ 0 & j > -h_3 \end{cases} \tag{1}$$

where $a_\rho \in E^\times$ is what we call the Hodge parameter of ρ. We denote such ρ by $V(\underline{\alpha}, h, a_\rho)$. The set of non-critical crystalline representations of weights h and φ-eigenvalues $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is exactly $\{V(\underline{\alpha}, h, a)\}_{a \in E^\times}$. In this note, for $a \in E^\times$, we associate a locally analytic representation $\pi(\underline{\alpha}, h, a)$ of $\mathrm{GL}_3(\mathbb{Q}_p)$. Here is our main theorem.

Theorem 1.1. *(1) (Local correspondence) We have $\pi(\underline{\alpha}, h, a) \cong \pi(\underline{\alpha}, h, a')$ if and only if $a = a'$ if and only if $V(\underline{\alpha}, h, a) \cong V(\underline{\alpha}, h, a')$.*

(2) (Local-global compatibility) Assume $\rho = V(\underline{\alpha}, h, a_\rho)$ is automorphic for the setting of [11], and let $\widehat{\pi}(\rho)$ be the (globally) associated Banach representation of $\mathrm{GL}_3(\mathbb{Q}_p)$. Then for $a \in E^\times$,

$$\pi(\underline{\alpha}, h, a) \hookrightarrow \widehat{\pi}(\rho) \text{ if and only if } a = a_\rho.$$

Remark 1.2. *(1) Let Π_∞ be the patched Banach representation over the patched Galois deformation ring R_∞ (for a certain 3-dimensional mod p representation $\overline{\rho}$ of $\mathrm{Gal}_{\mathbb{Q}_p}$) of [11]. Our local-global compatibility result is in fact obtained for a more general setting. We show that if there is a maximal ideal $\mathfrak{m}$ of $R_\infty[1/p]$ associated to ρ (which implies ρ has mod p reduction isomorphic to $\overline{\rho}$) such that $\Pi_\infty[\mathfrak{m}]^{\mathrm{lalg}} \neq 0$, then for $a \in E^\times$,*

$$\pi(\underline{\alpha}, h, a) \hookrightarrow \Pi_\infty[\mathfrak{m}] \text{ if and only if } a = a_\rho.$$

The assumption $\Pi_\infty[\mathfrak{m}]^{\mathrm{lalg}} \neq 0$ is equivalent to that ρ appears on the patched eigenvariety associated to Π_∞ (where the equivalence is easy in our non-critical case, but noting it also holds in the critical case by [7] [9]).

(2) For the case where ρ is critical for some refinements or if ρ has irregular Sen weights, one can associate to ρ a semi-simple locally analytic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ that determines ρ as in

[3] [27]. Together with our construction of $\pi(\underline{\alpha},h,a)$, this establishes a one-to-one correspondence between 3-dimensional generic crystabelline $\mathrm{Gal}_{\mathbb{Q}_p}$-representations and their corresponding locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$.

(3) Let $\pi_{\mathrm{alg}}(\underline{\alpha},h)$ be the locally algebraic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ associated to $\underline{\alpha}$ and h (which is the $\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)$ in (2)). A key feature of $\pi(\underline{\alpha},h,a)$ is that it contains two copies of $\pi_{\mathrm{alg}}(\underline{\alpha},h)$, one in the socle, another one in the cosocle (see (4)):

$$\pi_{\mathrm{alg}}(\underline{\alpha},h)\cong \mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)}\,\pi(\underline{\alpha},h,a)\longrightarrow \pi(\underline{\alpha},h,a)\xrightarrow{\kappa}\!\!\!\!\rightarrow \mathrm{cosoc}_{\mathrm{GL}_3(\mathbb{Q}_p)}\pi(\underline{\alpha},h,a)\cong \pi_{\mathrm{alg}}(\underline{\alpha},h).$$

The Hodge parameter $a\in E^\times$ is encoded in the extension of (the cosocle) $\pi_{\mathrm{alg}}(\underline{\alpha},h)$ by $\mathrm{Ker}(\kappa)$. A similar pattern of extensions (with locally algebraic representations in the cosocle) has appeared in the semi-stable non-crystalline case (cf. [4][5], tracing back to [2]). This work grows out of the finding of the "surplus" locally algebraic constituents in the non-critical case in [15, § Appendix C]. Note that the existence of such a constituent was first proved by Hellmann-Hernandez-Schraen in the split case for $\mathrm{GL}_3(\mathbb{Q}_p)$.

(4) The representation $\pi(\underline{\alpha},h,a_\rho)$ is a proper subrepresentation of $\widehat{\pi}(\rho)^{\mathrm{an}}$ (the locally analytic subrepresentation of $\widehat{\pi}(\rho)$), which has many more constituents. In fact, some of these constituents are already described in [10, § 5.3]. We remark that even when amalgamating our representation $\pi(\underline{\alpha},h,a_\rho)$ with $\pi(\rho)^{\mathrm{fs}}$ of [10], the result should still give only a proper subrepresentation of $\widehat{\pi}(\rho)^{\mathrm{an}}$. For example, one may expect that $\widehat{\pi}(\rho)^{\mathrm{an}}$ contains some (mysterious) supersingular constituent(s) (cf. [15]).

(5) We finally remark the results in the note are obtained for (more general) crystabelline representations.

We explain the construction of $\pi(\underline{\alpha},h,a)$ and the proof of Theorem 1.1. We first give a reinterpretation of the Hodge parameter. Let $D:=D_{\mathrm{rig}}(\rho)$ be the associated (φ,Γ)-module of rank 3 over the Robba ring $\mathcal{R}_E$. Recall an ordering of $\underline{\alpha}$ corresponds to a non-critical triangulation of D. Let D_1 (resp. C_1) be the rank 2 saturated (φ,Γ)-submodule (resp. quotient) of D corresponding to α_1 and α_2. So D_1 and C_1 depend on the choice of two φ-eigenvalues, but in fact any choice will work. Then D has the following two forms (where a line means a non-split extension with the left object the sub, and the right the quotient)

$$\Big[\underbrace{\mathcal{R}_E(\mathrm{unr}(\alpha_1)z^{h_1})\text{—}\mathcal{R}_E(\mathrm{unr}(\alpha_2)z^{h_2})}_{D_1}\text{—}\mathcal{R}_E(\mathrm{unr}(\alpha_3)z^{h_3})\Big]$$

$$\Big[\mathcal{R}_E(\mathrm{unr}(\alpha_3)z^{h_1})\text{—}\underbrace{\mathcal{R}_E(\mathrm{unr}(\alpha_1)z^{h_2})\text{—}\mathcal{R}_E(\mathrm{unr}(\alpha_2)z^{h_3})}_{C_1}\Big].$$

Let $\iota_D\in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$ be the natural (injective) composition

$$\iota_D: D_1\longrightarrow D\longrightarrow\!\!\!\!\rightarrow C_1.$$

Proposition 1.3. *We have $\dim_E \mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)=2$, and D is determined by the line $E[\iota_D]\subset \mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$ (together with α_3).*

The E-vector space $\mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$ is in fact spanned by the two (non-injective) maps: $D_1\twoheadrightarrow \mathcal{R}_E(\mathrm{unr}(\alpha_i)z^{h_2})\hookrightarrow C_1$ for $i=1,2$. We give a quick explanation why it is the right parameter space for a_ρ in (1). An injection $\iota: D_1\hookrightarrow C_1$ induces a map of filtered φ-modules

$\iota : D_{\mathrm{cris}}(D_1) \to D_{\mathrm{cris}}(C_1)$. Denoting by $\mathrm{Fil}^{\max}$ the unique non-trivial Hodge filtration for $D_{\mathrm{cris}}(D_1)$ and $D_{\mathrm{cris}}(C_1)$, the map ι then carries exactly the information on the relative position of the two lines $\iota(\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_1))$ and $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(C_1)$ in $D_{\mathrm{cris}}(C_1)$. When $[\iota] \in E[\iota_D]$, we see the parameter for the relative position is just a_ρ (see Remark 2.3 for more details).

Let $\iota \in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1, C_1)$ be an injection. We first associate to ι a set of pairs of certain deformations of D_1 and C_1, which will be crucially used in our construction of $\pi(\underline{\alpha}, h, a)$. Let $\mathscr{I}_\iota$ be the set of pairs $(\widetilde{D}_1, \widetilde{C}_1)$, that we call *higher intertwining pairs* (see Theorem 1.5 (2) below), satisfying

- $\widetilde{D}_1 \in \iota^-(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1)) \subset \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1)$ and $\widetilde{C}_1 \in \iota^+(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1)) \subset \mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, C_1)$, where ι^- and ι^+ denote the natural pull-back and push-forward maps respectively,
- there exists $\tilde{\iota} : \widetilde{D}_1 \hookrightarrow \widetilde{C}_1$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1 & \longrightarrow & \widetilde{D}_1 & \longrightarrow & D_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \tilde{\iota}} & & \downarrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & \widetilde{C}_1 & \longrightarrow & C_1 & \longrightarrow & 0. \end{array}$$

The following proposition summarizes some properties of $\iota^-(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1))$ and $\iota^+(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1))$.

Proposition 1.4. *For an injection $\iota \in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1, C_1)$,*

$$\dim_E \iota^-(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1)) = \dim_E \iota^+(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1)) = 3.$$

Moreover, $\iota^-(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1)) \supset \mathrm{Ext}^1_g(D_1, D_1)$ (resp. $\iota^+(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1)) \supset \mathrm{Ext}^1_g(C_1, C_1)$), the subspace of de Rham deformations, and any trianguline deformation contained in $\iota^-(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1))$ (resp. $\iota^+(\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, D_1))$) is de Rham.

The motivation to study such pairs is from the following theorem.

Theorem 1.5. *(1) For injections ι, $\iota' \in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1, C_1)$, $[\iota] \in E[\iota']$ if and only if $\mathscr{I}_\iota = \mathscr{I}_{\iota'}$.*

(2) Let D be given as above Proposition 1.3, then $\mathscr{I}_{\iota_D}$ is equal to the set of pairs $(\widetilde{D}_1, \widetilde{C}_1) \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1) \times \mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, C_1)$ such that there exists a deformation $\widetilde{D}$ of D over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ which sits in both of the following two exact sequences (of (φ, Γ)-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$)

$$0 \longrightarrow \widetilde{D}_1 \longrightarrow \widetilde{D} \longrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\mathrm{unr}(\alpha_3) z^{h_3}) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\mathrm{unr}(\alpha_3) z^{h_1}) \longrightarrow \widetilde{D} \longrightarrow \widetilde{C}_1 \longrightarrow 0.$$

In fact, we show that ι can be detected by a single element $(\widetilde{D}_1, \widetilde{C}_1) \in \mathscr{I}_\iota$ with $\widetilde{D}_1$ *non-trianguline*.

Now we move to the $\mathrm{GL}_3(\mathbb{Q}_p)$-side. We introduce a bit more notation. Let $\lambda := (h_1 - 2, h_2 - 1, h_3)$, $L(\lambda)$ be the algebraic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ of highest weight λ. Let $\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda) := (\mathrm{Ind}_{B^-}^{\mathrm{GL}_3} \mathrm{unr}(p^2\alpha_1) \boxtimes \mathrm{unr}(p\alpha_2) \boxtimes \mathrm{unr}(\alpha_3))^\infty \otimes_E L(\lambda)$, which is no other than the locally algebraic representation associated to D. By the classical intertwining property, the smooth induction remains unchanged if one alters the ordering of α_i. We can also consider the locally analytic parabolic induction, and we have

$$\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda) \longrightarrow \big(\mathrm{Ind}_{B^-}^{\mathrm{GL}_3} \mathrm{unr}(p^2\alpha_1) z^{h_1-2} \boxtimes \mathrm{unr}(p\alpha_2) z^{h_2-1} \boxtimes \mathrm{unr}(\alpha_3) z^{h_3}\big)^{\mathrm{an}} =: I(\underline{\alpha}, \lambda). \qquad (2)$$

The representation $I(\underline{\alpha}, \lambda)$ now depends on the ordering of $\underline{\alpha}$. For w running through S_3, we obtain six locally analytic principal series $I(w(\underline{\alpha}), \lambda)$. The structure of $I(w(\underline{\alpha}), \lambda)$ is clear by [25] (which has the same pattern as the dual Verma module). For our application, we are mostly interested in the constituents *right after* the locally algebraic $\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)$: the socle of $I(w(\underline{\alpha}), \lambda)/\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)$ has the form $\mathscr{C}(s_1, w) \oplus \mathscr{C}(s_2, w)$ (which are distinct and topologically irreducible). We refer to § 3.1 for the precise definition. These representations are not all distinct when varying w: $\mathscr{C}(s_i, w) \cong \mathscr{C}(s_j, w')$ if and only if $s_i = s_j$ and $w' \in \{w, s_k w\}$ with $s_k \neq s_i$. Let $\mathcal{S} := \{\mathscr{C}(s_i, w)\}$, where w has minimal length in $\{w, s_k w\}$ for $s_k \neq s_i$. So $\sharp \mathcal{S} = 6$. By amalgamating the six $I(w(\underline{\alpha}), \lambda)$ as much as possible, the resulting representation contains a unique subrepresentation $\pi_1(\underline{\alpha}, \lambda)$ of the form $[\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda) — \oplus_{\mathscr{C} \in \mathcal{S}} \mathscr{C}]$ (with socle $\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)$).

We discuss a bit more on parabolic inductions. Let $\pi(D_1)$ be the locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to D_1, $\pi_{\mathrm{alg}}(D_1)$ be its locally algebraic vector. Consider the parabolic induction $\big(\mathrm{Ind}_{P_1^-}^{\mathrm{GL}_3}(\pi(D_1) \otimes_E \varepsilon^{-1} \circ \det) \boxtimes \mathrm{unr}(\alpha_3) z^{h_3}\big)^{\mathrm{an}}$, whose socle is $\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)$. Again we just look at the constituents right after the locally algebraic $\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)$, and it turns out that we encounter exactly three of those in $\mathcal{S}$: $\{\mathscr{C}(s_1, 1), \mathscr{C}(s_1, s_1), \mathscr{C}(s_2, 1)\} =: \mathcal{S}^-$. And the parabolic induction contains a unique subrepresentation $\pi_1(\underline{\alpha}, \lambda)^-$ of the form $[\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda) — \oplus_{\mathscr{C} \in \mathcal{S}^-} \mathscr{C}]$. Replacing D_1 by C_1 and P_1 by P_2, and by a similar discussion, $(\mathrm{Ind}_{P_2^-}^{\mathrm{GL}_3} \mathrm{unr}(\alpha_3) z^{h_1} \varepsilon^{-2} \boxtimes \pi(C_1))^{\mathrm{an}}$ gives exactly the other three constituents $\mathcal{S}^+ := \{\mathscr{C}(s_2, s_2), \mathscr{C}(s_2, s_2 s_1), \mathscr{C}(s_1, s_1 s_2)\}$, and contains a unique representation $\pi_1(\underline{\alpha}, \lambda)^+$ of the form $[\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda) — \oplus_{\mathscr{C} \in \mathcal{S}^+} \mathscr{C}]$. We have hence

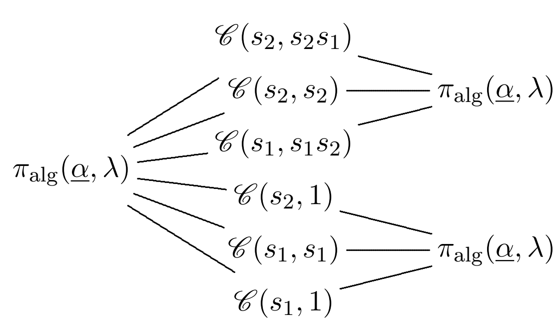
$$\pi_1(\underline{\alpha}, \lambda) \cong \pi_1(\underline{\alpha}, \lambda)^- \oplus_{\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)} \pi_1(\underline{\alpha}, \lambda)^+.$$

Using the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and taking the corresponding parabolic inductions, we can obtain two compositions (which are injective)

$$\begin{aligned} j^- : \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1) &\xrightarrow{\mathrm{pLL}} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \\ &\xrightarrow{\mathrm{Ind}} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda), \pi_1(\underline{\alpha}, \lambda)^-) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda), \pi_1(\underline{\alpha}, \lambda)), \end{aligned}$$

$$\begin{aligned} j^+ : \mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, C_1) &\xrightarrow{\mathrm{pLL}} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(C_1), \pi(C_1)) \\ &\xrightarrow{\mathrm{Ind}} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda), \pi_1(\underline{\alpha}, \lambda)^+) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda), \pi_1(\underline{\alpha}, \lambda)). \end{aligned}$$

We can now give the definition of $\pi(\underline{\alpha}, h, a)$ in Theorem 1.1, but we change the notation to $\pi(\underline{\alpha}, \lambda, \iota)$ to be consistent with the above discussion. Let $(\widetilde{D}_1, \widetilde{C}_1) \in \mathscr{I}_\iota$ with $\widetilde{D}_1$ non-trianguline (which actually implies $\widetilde{C}_1$ non-trianguline either). Consider $j^-(\widetilde{D}_1) \oplus_{\pi_1(\underline{\alpha}, \lambda)} j^+(\widetilde{C}_1)$, which is an extension of $\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)^{\oplus 2}$ by $\pi_1(\underline{\alpha}, \lambda)$ with the following form



(3)

Note that any subextension of $\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)$ by $\pi_1(\underline{\alpha},\lambda)$, of $j^-(\widetilde{D}_1)\oplus_{\pi_1(\underline{\alpha},\lambda)} j^+(\widetilde{D}_1)$, is *non-split*. It follows from the fact that $j^-(\widetilde{D}_1)$ and $j^+(\widetilde{C}_1)$ are linearly independent for non-trianguline $\widetilde{D}_1$ and $\widetilde{C}_1$, simply because $\mathcal{S}^-$ and $\mathcal{S}^+$ are disjoint (as illustrated in (3)). Roughly speaking, this implies the Galois intertwining pair $(\widetilde{D}_1,\widetilde{C}_1)$ (for ι) does *not* intertwine on the automorphic side. One can show that there is a unique extension, which is our $\pi(\underline{\alpha},\lambda,\iota)$, of $\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)$ by $\pi_1(\underline{\alpha},\lambda)$ satisfying that $\pi(\underline{\alpha},\lambda,\iota)\subset j^-(\widetilde{D}_1')\oplus_{\pi_1(\underline{\alpha},\lambda)} j^+(\widetilde{C}_1')$ for all $(\widetilde{D}_1',\widetilde{C}_1')\in\mathscr{I}_\iota$ with $\widetilde{D}_1'$ non-trianguline. In fact, by suitably normalizing the maps j^- and j^+, $\pi(\underline{\alpha},\lambda,\iota)$ can be defined to be the representation associated to $j^-(\iota^-([M]))-j^+(\iota^+([M]))$ for a non-de Rham $M\in\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1,D_1)$. Moreover, one can show that $\pi(\underline{\alpha},\lambda,\iota)$ determines $\mathscr{I}_\iota$ hence ι. This proves Theorem 1.1 (1). By construction, $\pi(\underline{\alpha},\lambda,\iota)$ has the following form

$$\begin{array}{ccccc} & & \mathscr{C}(s_2,s_2s_1) & & \\ & & \mathscr{C}(s_2,s_2) & & \\ & & \mathscr{C}(s_1,s_1s_2) & & \\ \pi_{\mathrm{alg}}(\underline{\alpha},\lambda) & & \mathscr{C}(s_2,1) & & \pi_{\mathrm{alg}}(\underline{\alpha},\lambda) \\ & & \mathscr{C}(s_1,s_1) & & \\ & & \mathscr{C}(s_1,1) & & \end{array} \tag{4}$$

We discuss our local-global compatibility result. We use the setting in Remark 1.2 (1). Let $D:=D_{\mathrm{rig}}(\rho)$. We only explain why $\pi(\underline{\alpha},\lambda,\iota_D)\hookrightarrow\Pi_\infty[\mathfrak{m}]$. To $(\widetilde{D}_1,\widetilde{C}_1)\in\mathscr{I}_{\iota_D}$ with $\widetilde{D}_1$ non-trianguline, Theorem 1.5 (2) associates a deformation $\widetilde{D}$ of D. Let $\mathcal{I}$ be a "thickening" ideal of $\mathfrak{m}$ associated to $\widetilde{D}$, so there is an exact sequence

$$0\longrightarrow\Pi_\infty[\mathfrak{m}]\longrightarrow\Pi_\infty[\mathcal{I}]\longrightarrow\Pi_\infty[\mathfrak{m}]. \tag{5}$$

Based on some local-global compatibility results on the Jacquet-Emerton modules of $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$, one can show the two presentations of $\widetilde{D}$ in Theorem 1.5 (2) give rise to two injections $j^-(\widetilde{D}_1)\hookrightarrow\Pi_\infty[\mathcal{I}]$ and $j^+(\widetilde{D}_1)\hookrightarrow\Pi_\infty[\mathcal{I}]$, which amalgamate to an injection (as $j^-(\widetilde{D}_1)$ and $j^+(\widetilde{C}_1)$ don't intertwine)

$$j^-(\widetilde{D}_1)\oplus_{\pi_1(\underline{\alpha},\lambda)} j^+(\widetilde{C}_1)\longhookrightarrow\Pi_\infty[\mathcal{I}].$$

Moreover, one can show that the two copies of $\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)$ in the cosocle of $j^-(\widetilde{D}_1)\oplus_{\pi_1(\underline{\alpha},\lambda)} j^+(\widetilde{C}_1)$ (see (3)) are both sent to the single $\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)$ in the second $\Pi_\infty[\mathfrak{m}]$ in (5). Consequently, a certain subextension of $\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)$ by $\pi_1(\underline{\alpha},\lambda)$, of $j^-(\widetilde{D}_1)\oplus_{\pi_1(\underline{\alpha},\lambda)} j^+(\widetilde{C}_1)$, has to inject into $\Pi_\infty[\mathfrak{m}]$. Letting $(\widetilde{D}_1,\widetilde{C}_1)$ vary, it is not so difficult to see the subextension has to be $\pi(\underline{\alpha},\lambda,\iota_D)$. See the proof of Theorem 4.14 for details.

Remark that our arguments in fact do not rely on the full strength of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (for example, we don't use Colmez's functor). Most of the arguments can generalize to $\mathrm{GL}_n(\mathbb{Q}_p)$ (or even $\mathrm{GL}_n(K)$ for a finite extension K of $\mathbb{Q}_p$). We will report the GL_n-case in an upcoming work, which of course covers the results in the note. However, we find it convenient (for both the reader and the author) to first discuss the $\mathrm{GL}_3(\mathbb{Q}_p)$-case.

We refer to the body of the context for more detailed and precise statements with slightly different notation.

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2 Hodge filtration and higher intertwining

Let D be a (φ,Γ)-module of rank 3 over $\mathcal{R}_E$, the E-coefficient Robba ring for $\mathbb{Q}_p$. Assume D is crystabelline of regular Hodge-Tate-Sen weights $h=(h_1>h_2>h_3)$. We discuss the relation between the Hodge filtration of D and certain paraboline deformations of D. Throughout the section, we write Ext^i (and $\mathrm{Hom}=\mathrm{Ext}^0$) without "$(\varphi,\Gamma)$" in the subscript for the i-th extension group of (φ,Γ)-modules (cf. [23]).

2.1 Hodge filtration

As D is crystabelline, there exist smooth characters ϕ_1,ϕ_2,ϕ_3 of $\mathbb{Q}_p^\times$ such that $D[\frac{1}{t}]\cong\oplus_{i=1}^3\mathcal{R}_E(\phi_i)[\frac{1}{t}]$. We assume D is *generic*, that is $\phi_i\phi_j^{-1}\neq 1,|\cdot|^{\pm 1}$ for $i\neq j$. An ordering of ϕ_1, ϕ_2, ϕ_3 is refereed to as a *refinement* of D. For $w\in S_3$, the refinement $(\phi_{w^{-1}(1)},\phi_{w^{-1}(2)},\phi_{w^{-1}(3)})$ is called *non-critical* if D admits a triangulation of parameter $w(\underline{\phi})z^h$, where $w(\underline{\phi})=\phi_{w^{-1}(1)}\boxtimes\phi_{w^{-1}(2)}\boxtimes\phi_{w^{-1}(3)}$ is the associated smooth character of $T(\mathbb{Q}_p)$. We recall it means D is isomorphic to a successive extension of $\mathcal{R}_E(\phi_{w^{-1}(i)}z^{h_i})$ for $i=1,2,3$. We also call $w(\underline{\phi})$ a refinement of D. Throughout the note, we assume the following hypothesis

Hypothesis 2.1. *Assume all the refinements of D are non-critical.*

Let D_1 (resp. C_1) be the submodule of D (resp. the quotient of D) given by an extension of $\mathcal{R}_E(z^{h_2}\phi_2)$ (resp. $\mathcal{R}_E(z^{h_3}\phi_2)$) by $\mathcal{R}_E(z^{h_1}\phi_1)$ (resp. by $\mathcal{R}_E(z^{h_2}\phi_1)$). By the assumption, it is easy to see both D_1 and C_1 are non-split. Denote by ι_D the composition $D_1\hookrightarrow D\twoheadrightarrow C_1$, and it is clear that ι_D is injective (using $\mathrm{Hom}(D_1,\mathcal{R}_E(z^{h_1}\phi_3))=0$).

Proposition 2.2. *(1)* $\dim_E\mathrm{Hom}(D_1,C_1)=2$.

(2) The cup-product

$$\mathrm{Ext}^1(\mathcal{R}_E(\phi_3z^{h_3}),D_1)\times\mathrm{Hom}(D_1,C_1)\longrightarrow\mathrm{Ext}^1(\mathcal{R}_E(\phi_3z^{h_3}),C_1)$$

is non-degenerate. Moreover, $[D]$ is exactly the line orthogonal to $[\iota_D]$. In particular, D is determined by $\underline{\phi}$, h and ι_D.

Proof. Consider $C_1\otimes_{\mathcal{R}_E}D_1^\vee$, which is isomorphic to a successive extension of $\mathcal{R}_E(\phi_i\phi_j^{-1}z^{h_{i+1}-h_j})$ for $i,j\in\{1,2\}$. By dévissage, it is easy to see $\dim_E\mathrm{Hom}(D_1,C_1)\leq 2$. For $i=1,2$, let

$$\alpha_i:=D_1\longrightarrow\!\!\!\!\!\rightarrow\mathcal{R}_E(z^{h_2}\phi_i)\hookrightarrow C_1 \tag{6}$$

which are obviously linearly independent. (1) follows.

We have a commutative diagram of cup-products

$$\begin{array}{ccccc}
\mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_1 z^{h_2})) & \times & \mathrm{Hom}(\mathcal{R}_E(\phi_1 z^{h_2}), \mathcal{R}_E(\phi_1 z^{h_2})) & \longrightarrow & \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_1 z^{h_2})) \\
\uparrow & & \sim\downarrow & & \| \\
\mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D_1) & \times & \mathrm{Hom}(D_1, \mathcal{R}_E(\phi_1 z^{h_2})) & \longrightarrow & \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_1 z^{h_2})) \\
\| & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D_1) & \times & \mathrm{Hom}(D_1, C_1) & \longrightarrow & \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), C_1).
\end{array}$$

The top pairing (of one dimensional spaces) is trivially perfect. We deduce

$$[\alpha_1]^\perp = \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_2 z^{h_1}))(\subset \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D_1)).$$

Similarly, we have $[\alpha_2]^\perp = \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_1 z^{h_1}))$. In particular, $[\alpha_1]^\perp$ and $[\alpha_2]^\perp$ are linearly disjoint. The first part of (2) follows. As ι_D factors through D, the map induced by the pairing $\langle -, \iota_D \rangle$ is given by the following composition

$$\mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D_1) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), C_1).$$

The first map sends $[D]$ to zero, hence $\langle D, \iota_D \rangle = 0$. This finishes the proof. □

Remark 2.3. *By the proposition, the map ι_D determines the Hodge filtration of D. We can actually see this in an explicit way. For simplicity, we assume D is crystalline. In this case, $\phi_i = \mathrm{unr}(\alpha_i)$ where α_i are the eigenvalues of φ on $D_{\mathrm{cris}}(D)$. Let $e_i \in D_{\mathrm{cris}}(D)$ be an α_i-eigenvector. As D_1 is non-split, by multiplying e_1, e_2 by non-zero scalars, we can and do assume $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_1) = \mathrm{Fil}^j D_{\mathrm{cris}}(D_1)$, $-h_1 < j \leq -h_2$, is generated by $e_1 + e_2$. As D is non-critical for all refinements, by multiplying e_3 by a non-zero scalar, we can and do assume $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D) = \mathrm{Fil}^j D_{\mathrm{cris}}(D)$, $-h_2 < j \leq -h_3$, is generated by $e_1 + a_D e_2 + e_3$ for some $a_D \in E^\times$. The Hodge filtration on $D_{\mathrm{cris}}(D)$ is in fact determined by the parameter a_D:*

$$\mathrm{Fil}^j D_{\mathrm{cris}}(D) = \begin{cases} D_{\mathrm{cris}}(D) & j \leq -h_1 \\ E(e_1 + e_2) \oplus E(e_1 + a_D e_2 + e_3) & -h_1 < j \leq -h_2 \\ E(e_1 + a_D e_2 + e_3) & -h_2 < j \leq -h_3 \\ 0 & j > -h_3 \end{cases}$$

With these basis, $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(C_1) = \mathrm{Fil}^j D_{\mathrm{cris}}(C_1)$, $-h_2 < j \leq -h_3$, is generated by $e_1 + a_D e_2$. The map ι_D induces a map $D_{\mathrm{cris}}(D_1) \to D_{\mathrm{cris}}(C_1)$ which sends exactly e_i to e_i. We see a_D can be read out from the relative position of the two lines $\iota_D(\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_1))$ and $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(C_1)$. It is then not difficult to see a_D and ι_D determine each other.

2.2 Higher intertwining pairs

We show that ι_D in $\mathrm{Hom}(D_1, C_1)$ can be detected by a set of pairs of certain deformations of D_1 and C_1, associated to D.

For de Rham (φ, Γ)-modules M and N, let $\mathrm{Ext}^1_g(M, N)$ denote the subspace of de Rham extensions. We identify each element in $\mathrm{Ext}^1(M, M)$ with a (φ, Γ)-module $\widetilde{M}$ over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ where ϵ acts via $\widetilde{M} \twoheadrightarrow M \xrightarrow{\mathrm{id}} M \hookrightarrow M$. Let $\mathrm{Ext}^1_{g'}(M, M)$ be the subspace of de Rham deformations up to twist by characters (over $E[\epsilon]/\epsilon^2$).

We quickly recall some facts on the deformations of D_1 (similar statements holding for C_1 as well). Let $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) be the filtration $\mathcal{R}_E(\phi_1 z^{h_1}) \subset D_1$ (resp. $\mathcal{R}_E(\phi_2 z^{h_1}) \subset D_1$), and $\mathrm{Ext}^1_{\mathscr{F}_i}(D_1, D_1)$ be subspace of trianguline deformations with respect to the filtration $\mathscr{F}_i$. For $\widetilde{D}_1 \in \mathrm{Ext}^1_{\mathscr{F}_i}(D_1, D_1)$ (viewed as a (φ,Γ)-module over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$), there exist characters $\widetilde{\delta}_{\mathscr{F}_i,1}, \widetilde{\delta}_{\mathscr{F}_i,2} : \mathbb{Q}_p^\times \to (E[\epsilon]/\epsilon^2)^\times$, $\widetilde{\delta}_{\mathscr{F}_i,1} \equiv \delta_{\mathscr{F}_i,1} := \phi_i z^{h_1} \pmod{\epsilon}$, and $\widetilde{\delta}_{\mathscr{F}_i,2} \equiv \delta_{\mathscr{F}_i,2} := \phi_j z^{h_2}$ (for $j \neq i$) such that $\widetilde{D}_1$ is isomorphic, as (φ,Γ)-module over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$, to an extension of $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_{\mathscr{F}_i,2})$ by $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_{\mathscr{F}_i,1})$. We call the character $\widetilde{\delta}_{\mathscr{F}_i} := \widetilde{\delta}_{\mathscr{F}_i,1} \boxtimes \widetilde{\delta}_{\mathscr{F}_i,2}$ of $T(\mathbb{Q}_p)$ a trianguline parameter of $\widetilde{D}_1$. The following proposition is well-known.

Proposition 2.4. *(1)* $\dim_E \mathrm{Ext}^1(D_1, D_1) = 5$, $\dim_E \mathrm{Ext}^1_g(D_1, D_1) = 2$ *and* $\dim_E \mathrm{Ext}^1_{\mathscr{F}_i}(D_1, D_1) = 4$ *for* $i = 1, 2$.

(2) There is a natural exact sequence

$$0 \to \mathrm{Ext}^1_{g'}(D_1, D_1) \to \mathrm{Ext}^1_{\mathscr{F}_1}(D_1, D_1) \oplus \mathrm{Ext}^1_{\mathscr{F}_2}(D_1, D_1) \to \mathrm{Ext}^1(D_1, D_1) \to 0.$$

(3) For $i = 1, 2$, *the map* $\mathrm{Ext}^1_{\mathscr{F}_i}(D_1, D_1) \to \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i})$, *sending* $\widetilde{D}_1$ *to its trianguline parameter, is a bijection, where* $\delta_{\mathscr{F}_i} := \delta_{\mathscr{F}_i,1} \boxtimes \delta_{\mathscr{F}_i,2}$.

Remark 2.5. *For* $\widetilde{D}_1 \in \mathrm{Ext}^1_{g'}(D_1, D_1)$, *there exist a continuous character* $\psi \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$ *and smooth characters* $\psi_i \in \mathrm{Hom}_{\mathrm{sm}}(\mathbb{Q}_p^\times, E)$ *such that both* $\phi_1 z^{h_1}(1+(\psi_1+\psi/2)\epsilon) \boxtimes \phi_2 z^{h_2}(1+(\psi_2+\psi/2)\epsilon)$ *and* $\phi_2 z^{h_1}(1+(\psi_2+\psi/2)\epsilon) \boxtimes \phi_1 z^{h_2}(1+(\psi_1+\psi/2)\epsilon)$ *are trianguline parameters of* $\widetilde{D}_1$. *We denote by*

$$\kappa : \mathrm{Ext}^1_{g'}(D_1, D_1) \longrightarrow \mathrm{Hom}(\mathbb{Q}_p^\times, E) \times \mathrm{Hom}_{\mathrm{sm}}(\mathbb{Q}_p^\times, E)^{\oplus 2} \tag{7}$$

the map sending $\widetilde{D}_1$ *to* (ψ, ψ_1, ψ_2). *Remark that the map is compatible with the map in (3).*

Now let $\mathscr{F}$ be the filtration $D_1 \subset D$. A deformation $\widetilde{D}$ of D over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ is called an $\mathscr{F}$-*deformation*, if there exist a deformation $\widetilde{D}_1$ of D_1 over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ and a deformation $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_3 z^{h_3}(1+\psi\epsilon))$ of $\mathcal{R}_E(\phi_3 z^{h_3})$ over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ (where $\psi \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$), such that $\widetilde{D}$ is isomorphic to an extension of $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_3 z^{h_3}(1+\psi\epsilon))$ by $\widetilde{D}_1$. Denote by $\mathrm{Ext}^1_{\mathscr{F}}(D, D) \subset \mathrm{Ext}^1(D, D)$ the subspace of $\mathscr{F}$-deformations.

Lemma 2.6. *The subspace* $\mathrm{Ext}^1_{\mathscr{F}}(D, D)$ *is the kernel of the following composition of surjective maps*

$$\mathrm{Ext}^1(D, D) \twoheadrightarrow \mathrm{Ext}^1(D_1, D) \twoheadrightarrow \mathrm{Ext}^1(D_1, \mathcal{R}_E(\phi_3 z^{h_3})). \tag{8}$$

In particular, $\dim_E \mathrm{Ext}^1_{\mathscr{F}}(D, D) = 8$.

Proof. The surjectivity follows easily from dévissage and the fact D is generic. The first part follows by definition. We have $\dim_E \mathrm{Ext}^1(D, D) = 10$ and $\dim_E \mathrm{Ext}^1(D_1, \mathcal{R}_E(\phi_3 z^{h_3})) = 2$. The second part follows. □

Proposition 2.7. *There is a natural exact sequence* ($\mathcal{R}$ *for* $\mathcal{R}_E$)

$$0 \to \mathrm{Ext}^1(\mathcal{R}(\phi_3 z^{h_3}), D_1)/E([D]) \xrightarrow{j} \mathrm{Ext}^1_{\mathscr{F}}(D, D) \to \mathrm{Ext}^1(D_1, D_1) \times \mathrm{Ext}^1(\mathcal{R}(\psi_3 z^{h_3}), \mathcal{R}(\psi_3 z^{h_3})) \to 0.$$

Proof. By Lemma 2.6 and dévissage, there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_3 z^{h_3})) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D_1) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D) \longrightarrow \mathrm{Ext}^1_{\mathscr{F}}(D, D) \longrightarrow \mathrm{Ext}^1(D_1, D_1) \to 0.$$

The image of the first map is $E([D])$, and the cokernel of the second map is $\mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_3 z^{h_3}))$. Letting V be the cokernel of j, we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_3 z^{h_3})) \longrightarrow V \longrightarrow \mathrm{Ext}^1(D_1, D_1) \longrightarrow 0.$$

Meanwhile, the composition (8) factors through

$$\mathrm{Ext}^1(D, D) \twoheadrightarrow \mathrm{Ext}^1(D, \mathcal{R}_E(\phi_3 z^{h_3}) \twoheadrightarrow \mathrm{Ext}^1(D_1, \mathcal{R}_E(\phi_3 z^{h_3})).$$

We deduce an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(D_1, D_1) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), D_1) \\ \longrightarrow \mathrm{Ext}^1(D, D_1) \longrightarrow \mathrm{Ext}^1_{\mathscr{F}}(D, D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_3 z^{h_3})) \longrightarrow 0. \end{aligned} \quad (9)$$

Hence V is actually split. The proposition follows. □

Remark 2.8. *(1) We identify* $\mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_3 z^{h_3}))$ *with* $\mathrm{Hom}(\mathbb{Q}_p^\times, E)$*, and denote by* $\kappa_{\mathscr{F}}$ *the induced surjection*

$$\kappa_{\mathscr{F}} = (\kappa_{\mathscr{F},1}, \kappa_{\mathscr{F},2}) : \mathrm{Ext}^1_{\mathscr{F}}(D, D) \twoheadrightarrow \mathrm{Ext}^1(D_1, D_1) \times \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_3}), \mathcal{R}_E(\phi_3 z^{h_3})).$$

For $\widetilde{D} \in \mathrm{Ext}^1_{\mathscr{F}}(D, D)$ *of the form* $[\widetilde{D}_1 — \mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_3 z^{h_3}(1+\psi\epsilon))]$*,* $\kappa_{\mathscr{F}}$ *sends* $\widetilde{D}$ *to* $(\widetilde{D}_1, \psi)$*.*

(2) Let $\mathscr{G}$ *be the filtration* $\mathcal{R}_E(\phi_3 z^{h_1}) \subset D$*, and fix* $\mathcal{R}_E(\phi_3 z^{h_1}) \hookrightarrow D \twoheadrightarrow C_1$*. We define* $\mathscr{G}$*-deformations of* D *in a similar way, and denote by* $\mathrm{Ext}^1_{\mathscr{G}}(D, D)$ *the subspace of* $\mathscr{G}$*-deformations, which is of dimension* 8*. Similarly, we have an exact sequence*

$$0 \to \mathrm{Ext}^1(C_1, \mathcal{R}_E(\phi_3 z^{h_1}))/E([D]) \to \mathrm{Ext}^1_{\mathscr{G}}(D, D) \xrightarrow{\kappa_{\mathscr{G}}} \mathrm{Ext}^1(C_1, C_1) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) \to 0.$$

For $\iota \in \mathrm{Hom}(D_1, C_1)$. Consider the pull-back and push-forward maps:

$$\iota^- : \mathrm{Ext}^1(C_1, D_1) \longrightarrow \mathrm{Ext}^1(D_1, D_1), \;\; \iota^+ : \mathrm{Ext}^1(C_1, D_1) \longrightarrow \mathrm{Ext}^1(C_1, C_1).$$

Put

$$\mathrm{Ext}^1_{\iota}(D_1, D_1) := \iota^-(\mathrm{Ext}^1(C_1, D_1)), \;\; \mathrm{Ext}^1_{\iota}(C_1, C_1) := \iota^+(\mathrm{Ext}^1(C_1, D_1)).$$

Lemma 2.9. *For* $i \in \{1, 2\}$*, we have* $\dim_E \mathrm{Ext}^1_{\alpha_i}(D_1, D_1) = 2$*. Moreover,*

$$\mathrm{Ext}^1_{\alpha_1}(D_1, D_1) \cap \mathrm{Ext}^1_{\alpha_2}(D_1, D_1) = 0.$$

The same holds with D_1 *replaced by* C_1*.*

Proof. We only prove it for D_1, with C_1 being similar. The map α_i^- factors through

$$\mathrm{Ext}^1(C_1, D_1) \twoheadrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_i z^{h_2}), D_1) \hookrightarrow \mathrm{Ext}^1(D_1, D_1)$$

where the corresponding surjectivity and injectivity follow easily by dévissage. The first part then follows from the fact

$$\dim_E \mathrm{Ext}^1(\mathcal{R}_E(\phi_i z^{h_2}), D_1) = 2.$$

We also see $\mathrm{Ext}^1_{\alpha_i}(D_1, D_1)$ is just the kernel of $\mathrm{Ext}^1(D_1, D_1) \to \mathrm{Ext}^1(\mathcal{R}_E(\phi_j z^{h_1}), D_1)$ for $j \neq i$. We then easily deduce $\mathrm{Ext}^1_{\alpha_1}(D_1, D_1) \cap \mathrm{Ext}^1_{\alpha_2}(D_1, D_1) = 0$. □

Proposition 2.10. *Let $\iota \in \mathrm{Hom}(D_1, C_1)$ be an injection.*

(1) $\dim_E \mathrm{Ext}^1_\iota(D_1, D_1) = \dim_E \mathrm{Ext}^1_\iota(C_1, C_1) = 3$.

(2) $\mathrm{Ext}^1_g(D_1, D_1) \subset \mathrm{Ext}^1_\iota(D_1, D_1)$ *and* $\mathrm{Ext}^1_g(C_1, C_1) \subset \mathrm{Ext}^1_\iota(C_1, C_1)$. *Moreover, any trianguline deformation in* $\mathrm{Ext}^1_\iota(D_1, D_1)$ *(resp. in* $\mathrm{Ext}^1_\iota(C_1, C_1)$*) is de Rham.*

(3) For $\iota' \in \mathrm{Hom}(D_1, C_1)$, $\mathrm{Ext}^1_{\iota'}(D_1, D_1) = \mathrm{Ext}^1_\iota(D_1, D_1)$ *if and only if* $\mathrm{Ext}^1_{\iota'}(C_1, C_1) = \mathrm{Ext}^1_\iota(C_1, C_1)$ *if and only if* $\iota' = a\iota$ *for some* $a \in E^\times$.

Proof. We only prove it for D_1 with C_1 being similar. Note first $\dim_E \mathrm{Ext}^1(C_1, D_1) = 4$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1(C_1, \mathcal{R}_E(\phi_1 z^{h_1})) & \longrightarrow & \mathrm{Ext}^1(C_1, D_1) & \longrightarrow & \mathrm{Ext}^1(C_1, \mathcal{R}_E(\phi_2 z^{h_2})) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \iota^-} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}^1(D_1, \mathcal{R}_E(\phi_1 z^{h_1})) & \longrightarrow & \mathrm{Ext}^1(D_1, D_1) & \longrightarrow & \mathrm{Ext}^1(D_1, \mathcal{R}_E(\phi_2 z^{h_2})) & \longrightarrow & 0. \end{array} \tag{10}$$

By dévissage and using [6, Lem. 5.1.1], it is not difficult to see the right vertical map is injective. We can furthermore dévissage the left vertical map of (10):

$$\begin{array}{ccccc} \mathrm{Ext}^1(\mathcal{R}_E(\phi_2 z^{h_3}), \mathcal{R}_E(\phi_1 z^{h_1})) & \longrightarrow & \mathrm{Ext}^1(C_1, \mathcal{R}_E(\phi_1 z^{h_1})) & \longrightarrow & \mathrm{Ext}^1(\mathcal{R}_E(\phi_1 z^{h_2}), \mathcal{R}_E(\phi_1 z^{h_1})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}^1(\mathcal{R}_E(\phi_2 z^{h_2}), \mathcal{R}_E(\phi_1 z^{h_1})) & \xrightarrow{0} & \mathrm{Ext}^1(D_1, \mathcal{R}_E(\phi_1 z^{h_1})) & \longrightarrow & \mathrm{Ext}^1(\mathcal{R}_E(\phi_1 z^{h_1}), \mathcal{R}_E(\phi_1 z^{h_1})). \end{array}$$

As $h_2 < h_1$, the image of the right vertical map is exactly $\mathrm{Ext}^1_g(\mathcal{R}_E(\phi_1 z^{h_1}), \mathcal{R}_E(\phi_1 z^{h_1}))$, which is one dimensional. Together with $\dim_E \mathrm{Ext}^1(C_1, \mathcal{R}_E(\phi_2 z^{h_2})) = 2$, (1) follows.

For a (φ, Γ)-module M over $\mathcal{R}_E$, denote by $W^+_{\mathrm{dR}}(M)$ the associated $B^+_{\mathrm{dR},E}$-representation of $\mathrm{Gal}_{\mathbb{Q}_p}$, where $B^+_{\mathrm{dR},E} := B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$. We have a tautological exact sequence

$$H^1_g(D_1 \otimes_{\mathcal{R}_E} C_1^\vee) \hookrightarrow H^1(D_1 \otimes_{\mathcal{R}_E} C_1^\vee) \longrightarrow H^1(\mathrm{Gal}_K, W^+_{\mathrm{dR}}(D_1 \otimes_{\mathcal{R}_E} C_1^\vee)).$$

It is not difficult to see $\dim_E H^1(\mathrm{Gal}_K, W^+_{\mathrm{dR}}(D_1 \otimes_{\mathcal{R}_E} C_1^\vee)) = 1$, hence $\dim_E \mathrm{Ext}^1_g(C_1, D_1) = \dim_E H^1_g(D_1 \otimes_{\mathcal{R}_E} C_1^\vee) \geq 3$. As ι^- obviously induces

$$\iota^-_g : \mathrm{Ext}^1_g(C_1, D_1) \longrightarrow \mathrm{Ext}^1_g(D_1, D_1),$$

by comparing the dimensions, we see $\mathrm{Ker}\, \iota^-_g = \mathrm{Ker}\, \iota^-$ is one dimensional and ι^-_g is surjective (and $\dim_E \mathrm{Ext}^1_g(C_1, D_1) = 3$).

We have seen $\mathrm{Ext}^1_g(D_1, D_1) \subset \mathrm{Ext}^1_\iota(D_1, D_1) \cap \mathrm{Ext}^1_{\mathscr{F}_1}(D_1, D_1)$. If it is not an equality, by comparing the dimension, $\mathrm{Ext}^1_\iota(D_1, D_1) \cap \mathrm{Ext}^1_{\mathscr{F}_1}(D_1, D_1) = \mathrm{Ext}^1_\iota(D_1, D_1)$. Let $\widetilde{D}_1 \in \mathrm{Ext}^1_\iota(D_1, D_1) \cap \mathrm{Ext}^1_{\mathscr{F}_1}(D_1, D_1)$. By (10), the image of $\widetilde{D}_1$ in $\mathrm{Ext}^1(D_1, \mathcal{R}_E(\phi_2 z^{h_2}))$ lies in $\mathrm{Ext}^1(C_1, \mathcal{R}_E(\phi_2 z^{h_2})) \cap \mathrm{Ext}^1(\mathcal{R}_E(\phi_2 z^{h_2}), \mathcal{R}_E(\phi_2 z^{h_2}))$, which, by [6, Lem. 5.5.9], is isomorphic to the one dimensional $\mathrm{Ext}^1_g(\mathcal{R}_E(\phi_2 z^{h_2}), \mathcal{R}_E(\phi_2 z^{h_2}))$ hence is not equal to the whole $\mathrm{Ext}^1(C_1, \mathcal{R}_E(\phi_2 z^{h_2}))$. Thus we have $\mathrm{Ext}^1_\iota(D_1, D_1) \cap \mathrm{Ext}^1_{\mathscr{F}_1}(D_1, D_1) \neq \mathrm{Ext}^1_\iota(D_1, D_1)$. The same holds with $\mathscr{F}_1$ replaced by $\mathscr{F}_2$, and (2) follows.

Finally, consider the cup-product

$$\mathrm{Ext}^1(C_1, D_1) \times \mathrm{Hom}(D_1, C_1) \longrightarrow \mathrm{Ext}^1(D_1, D_1).$$

Suppose $\iota' \notin E[\iota]$, then ι' and ι form a basis of $\mathrm{Hom}(D_1, C_1)$. If $\mathrm{Ext}^1_{\iota'}(D_1, D_1) = \mathrm{Ext}^1_{\iota}(D_1, D_1)$, we then easily deduce $\mathrm{Ext}^1_{\alpha_i}(D_1, D_1) \subset \mathrm{Ext}^1_{\iota}(D_1, D_1)$ for all $i = \{1, 2\}$. However, by Lemma 2.9, $\dim_E \big(\mathrm{Ext}^1_{\alpha_1}(D_1, D_1) + \mathrm{Ext}^1_{\alpha_2}(D_1, D_1)\big) = 4$, contradiction. □

Denote by $\kappa : \mathrm{Ext}^1_g(D_1, D_1) \to \mathrm{Hom}_{\mathrm{sm}}(\mathbb{Q}_p^\times, E)^{\oplus 2}$ the (bijective) map induced by (7). And we have a similar bijection $\kappa : \mathrm{Ext}^1_g(C_1, C_1) \twoheadrightarrow \mathrm{Hom}_{\mathrm{sm}}(\mathbb{Q}_p^\times, E)^2$.

Proposition 2.11. *For $M \in \mathrm{Ext}^1_g(C_1, D_1)$, $\kappa \circ \iota_g^-(M) = \kappa \circ \iota_g^+(M)$.*

Proof. By definition, there is a natural injection $\tilde{\iota} : \iota_g^-(M) \hookrightarrow \iota_g^+(M)$ which sits in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1 & \longrightarrow & \iota_g^-(M) & \longrightarrow & D_1 & \longrightarrow & 0 \\ & & \iota\big\downarrow & & \tilde{\iota}\big\downarrow & & \iota\big\downarrow & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & \iota_g^+(M) & \longrightarrow & C_1 & \longrightarrow & 0. \end{array}$$

It is easy to see $\tilde{\iota}$ is moreover $\mathcal{R}_{E[\epsilon]/\epsilon^2}$-linear if $\iota_g^-(M)$ and $\iota_g^+(M)$ are equipped with the natural $\mathcal{R}_{E[\epsilon]/\epsilon^2}$-action. Suppose $\kappa \circ \iota_g^-(M) = (\psi_1, \psi_2)$ and $\kappa \circ \iota_g^+(M) = (\psi_1', \psi_2')$. Then $\iota_g^-(M)$ (resp. $\iota_g^+(M)$) is isomorphic, as (φ, Γ)-module over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$, to an extension of $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_2 z^{h_2}(1 + \psi_2\epsilon))$ $\big($resp. of $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_2 z^{h_3}(1 + \psi_2'\epsilon))\big)$ by $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_1 z^{h_1}(1 + \psi_1\epsilon))$ $\big($resp. $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_1 z^{h_2}(1 + \psi_1'\epsilon))\big)$. It is not difficult to see $\tilde{\iota}$ induces injections $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_i z^{h_i}(1 + \psi_i\epsilon)) \hookrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\phi_i z^{h_{i+1}}(1 + \psi_i'\epsilon))$ of (φ, Γ)-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$. Hence $\psi_i = \psi_i'$ for $i = 1, 2$. □

For an injection $\iota \in \mathrm{Hom}(D_1, C_1)$, we define $\mathscr{I}_\iota$ to be following set:

$$\{(\widetilde{D}_1, \widetilde{C}_1) \in \mathrm{Ext}^1_\iota(D_1, D_1) \times \mathrm{Ext}^1_\iota(C_1, C_1) \mid \exists M \in \mathrm{Ext}^1(C_1, D_1) \text{ with } \iota^-(M) = \widetilde{D}_1, \iota^+(M) = \widetilde{C}_1\}.$$

If $\iota = \iota_D$ for some D as in § 2.1, we write $\mathscr{I}_D := \mathscr{I}_{\iota_D}$. By Proposition 2.10 (3) and Proposition 2.2 (2), we have:

Corollary 2.12. *We have $\mathscr{I}_\iota = \mathscr{I}_{\iota'}$ if and only if $\iota' = a\iota$ for some $a \in E^\times$. In particular, for non critical D and D', which both have Sen weights (h_1, h_2, h_3) and a refinement ϕ, then $\mathscr{I}_D = \mathscr{I}_{D'}$ if and only if $D \cong D'$.*

Now let D be as in § 2.1, and $\iota_D : D_1 \hookrightarrow C_1$ be the associated injection.

Theorem 2.13 (Higher intertwining)**.** *Let $\widetilde{D} \in \mathrm{Ext}^1_{\mathscr{F}}(D, D)$ with $\kappa_{\mathscr{F}}(\widetilde{D}) = (\widetilde{D}_1, \psi)$. The followings are equivalent:*

1. *$\widetilde{D} \in \mathrm{Ext}^1_{\mathscr{F}}(D, D) \cap \mathrm{Ext}^1_{\mathscr{G}}(D, D)$.*
2. *$\widetilde{D}_1 \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(1 - \psi\epsilon) \in \mathrm{Ext}^1_{\iota_D}(D_1, D_1)$.*

Moreover, if the equivalent conditions hold, then $\kappa_{\mathscr{G},2}(\widetilde{D}) = \psi$ and there exists $M \in \mathrm{Ext}^1(C_1, D_1)$ such that $\widetilde{D}_1 = \iota_D^-(M) \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(1 + \psi\epsilon)$ and $\kappa_{\mathscr{G},1}(\widetilde{D}) = \iota_D^+(M) \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(1 + \psi\epsilon)$.

Proof. Twisting by $1 - \psi\epsilon$, we assume $\kappa_{\mathscr{F},2}(\widetilde{D}) = 0$. Using a similar statement in Lemma 2.6 (for $\mathscr{G}$), $\widetilde{D} \in \mathrm{Ext}^1_{\mathscr{G}}(D, D)$ if and only if it lies in the kernel of the composition

$$\mathrm{Ext}^1(D, D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_1}), D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_1}), C_1). \tag{11}$$

As $\kappa_{\mathscr{F},2}(\widetilde{D}) = 0$, $\widetilde{D}$ lies in the image of $\mathrm{Ext}^1(D, D_1) \to \mathrm{Ext}^1_{\mathscr{F}}(D, D)$ (see (9)), and we let $M_1 \in \mathrm{Ext}^1(D, D_1)$ be a preimage of $\widetilde{D}$. Consider the composition

$$\mathrm{Ext}^1(D, D_1) \hookrightarrow \mathrm{Ext}^1(D, D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_1}), D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_1}), C_1).$$

It is straightforward to see it is equal to the composition

$$\mathrm{Ext}^1(D, D_1) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_1}), D_1) \xrightarrow{\iota_D} \mathrm{Ext}^1(\mathcal{R}_E(\phi_3 z^{h_1}), C_1). \tag{12}$$

So $\widetilde{D}$ lies in the kernel of (11) if and only if M_1 is sent to zero via (12). However, using dévissage and [6, Lem. 5.1.1], the push-forward map ι_D in (12) is injective. We see (under the assumption $\psi = 0$) the condition 1 is equivalent to that M_1 lies in the kernel of the first map of (12), that is equal to $\mathrm{Ext}^1(C_1, D_1)$ by dévissage. This is furthermore equivalent to that $\widetilde{D}_1$ lies in the image of the composition

$$\mathrm{Ext}^1(C_1, D_1) \hookrightarrow \mathrm{Ext}^1(D, D_1) \longrightarrow \mathrm{Ext}^1(D_1, D_1), \tag{13}$$

which is no other than the pull-back map induced by ι_D. The other parts are straightforward. □

3 Locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$

Let D be as in § 2.1, and let $\lambda = (\lambda_1, \lambda_2, \lambda_3) := (h_1 - 2, h_2 - 1, h_3)$. We associate to D a locally analytic representation $\pi(\underline{\phi}, \lambda, \iota_D)$ of $\mathrm{GL}_3(\mathbb{Q}_p)$. We show that the construction gives a one-to-one correspondence between $\{\pi(\underline{\phi}, \lambda, \iota_D)\}$ and $\{D\}$.

3.1 Preliminaries and notation

For a smooth character $\delta = \delta_1 \boxtimes \delta_2 \boxtimes \delta_3 : T(\mathbb{Q}_p) \to E^\times$, denote by $\jmath(\delta) := \delta_1 |\cdot|^{-2} \boxtimes \delta_2 |\cdot|^{-1} \boxtimes \delta_3$, where $|\cdot|$ is the p-adic norm with $|p| = p^{-1}$. Let $\varepsilon := z|\cdot| : \mathbb{Q}_p^\times \to E^\times$ be the cyclotomic character.

Let T be the torus subgroup of GL_3 , $B \supset T$ be the Borel subgroup of upper triangular matrices. For a standard parabolic subgroup P of GL_3 (containing B), denote by P^- its opposite parabolic subgroup. For a weight μ of $\mathfrak{gl}_3$, denote by $M^-(\mu) := \mathrm{U}(\mathfrak{gl}_3) \otimes_{\mathrm{U}(\mathfrak{b}^-)} \mu$ (with $\mathfrak{b}^-$ the Lie algebra of B^-), and let $L^-(\mu)$ be its unique simple quotient. If μ is anti-dominant, then $L^-(\mu)$ is finite dimensional isomorphic to $L(-\mu)^\vee$, where $L(-\mu)$ is the algebraic representation of highest weight $-\mu$ with respect to B. We use the same notation for GL_2 when there is no ambiguity.

Let D be as in the Section 2.1. For a refinement $w(\underline{\phi})$ of D, consider the locally algebraic representation $(\mathrm{Ind}_{B^-}^{\mathrm{GL}_3} \jmath(w(\underline{\phi})))^\infty \otimes_E L(\lambda)$, which turns out to be all isomorphic, that we denote by $\pi_{\mathrm{alg}}(\underline{\phi}, \lambda)$. In fact, the smooth induction $(\mathrm{Ind}_{B^-}^{\mathrm{GL}_3} \jmath(\underline{\phi}))^\infty$ is just the representation corresponding to the Weil-Deligne representation $\oplus_{i=1}^3 \phi_3$ via the classical local Langlands correspondence.

For $w \in S_3$ and a simple reflection s_i, put (where "$\mathcal{F}$" is the notation for Orlik-Strauch representations as in [25])

$$\mathscr{C}(s_i, w) := \mathcal{F}_{B^-}^{\mathrm{GL}_3}\big(L^-(-s_i \cdot \lambda), \jmath(w(\underline{\phi}))\big) \cong \mathcal{F}_{P_j^-}^{\mathrm{GL}_3}\big(L^-(-s_i \cdot \lambda), (\mathrm{Ind}_{B^- \cap L_j}^{L_j} \jmath(w(\underline{\phi})))^\infty\big),$$

where $j \neq i$, $P_1^- = \begin{pmatrix} \mathrm{GL}_2 & 0 \\ * & \mathrm{GL}_1 \end{pmatrix}$ and $P_2^- = \begin{pmatrix} \mathrm{GL}_1 & 0 \\ * & \mathrm{GL}_2 \end{pmatrix}$. By [25, Thm.], $\mathscr{C}(s_i, w) \cong \mathscr{C}(s_k, w')$ if and only if $s_i = s_k$ and $w' = s_j w$ with $s_j \neq s_i$. Letting s_i and w vary and assuming w has minimal length in the class $\{w, s_j w\}_{s_j \neq s_i}$, there are exactly 6 distinct representations $\mathcal{S} := \{\mathscr{C}(s_1, 1), \mathscr{C}(s_1, s_1), \mathscr{C}(s_1, s_1 s_2), \mathscr{C}(s_2, 1), \mathscr{C}(s_2, s_2), \mathscr{C}(s_2, s_1 s_2)\}$.

For $w\in S_3$, consider the principal series

$$(\mathrm{Ind}_{B^-}^{\mathrm{GL}_3} \jmath(w(\underline{\phi}))z^{\lambda})^{\mathrm{an}} \cong \mathcal{F}_{B^-}^{\mathrm{GL}_3}\big(M^-(-\lambda), \jmath(w(\underline{\phi}))\big).$$

It contains a unique subrepresentation $\pi_1(\underline{\phi},\lambda,w)$ of the form $[\pi_{\mathrm{alg}}(\underline{\phi},\lambda)\text{—}(\mathscr{C}(s_1,w)\oplus \mathscr{C}(s_2,w))]$, where each sub-extension $[\pi_{\mathrm{alg}}(\underline{\phi},\lambda)\text{—}\mathscr{C}(s_i,w)]$ is non-split. Let $\pi_1(\underline{\phi},\lambda)$ be the unique quotient of $\oplus_{w\in S_3}\pi_1(\underline{\phi},\lambda,w)$ of socle $\pi_{\mathrm{alg}}(\underline{\phi},\lambda)$, which is an extension of $\oplus_{\mathscr{C}\in\mathcal{S}}\mathscr{C}$ by $\pi_{\mathrm{alg}}(\underline{\phi},\lambda)$ with each sub-extension $[\pi_{\mathrm{alg}}(\underline{\phi},\lambda)\text{—}\mathscr{C}]$ non-split.

Now let $D_1\subset D$ be as in § 2.1, and $\pi(D_1)$ be the locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to D_1. We recall the structure of $\pi(D_1)$. Let $\lambda^1:=(h_1-1,h_2)$. For $i,j\in\{1,2\}$, $i\neq j$, let $I_{\mathscr{F}_i}:=(\mathrm{Ind}_{B^-}^{\mathrm{GL}_2} z^{\lambda^1}(\phi_i|\cdot|^{-1}\boxtimes\phi_j))^{\mathrm{an}}$, and $I_{\mathscr{F}_i,0}:=(\mathrm{Ind}_{B^-}^{\mathrm{GL}_2} z^{\lambda^1}(\phi_i|\cdot|^{-1}\boxtimes\phi_j))^{\mathrm{lalg}}\cong (\mathrm{Ind}_{B^-}^{\mathrm{GL}_2}\phi_i|\cdot|^{-1}\boxtimes\phi_j)^{\infty}\otimes_E L(\lambda^1)$. We have $I_{\mathscr{F}_1,0}\cong I_{\mathscr{F}_2,0}=:\pi_{\mathrm{alg}}(D_1)$. We will use the notation $I_{\mathscr{F}_i,0}$ when we want to emphasize its relation with $I_{\mathscr{F}_i}$. The representation $\pi_{\mathrm{alg}}(D_1)$ is in fact the locally algebraic subrepresentation of $\pi(D_1)$. As D_1 is non-split, $\pi(D_1)\cong I_{\mathscr{F}_1}\oplus_{\pi_{\mathrm{alg}}(D_1)} I_{\mathscr{F}_2}$. Letting $\mathscr{C}(s,\mathscr{F}_i):=(\mathrm{Ind}_{B^-}^{\mathrm{GL}_2} z^{s\cdot\lambda^1}(\phi_i|\cdot|^{-1}\boxtimes\phi_j))^{\mathrm{an}}$ for $i,j\in\{1,2\}$, $i\neq j$ (recall $\mathscr{F}_i$ denotes a refinement of D_1), then $\pi(D_1)$ has the form $\big[\pi_{\mathrm{alg}}(D_1)\text{—}\mathscr{C}(s,\mathscr{F}_1)\oplus\mathscr{C}(s,\mathscr{F}_2)\big]$.

Consider $(\mathrm{Ind}_{P_1^-}^{\mathrm{GL}_3}(\pi(D_1)\otimes_E\varepsilon^{-1}\circ\det)\boxtimes\phi_3 z^{h_3})^{\mathrm{an}}$ which is isomorphic to

$$(\mathrm{Ind}_{B^-}^{\mathrm{GL}_3}\jmath(\underline{\phi})z^{\lambda})^{\mathrm{an}}\oplus_{(\mathrm{Ind}_{P_1^-}^{\mathrm{GL}_3}(\pi_{\mathrm{alg}}(D_1)\otimes_E\varepsilon^{-1}\circ\det)\boxtimes\phi_3 z^{h_3})^{\mathrm{an}}}(\mathrm{Ind}_{B^-}^{\mathrm{GL}_3}\jmath(s_1(\underline{\phi}))z^{\lambda})^{\mathrm{an}}.$$

By [25, Thm.], we see that the constituents $\mathscr{C}\in\mathcal{S}$, that the induction contains, are exactly $\mathcal{S}_{\mathscr{F}}:=\{\mathscr{C}(s_1,1),\mathscr{C}(s_1,s_1),\mathscr{C}(s_2,1)\}$, each with multiplicity one. Moreover, it contains a unique subrepresentation $\pi_1(\underline{\phi},\lambda)^-$, that is an extension of $\mathscr{C}(s_1,1)\oplus\mathscr{C}(s_1,s_1)\oplus\mathscr{C}(s_2,1)$ by $\pi_{\mathrm{alg}}(\underline{\phi},\lambda)$, with each sub-extension $[\pi_{\mathrm{alg}}(\underline{\phi},\lambda)\text{—}\mathscr{C}]$ non-split.

Similarly, let $\pi(C_1)$ be the locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to C_1, and $\mathcal{S}_{\mathscr{G}}:=\{\mathscr{C}(s_2,s_2),\mathscr{C}(s_2,s_2s_1),\mathscr{C}(s_1,s_1s_2)\}$. Then $\mathscr{C}\in\mathcal{S}$ appears in $(\mathrm{Ind}_{P_2^-}^{\mathrm{GL}_3}\phi_3 z^{h_1-2}|\cdot|^{-2}\boxtimes\pi(C_1))^{\mathrm{an}}$ if and only if $\mathscr{C}\in\mathcal{S}_{\mathscr{G}}$. And if so, it has multiplicity one. Moreover, $(\mathrm{Ind}_{P_2^-}^{\mathrm{GL}_3}\phi_3 z^{h_1-2}|\cdot|^{-2}\boxtimes\pi(C_1))^{\mathrm{an}}$ contains a unique subrepresentation $\pi_1(\underline{\phi},\lambda)^+$, that is an extension of $\oplus_{\mathscr{C}\in\mathcal{S}_{\mathscr{G}}}\mathscr{C}$ by $\pi_{\mathrm{alg}}(\underline{\phi},\lambda)$, with each subextension $[\pi_{\mathrm{alg}}(\underline{\phi},\lambda)\text{—}\mathscr{C}]$ non split. By the discussions, we have

Lemma 3.1. $\pi_1(\underline{\phi},\lambda)\cong\pi_1(\underline{\phi},\lambda)^+\oplus_{\pi_{\mathrm{alg}}(\underline{\phi},\lambda)}\pi_1(\underline{\phi},\lambda)^-$.

3.2 Extension groups

We collect some facts on the extension groups of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and of $\mathrm{GL}_3(\mathbb{Q}_p)$. We invite the reader to compare these with the extension groups of (φ,Γ)-modules considered in § 2.

3.2.1 $\mathrm{GL}_2(\mathbb{Q}_p)$.

The following proposition is well-known (e.g. by an easier variation of the arguments in [5, § 3.2.2] or the proof of Proposition 3.3 below).

Proposition 3.2. *(1) We have a natural exact sequence*

$$0\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1),\pi(D_1))\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1),\pi(D_1))\\ \longrightarrow \oplus_{i=1}^2\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1),\mathscr{C}(s,\mathscr{F}_i))\longrightarrow 0.$$

Moreover, we have $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi_{\mathrm{alg}}(D_1)) = 3$*,* $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) = 5$*, and* $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \mathscr{C}(s, \mathscr{F}_i)) = 1$*.*

(2) For $i \in \{1, 2\}$*, there is a natural exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi_{\mathrm{alg}}(D_1)) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), I_{\mathscr{F}_i}) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \mathscr{C}(s, \mathscr{F}_i)) \to 0,$$

hence $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), I_{\mathscr{F}_i}) = 4$*.*

For $i \in \{1, 2\}$, consider the following composition (recalling $\delta_{\mathscr{F}_i} = \phi_i z^{h_1} \boxtimes \phi_j z^{h_2}$ for $j \in \{1, 2\}$, $j \neq i$)

$$\zeta_i : \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(I_{\mathscr{F}_i}, I_{\mathscr{F}_i}) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), I_{\mathscr{F}_i}), \tag{14}$$

where the first map is obtained by applying the functor $(\mathrm{Ind}_{B^-}^{\mathrm{GL}_2} - \otimes_E (\varepsilon^{-1} \boxtimes 1))^{\mathrm{an}}$ and the second is the pull-back map (with respect to a fixed embedding $\pi_{\mathrm{alg}}(D_1) \hookrightarrow I_{\mathscr{F}_i}$). Using Schraen's spectral sequence [26], it is not difficult to see the composition is bijective. The composition can also be obtained by Emertons's functor $I_{B^-}^{\mathrm{GL}_2}$ in [18]. Recall for $\widetilde{\delta}_{\mathscr{F}_i} \in \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i})$, $I_{B^-}^{\mathrm{GL}_2}(\widetilde{\delta}_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1))$ is the closed subrepresentation of $(\mathrm{Ind}_{B^-}^{\mathrm{GL}_2} \widetilde{\delta}_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1))^{\mathrm{an}}$ generated by (where δ_B denotes the modulus character of $B(\mathbb{Q}_p)$)

$$\widetilde{\delta}_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1)\delta_B \longrightarrow J_B\big((\mathrm{Ind}_{B^-}^{\mathrm{GL}_2} \widetilde{\delta}_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1))^{\mathrm{an}}\big) \hookrightarrow (\mathrm{Ind}_{B^-}^{\mathrm{GL}_2} \widetilde{\delta}_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1))^{\mathrm{an}}.$$

There is a natural surjection $I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1)\big) \twoheadrightarrow I_{\mathscr{F}_i, 0} \cong \pi_{\mathrm{alg}}(D_1)$, and let W be its kernel, which is clearly a closed subrepresentation of $I_{\mathscr{F}_i}$. Then the image of $\widetilde{\delta}_{\mathscr{F}_i}$ under (14) is just the push-forward of $I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1)\big)$ along $W \hookrightarrow I_{\mathscr{F}_i}$. Finally, the inverse of (14) can be described using the Jacquet-Emerton functor: for any extension $\pi \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), I_{\mathscr{F}_i})$, its Jacquet-Emerton module $J_B(\pi)$ (cf. [18]) contains a unique self-extension of $\delta_{\mathscr{F}_i}(\varepsilon^{-1} \boxtimes 1)\delta_B$, which is just the inverse image of π twisted by $(\varepsilon^{-1} \boxtimes 1)\delta_B$.

Denote by $\mathrm{Ext}^1_{g'}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \subset \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i})$ the subspace of extensions which are locally algebraic up to twist by a certain character of $Z(\mathbb{Q}_p)$ (over $E[\epsilon]/\epsilon^2$). Then ζ_i restricts to a bijection

$$\zeta_i : \mathrm{Ext}^1_{g'}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi_{\mathrm{alg}}(D_1)).$$

Let ξ denote the map

$$\begin{aligned} \xi : \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_1}, \delta_{\mathscr{F}_1}) \quad &\xrightarrow{\sim} \quad \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_2}, \delta_{\mathscr{F}_2}) \\ (\phi_1 z^{h_1}(1 + \psi_1 \epsilon)) \boxtimes (\phi_2 z^{h_2}(1 + \psi_2 \epsilon)) \quad &\mapsto \quad (\phi_2 z^{h_1}(1 + \psi_2 \epsilon)) \boxtimes (\phi_1 z^{h_1}(1 + \psi_1 \epsilon)). \end{aligned}$$

By the classical intertwining, we have

$$\zeta_1 = \zeta_2 \circ \xi : \mathrm{Ext}^1_{g'}(\delta_{\mathscr{F}_1}, \delta_{\mathscr{F}_1}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi_{\mathrm{alg}}(D_1)). \tag{15}$$

Denote by $\mathrm{Ext}^1_{\mathscr{F}_i}(\pi_{\mathrm{alg}}(D_1), \pi(D_1))$ $\big($resp. $\mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(D_1), \pi(D_1))$, resp, $\mathrm{Ext}^1_{g}(\pi_{\mathrm{alg}}(D_1), \pi(D_1))\big)$ the image via the push-forward (injective) map of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), I_{\mathscr{F}_i})$ $\big($resp. of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi_{\mathrm{alg}}(D_1))$, resp. of $\mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(D_1), \pi_{\mathrm{alg}}(D_1))$, "lalg" denoting the locally algebraic extensions$\big)$. Then we have an exact sequence (compare with Proposition 2.4 (2))

$$0 \to \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \to \oplus_{i=1,2} \mathrm{Ext}^1_{\mathscr{F}_i}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \to 0. \tag{16}$$

3.2.2 $\mathrm{GL}_3(\mathbb{Q}_p)$.

In this section, to distinguish, we use Z, T, B to denote subgroups of GL_3, and Z_1, T_1, B_1 the corresponding subgroups of GL_2.

Proposition 3.3. *(1) There is a natural exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)) \longrightarrow \oplus_{\mathscr{C}\in\mathcal{S}} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}) \longrightarrow 0, \quad (17)$$

where $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) = 4$, $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}) = 1$ *for* $\mathscr{C} \in \mathcal{S}$, *and (hence)* $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)) = 10$.

(2) There is an exact sequence (where the same holds with $-$, $\mathcal{S}_{\mathscr{F}}$ *replaced by* $+$, $\mathcal{S}_{\mathscr{G}}$*)*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)^-) \longrightarrow \oplus_{\mathscr{C}\in\mathcal{S}_{\mathscr{F}}} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}) \longrightarrow 0,$$

with $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)^-) = 7$. *In particular,*

$$\mathrm{Ext}^1_{\mathrm{GL}_3}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)^+) \cap \mathrm{Ext}^1_{\mathrm{GL}_3}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)^-).$$

(3) For $w \in S_3$, $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda,w)) = 6$, *and there is an exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda,w)) \longrightarrow \oplus_{i=1,2} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}(s_i,w)) \longrightarrow 0.$$

Proof. All the sequences are obtained by dévissage. And it suffices to show the last maps are surjective. We use Ext^i_Z to denote the subgroup of extensions with a fixed central character. By [26, § 4.3], $\mathrm{Ext}^i_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) \cong \mathrm{Ext}^i_{Z,\mathrm{lalg}}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda))$ for $i \leq 2$. Hence

$$\dim_E \mathrm{Ext}^i_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) = \begin{cases} 2 & i=1 \\ 0 & i=2 \end{cases}. \quad (18)$$

By dévissage, we get an exact sequence

$$0 \to \mathrm{Ext}^1_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) \to \mathrm{Ext}^1_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)) \to \oplus_{\mathscr{C}\in\mathcal{S}} \mathrm{Ext}^1_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}) \to 0.$$

By an easier variation of the proof of [14, Lem. 2.28], for $\mathscr{C} \in \mathcal{S}$,

$$\dim_E \mathrm{Ext}^1_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}) = \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}) = 1.$$

Together with the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)) & \longrightarrow & \oplus_{\mathscr{C}\in\mathcal{S}} \mathrm{Ext}^1_Z(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)) & \longrightarrow & \oplus_{\mathscr{C}\in\mathcal{S}} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \mathscr{C}), \end{array}$$

(the surjectivity in) (17) follows. By (18) and similar arguments in [5, Lem. 3.16],

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_{\mathrm{alg}}(\underline{\phi},\lambda)) = 4.$$

(1) follows. (2) and (3) follow by similar arguments. □

Remark 3.4. *We have*

$$\zeta_w : \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\jmath(w(\underline{\phi}))z^\lambda, \jmath(w(\underline{\phi}))z^\lambda\big) \xrightarrow{\mathrm{Ind}} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big((\mathrm{Ind}_{B^-}^{\mathrm{GL}_3}\jmath(w(\underline{\phi}))z^\lambda)^{\mathrm{an}}, (\mathrm{Ind}_{B^-}^{\mathrm{GL}_3}\jmath(w(\underline{\phi}))z^\lambda)^{\mathrm{an}}\big)$$
$$\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), (\mathrm{Ind}_{B^-}^{\mathrm{GL}_3}\jmath(w(\underline{\phi}))z^\lambda)^{\mathrm{an}}\big) \xleftarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda,w))$$

where the composition of the first two maps is an isomorphism by [26, (4.38)], and the third isomorphism follows from [14, Lem. 2.26]. Similarly as in the discussion below Proposition 3.2, the inverse of ζ_w can be described by applying Jacquet-Emerton functor.

For $w \in S_3$, we put $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda))$ to be the image of

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda,w)) \longhookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)).$$

By Remark 3.4, we have a natural isomorphism

$$\zeta_w : \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\jmath(w(\underline{\phi}))z^\lambda, \jmath(w(\underline{\phi}))z^\lambda\big) \xrightarrow{\sim} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)).$$

We denote by $\mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda))$ (resp. $\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda))$) the image of

$$i^- : \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)^-) \longhookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda))$$

$$\big(\text{resp. } i^+ : \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)^+) \longhookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda))\big).$$

By Schraen's spectral sequence [26, (4.38)], there is a natural isomorphism

$$\mathrm{Ext}^1_{L_1(\mathbb{Q}_p)}\big((\pi_{\mathrm{alg}}(D_1)\otimes_E \varepsilon^{-1}\circ\det)\boxtimes\phi_3 z^{h_3}, (\pi(D_1)\otimes_E\varepsilon^{-1}\circ\det)\boxtimes\phi_3 z^{h_3}\big)$$
$$\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), (\mathrm{Ind}_{P_1^-}^{\mathrm{GL}_3}(\pi(D_1)\otimes_E\varepsilon^{-1}\circ\det)\boxtimes\phi_3 z^{h_3})^{\mathrm{an}}\big)$$
$$\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)^-), \quad (19)$$

where the first map is given by $(\mathrm{Ind}_{P_1^-}^{\mathrm{GL}_3} -)^{\mathrm{an}}$ composed with the pull-back map, and the second map is the restriction map, being an isomorphism by [14, Lem. 2.26]. There is a natural bijection

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) \xrightarrow{\sim}$$
$$\mathrm{Ext}^1_{L_1(\mathbb{Q}_p)}\big((\pi_{\mathrm{alg}}(D_1)\otimes_E\varepsilon^{-1}\circ\det)\boxtimes\phi_3 z^{h_3}, (\pi(D_1)\otimes_E\varepsilon^{-1}\circ\det)\boxtimes\phi_3 z^{h_3}\big), \quad (20)$$

sending (π,ψ) to $\pi\otimes_{E[\epsilon]/\epsilon^2}\phi_3 z^{h_3}(1+\psi\epsilon)$ (noting the elements in both sides can be equipped with a natural $E[\epsilon]/\epsilon^2$-structure). Taking the composition of (19) (20) and i^-, we obtain an isomorphism

$$j^- : \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)).$$

Similarly, we have an isomorphism

$$j^+ : \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(C_1), \pi(C_1)) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda), \pi_1(\underline{\phi},\lambda)).$$

For $i = 1, 2$, the refinement $\mathscr{F}_i$ on D_1 (resp. $\mathscr{G}_i$ on C_1) corresponds to a refinement, still denoted by $\mathscr{F}_i$ (resp. $\mathscr{G}_i$), on D.

Proposition 3.5. *For $i = 1, 2$, let $w \in S_3$ such that $w(\underline{\phi})$ is the refinement $\mathscr{F}_i$ (resp. $\mathscr{G}_i$), the map j^- (resp. j^+) restricts to an isomorphism*

$$j^- : \mathrm{Ext}^1_{\mathscr{F}_i}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) \xrightarrow{\sim} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$$

$$\big(\textit{resp. } j^+ : \mathrm{Ext}^1_{\mathscr{G}_i}(\pi_{\mathrm{alg}}(C_1), \pi(C_1)) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) \xrightarrow{\sim} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))\big).$$

Proof. By the explicit construction (noting elements on the both sides come from Borel induced representations), we obtain the maps in the proposition. By Remark 3.4 and (14), we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_{T_1(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) & \xrightarrow[\sim]{(\zeta_i, \mathrm{id})} & \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \times \mathrm{Hom}(\mathbb{Q}_p^\times, E) \\ \sim\downarrow & & j^-\downarrow \\ \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\jmath(w(\underline{\phi}))z^\lambda, \jmath(w(\underline{\phi}))z^\lambda) & \xrightarrow[\sim]{\zeta_w} & \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda)), \end{array}$$

where the left vertical map sends $(\delta_{\mathscr{F}_i}((1+\psi_1\epsilon) \boxtimes (1+\psi_2\epsilon)), \psi_3)$ to $\jmath(w(\underline{\phi}))z^\lambda((1+\psi_1\epsilon) \boxtimes (1+\psi_2\epsilon) \boxtimes (1+\psi_3\epsilon))$. The map j^+ also sits in a similar diagram. The proposition follows. □

3.3 p-adic Langlands correspondence for $\mathrm{GL}_3(\mathbb{Q}_p)$ in the crystabelline case

We construct a $\mathrm{GL}_3(\mathbb{Q}_p)$-representation that determines and depends on D.

3.3.1 p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ revisited

We collect some facts on the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let D_1 be as in § 2, and recall $\delta_{\mathscr{F}_1} = \phi_1 z^{h_1-1} \boxtimes \phi_2 z^{h_2}$, $\delta_{\mathscr{F}_2} = \phi_2 z^{h_1-1} \boxtimes \phi_1 z^{h_2}$. We fix $\pi_{\mathrm{alg}}(D_1) \hookrightarrow \pi(D_1)$ and $\pi_{\mathrm{alg}}(D_1) \hookrightarrow I_{\mathscr{F}_i}$ (hence $I_{\mathscr{F}_i} \hookrightarrow \pi(D_1)$). We have isomorphisms

$$\mathrm{rec}_i : \mathrm{Ext}^1_{\mathscr{F}_i}(D_1, D_1) \xrightarrow{\sim} \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \xrightarrow[\sim]{\zeta_i} \mathrm{Ext}^1_{\mathscr{F}_i}(\pi_{\mathrm{alg}}(D_1), \pi(D_1))$$

which restricts to isomorphisms

$$\mathrm{Ext}^1_{g'}(D_1, D_1) \xrightarrow{\sim} \mathrm{Ext}^1_{g'}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \xrightarrow[\sim]{\zeta_i} \mathrm{Ext}^1_{g,Z}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi(D_1)).$$

The following lemma is a direct consequence of (15) and Remark 2.5.

Lemma 3.6. *For $i = 1, 2$, $\mathrm{rec}_i|_{\mathrm{Ext}^1_{g'}(D_1, D_1)}$ are equal .*

By Lemma 3.6 and Proposition 2.4 (2), $(\mathrm{rec}_1, \mathrm{rec}_2)$ "glue" to a bijection

$$\mathrm{rec} : \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)). \tag{21}$$

3.3.2 Crystabelline correspondence for $\mathrm{GL}_3(\mathbb{Q}_p)$

Let D be as in § 2, and $\iota_D \in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1, C_1)$ be associated to D, and $\mathscr{I}_D$ be the associated set of higher intertwining pairs. Recall ι_D is determined by $\mathscr{I}_D$. We associate to $(\underline{\phi}, \lambda, \iota_D)$ a locally analytic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$.

Consider the compositions

$$i^- : \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1) \xrightarrow[\sim]{\mathrm{rec}} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \xrightarrow{j^-} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda)),$$

$$i^+ : \mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1, C_1) \xrightarrow[\sim]{\mathrm{rec}} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(C_1), \pi(C_1)) \xrightarrow{j^+} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda)).$$

Recall we have $\mathrm{Ext}^1_g(D_1, D_1) \xrightarrow[\sim]{\kappa} \mathrm{Hom}_{\mathrm{sm}}(\mathbb{Q}_p, E)^{\oplus 2}$ and $\mathrm{Ext}^1_g(C_1, C_1) \xrightarrow[\sim]{\kappa} \mathrm{Hom}_{\mathrm{sm}}(\mathbb{Q}_p, E)^{\oplus 2}$. By the explicit description of the maps (see Remark 2.5, Remark 3.4), we have:

Lemma 3.7. *We have $i^- \circ \kappa^{-1} = i^+ \circ \kappa^{-1}$. In particular, $i^-(\mathrm{Ext}^1_g(D_1, D_1)) = i^+(\mathrm{Ext}^1_g(C_1, C_1)) =: \mathrm{Ext}^1_{g,\phi_3}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$.*

Let $\iota : D_1 \hookrightarrow C_1$. Consider the restriction of i^- and i^+ to $\mathrm{Ext}^1_\iota(D_1, D_1)$ and $\mathrm{Ext}^1_\iota(C_1, C_1)$.

Lemma 3.8. *We have $i^-(\mathrm{Ext}^1_\iota(D_1, D_1)) \cap i^+(\mathrm{Ext}^1_\iota(C_1, C_1)) = \mathrm{Ext}^1_{g,\phi_3}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$.*

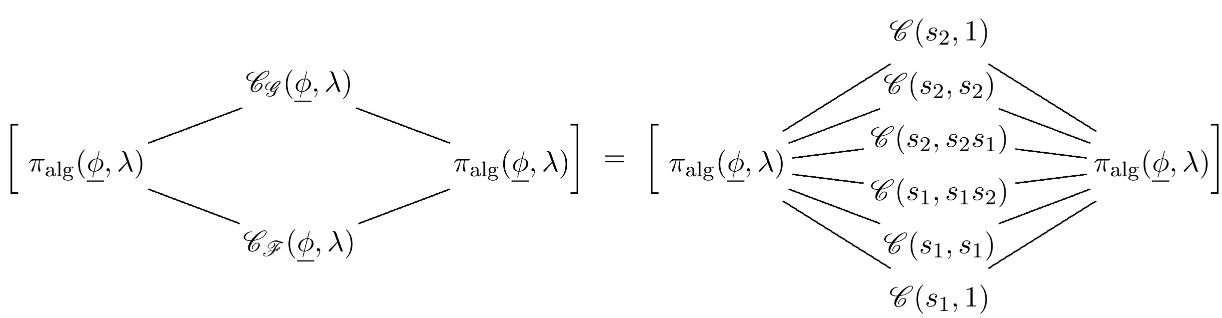
Proof. By Proposition 3.3 (2), $i^-(\mathrm{Ext}^1_\iota(D_1, D_1)) \cap i^+(\mathrm{Ext}^1_\iota(C_1, C_1)) \subset \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_{\mathrm{alg}}(\underline{\phi}, \lambda))$ hence is contained in $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$ for all $w \in S_3$. The lemma then follows from Proposition 3.5, Proposition 2.10 (2) and Lemma 3.8. □

Let $\mathscr{I}^0_\iota \subset \mathscr{I}_\iota$ be the subset consisting of non-de Rham pairs. Let $(\widetilde{D}_1, \widetilde{C}_1) \in \mathscr{I}^0_\iota$, and put $\pi(\underline{\phi}, \lambda, \iota) \in \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$ to be the isomorphism class of $i^-([\widetilde{D}_1]) - i^+([\widetilde{C}_1])$.

Lemma 3.9. *The representation $\pi(\underline{\phi}, \lambda, \iota)$ does not depend on the choice of $(\widetilde{D}_1, \widetilde{C}_1) \in \mathscr{I}^0_\iota$.*

Proof. Let $(\widetilde{D}'_1, \widetilde{C}'_1) \in \mathscr{I}^0_\iota$. By Proposition 2.10 (1) (2), there exist a de Rham pair $(x_0, y_0) \in \mathscr{I}_\iota$ and $a \in E^\times$ such that $[\widetilde{D}'_1] = [x_0] + a[\widetilde{D}_1]$ and $[\widetilde{C}'_1] = [y_0] + a[\widetilde{C}_1]$. By Proposition 2.11 and Lemma 3.7, $i^-([x_0]) = i^+[y_0]$, hence $i^-([\widetilde{D}'_1]) - i^+([\widetilde{C}'_1]) = a(i^-([\widetilde{D}_1]) - i^+([\widetilde{C}_1]))$. The lemma follows. □

By definition, $\pi(\underline{\phi}, \lambda, \iota)$ has the following structure



The representation $\pi(\underline{\phi}, \lambda, \iota)$ has the following "coordinate-free" description.

Proposition 3.10. *We have* $\pi(\underline{\phi},\lambda,\iota)=\cap_{(\widetilde{D}_1,\widetilde{C}_1)\in\mathscr{I}_\iota^0}\Big(i^-(\widetilde{D}_1)\oplus_{\pi_1(\underline{\phi},\lambda)}i^+(\widetilde{C}_1)\Big)$*, where the intersection is taken in the universal extension of* $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda),\pi_1(\underline{\phi},\lambda))\otimes_E\pi_{\mathrm{alg}}(\underline{\phi},\lambda)$ *by* $\pi_1(\underline{\phi},\lambda)$*, and we also use* $i^{\pm}(-)$ *to denote the representation corresponding to the elements in the extension group.*

Proof. The "$\subset$" part follows from Lemma 3.9. For "$\supset$", let $(\widetilde{D}_1,\widetilde{C}_1)\in\mathscr{I}_\iota^0$, $(x_0,y_0)\in\mathscr{F}_\iota$ be de Rham (and non-zero) and $(\widetilde{D}_1',\widetilde{C}_1')\in\mathscr{F}_\iota^0$ such that $[\widetilde{D}_1']=[\widetilde{D}_1]+[x_0]$, $[\widetilde{C}_1']=[\widetilde{C}_1]+[y_0$. If

$$\big(i^-(\widetilde{D}_1)\oplus_{\pi_1(\underline{\phi},\lambda)}i^+(\widetilde{C}_1)\big)\cap\big(i^-(\widetilde{D}_1')\oplus_{\pi_1(\underline{\phi},\lambda)}i^+(\widetilde{C}_1')\big)\supsetneq\pi(\underline{\phi},\lambda,\iota),$$

then $\big(i^-(\widetilde{D}_1)\oplus_{\pi_1(\underline{\phi},\lambda)}i^+(\widetilde{C}_1)\big)\cong\big(i^-(\widetilde{D}_1')\oplus_{\pi_1(\underline{\phi},\lambda)}i^+(\widetilde{C}_1')\big)$. However, by Proposition 3.3 (2), Lemma 3.7 and 3.8, $i^-([\widetilde{D}_1])$, $i^+([\widetilde{C}_1])$ and $i^-([x_0])=i^+([y_0])$ are linearly independent, a contradiction. ☐

Theorem 3.11. *For two injections* $\iota,\iota':D_1\hookrightarrow C_1$*,* $\pi(\underline{\phi},\lambda,\iota)\cong\pi(\underline{\phi},\lambda,\iota')$ *if and only if* $E[\iota]=E[\iota']\subset\mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$*. In particular, for* D *given as in § 2,* $\pi(\underline{\phi},\lambda,\iota_D)$ *depends on and uniquely determines* D*.*

Proof. Suppose $\pi(\underline{\phi},\lambda,\iota)\cong\pi(\underline{\phi},\lambda,\iota')$, and $E[\iota]\neq E[\iota']$. Let $(x,y)\in\mathscr{I}_\iota^0$, and $(x',y')\in\mathscr{I}_{\iota'}^0$ such that $[\pi(\underline{\phi},\lambda,\iota)]=i^-(x)-i^+(y)$ and $[\pi(\underline{\phi},\lambda,\iota')]=i^-(x')-i^+(y')$. Hence $i^-(x-x')=i^+(y'-y)$. Similarly as in the proof of Lemma 3.8, by Proposition 3.3 (2), this implies $i^-(x-x')\in\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\phi},\lambda),\pi_{\mathrm{alg}}(\underline{\phi},\lambda))$. By Proposition 3.5 (and the proof, see also Remark 3.4), this implies $x-x'\in\mathrm{Ext}^1_g(D_1,D_1)$ hence $\mathrm{Ext}^1_\iota(D_1,D_1)=\mathrm{Ext}^1_{\iota'}(D_1,D_1)$, contradicting Proposition 2.10 (3). ☐

Remark 3.12. *(1) If* D *is critical for some refinements or if* D *has irregular Sen weights, one can associate to* D *a semi-simple locally analytic representation of* $\mathrm{GL}_3(\mathbb{Q}_p)$ *that determines* D *as in [3] [27]. Together with our construction of* $\pi(\underline{\phi},\lambda,\iota)$*, this establishes a one-to-one correspondence between generic crystabelline* (φ,Γ)*-modules of rank 3 and their corresponding locally analytic representations of* $\mathrm{GL}_3(\mathbb{Q}_p)$*.*

(2) Note that $\pi(\underline{\phi},\lambda,\iota_D)$ *is only a small piece of the hypothetical full locally analytic* $\mathrm{GL}_3(\mathbb{Q}_p)$*-representation* $\pi(D)$ *associated to* D*. We quickly discuss the relation between* $\pi(\underline{\phi},\lambda,\iota_D)$ *with the wall-crossing of* $\pi(D)$ *considered in [15]. Let* $\theta_1:=(h_1-2,h_1-2,h_3)$ *which is a partially singular weight. Consider the wall-crossing* $R_{s_1}\pi(D):=T_{\theta_1}^{\lambda}T_{\lambda}^{\theta_1}\pi(D)$*. We have a natural map* $\pi(D)\xrightarrow{j}R_{s_1}\pi(D)$*. Let* $\pi_{s_2}^-(D):=\mathrm{Ker}(j)$ *and* $\pi_{s_2}^+(D):=\mathrm{Im}(j)$*. Since* D *is non-critical for all refinements, one may expect that* $\pi_{s_2}^{\pm}(D)$ *only depends on* $\underline{\phi}$ *and* λ*. Indeed, one may expect* $\pi_{s_2}^-(D)$ *to be the extension associated to* $\mathrm{Fil}^{\max}D_{\mathrm{pcr}}(D)$ *in [4, (EXT)]. One may furthermore expect that the Hodge parameter of* D *is encoded in* $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{s_2}^+(D),\pi_{s_2}^-(D))$*. Let*

$$\begin{aligned}\pi_{s_2}^-(\underline{\phi},\lambda)&:=\big[\pi_{\mathrm{alg}}(\underline{\phi},\lambda)\text{---}\big(\mathscr{C}(s_2,1)\oplus\mathscr{C}(s_2,s_2)\oplus\mathscr{C}(s_2,s_2s_1)\big)\big]\\ \pi_{s_2}^+(\underline{\phi},\lambda)&:=\big[\big(\mathscr{C}(s_1,1)\oplus\mathscr{C}(s_1,s_1)\oplus\mathscr{C}(s_1,s_1s_2)\big)\text{---}\pi_{\mathrm{alg}}(\underline{\phi},\lambda)\big].\end{aligned}$$

The (conjectural) injection $\pi(\underline{\phi},\lambda,\iota_D)\hookrightarrow\pi(D)$ *fits into the following commutative diagram*

$$\begin{array}{ccccccccc}0&\longrightarrow&\pi_{s_2}^-(\underline{\phi},\lambda)&\longrightarrow&\pi(\underline{\phi},\lambda,\iota_D)&\longrightarrow&\pi_{s_2}^+(\underline{\phi},\lambda)&\longrightarrow&0\\ &&\downarrow&&\downarrow&&\downarrow&&\\ 0&\longrightarrow&\pi_{s_2}^-(D)&\longrightarrow&\pi(D)&\longrightarrow&\pi_{s_2}^+(D)&\longrightarrow&0.\end{array}$$

The Hodge parameter of D*, presumed to be encoded in the mysterious* $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{s_2}^+(D),\pi_{s_2}^-(D))$*, turns out to live in the much more transparent* $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\pi_{s_2}^+(\underline{\phi},\lambda),\pi_{s_2}^-(\underline{\phi},\lambda))$*.*

4 Local-global compatibility

4.1 Universal extensions

Keep the notation. We collect some facts on the action of the universal deformation ring of D_1 on universal extensions of $\pi_{\mathrm{alg}}(D_1)$ by $\pi(D_1)$. Keep fixing injections $\pi_{\mathrm{alg}}(D_1) \hookrightarrow I_{\mathscr{F}_i} \hookrightarrow \pi(D_1)$.

Let R_{D_1} (resp. $R_{D_1,\mathscr{F}_i}$) be the universal deformation ring of deformations of D_1 (resp. trianguline deformations of D_1 with respect to $\mathscr{F}_i$) over artinian local E-algebras. It is known that R_{D_1} and $R_{D_1,\mathscr{F}_i}$ are both completed formally smooth local E-algebras, and there are natural quotient maps $R_{D_1} \twoheadrightarrow R_{D_1,\mathscr{F}_i}$. We let $\mathfrak{m}$ (resp. $\mathfrak{m}_i$) be the maximal ideal of R_{D_1} (resp. of $R_{D_1,\mathscr{F}_i}$).

Let $\widetilde{\pi}(D_1)^{\mathrm{univ}}$ be the universal extension of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \otimes_E \pi_{\mathrm{alg}}(D_1)$ by $\pi(D_1)$. Then $\widetilde{\pi}(D_1)^{\mathrm{univ}}$ can be equipped with a natural E-linear action of $R_{D_1}/\mathfrak{m}^2$ as follows. For $x \in \mathfrak{m}/\mathfrak{m}^2$, we view x as a map $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \to E$ using

$$(\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1) \xrightarrow{\mathrm{rec}} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)).$$

Then x acts on $\widetilde{\pi}(D_1)^{\mathrm{univ}}$ via

$$\widetilde{\pi}(D_1)^{\mathrm{univ}} \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \otimes_E \pi_{\mathrm{alg}}(D_1) \xrightarrow{x} \pi_{\mathrm{alg}}(D_1) \hookrightarrow \pi(D_1) \hookrightarrow \widetilde{\pi}(D_1)^{\mathrm{univ}}. \tag{22}$$

The lemma below follows by definition, where "$[\mathcal{J}]$" denotes the subspace annihilated by the ideal $\mathcal{J}$.

Lemma 4.1. *(1) $\widetilde{\pi}(D_1)^{\mathrm{univ}}[\mathfrak{m}] = \pi(D_1)$.*

(2) Let $v \in \operatorname{Spec} E[\epsilon]/\epsilon^2 \to \operatorname{Spf} R_{D_1}$ be a non-zero element in the tangent space of $\operatorname{Spf} R_{D_1}$ at $\mathfrak{m}$, $I_v \supset \mathfrak{m}^2$ be the associated ideal. Let $\widetilde{D}_{1,v}$ be the associated deformation of D_1 over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$. Then $\widetilde{\pi}(D_1)^{\mathrm{univ}}[I_v] \cong \mathrm{rec}(\widetilde{D}_{1,v})$.

For our application, we give an "alternative" construction of the $R_{D_1} \times \mathrm{GL}_2(\mathbb{Q}_p)$-module $\widetilde{\pi}(D_1)^{\mathrm{univ}}$ using trianguline deformations. Let $\mathfrak{m}_{\delta_{\mathscr{F}_i}}$ be the maximal ideal of $\widehat{T}$ at $\delta_{\mathscr{F}_i}$, and $\widehat{\mathcal{O}}_{\widehat{T},\mathfrak{m}_{\delta_{\mathscr{F}_i}}}$ be the completion of $\widehat{T}$ at $\mathfrak{m}_{\delta_{\mathscr{F}_i}}$. We have a natural isomorphism of completed E-algebras: $\widehat{\mathcal{O}}_{\widehat{T},\mathfrak{m}_{\delta_{\mathscr{F}_i}}} \xrightarrow{\sim} R_{D_1,\mathscr{F}_i}$ sending a trianguline deformation to its corresponding trianguline parameter. Let $\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ}}$ be the the universal deformation of $\delta_{\mathscr{F}_i}$ over $\widehat{\mathcal{O}}_{\widehat{T},\mathfrak{m}_{\delta_{\mathscr{F}_i}}}/\mathfrak{m}_{\delta_{\mathscr{F}_i}}^2 \cong R_{D_1,\mathscr{F}_i}/\mathfrak{m}_i^2$, which is a free $R_{D_1,\mathscr{F}_i}/\mathfrak{m}_i^2$-module of rank 1. Consider $\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee} := \mathrm{Hom}_E(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ}}, E)$, equipped with a natural action of $T(\mathbb{Q}_p) \times R_{D_1,\mathscr{F}_i}/\mathfrak{m}_i^2$ given by $(af)(x) = f(ax)$ for $a \in T(\mathbb{Q}_p) \times R_{D_1,\mathscr{F}_i}/\mathfrak{m}_i^2$. It is straightforward to see $\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}$ is isomorphic, as $T(\mathbb{Q}_p)$-representation, to the universal extension of $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \otimes_E \delta_{\mathscr{F}_i}$ by $\delta_{\mathscr{F}_i}$. And the $R_{D_1,\mathscr{F}_i}/\mathfrak{m}_i^2$-action on $\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}$ admits a similar description as in (22) (using Proposition 2.4 (3)). Consider the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1} \boxtimes 1)\big)$, which is equipped with an induced $R_{D_1,\mathscr{F}_i}/\mathfrak{m}_i^2$-action.

Lemma 4.2. *$I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1} \boxtimes 1)\big)$ is the universal extension of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(I_{\mathscr{F}_i,0}, I_{\mathscr{F}_i}) \otimes_E I_{\mathscr{F}_i,0}$ by $I_{\mathscr{F}_i}$. Moreover, the action of $x \in \mathfrak{m}_i/\mathfrak{m}_i^2$-action is given by*

$$I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1} \boxtimes 1)\big) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(I_{\mathscr{F}_i,0}, I_{\mathscr{F}_i}) \otimes_E I_{\mathscr{F}_i,0} \xrightarrow{x \circ \mathrm{rec}_i} I_{\mathscr{F}_i,0} \hookrightarrow I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1} \boxtimes 1)\big).$$

Proof. By the exact sequence

$$0 \longrightarrow \delta_{\mathscr{F}_i} \longrightarrow \widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee} \longrightarrow \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \otimes_E \delta_{\mathscr{F}_i} \longrightarrow 0,$$

and [14, Lem. 4.12], it is not difficult to see $I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big)$ sits in an exact sequence

$$0 \longrightarrow I_{\mathscr{F}_i} \longrightarrow I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big) \xrightarrow{f} \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \otimes_E I_{\mathscr{F}_i,0} \longrightarrow 0.$$

Indeed, the kernel of f is obviously a subrepresentation of $I_{\mathscr{F}_i}$, which can not be $I_{\mathscr{F}_i,0}$ as for ι_0, ι_i in Proposition 2.4, we have $\mathrm{Im}(\iota_i) \supsetneq \mathrm{Im}(\iota_0)$. Using the natural isomorphism $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \cong \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(I_{\mathscr{F}_i,0}, I_{\mathscr{F}_i})$ (cf. (14)) and the fact that $\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}$ is universal, the first part of the lemma follows. The second is straightforward to check. $\square$

In particular, $I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big)[\mathfrak{m}_i] = I_{\mathscr{F}_i}$. We equip $\pi(D_1)$ with an action of $R_{D_1,\mathscr{F}_i}$ via $R_{D_1,\mathscr{F}_i} \twoheadrightarrow R_{D_1,\mathscr{F}_i}/\mathfrak{m}_i$. The amalgamated sum for both the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and of $R_{D_1,\mathscr{F}_i}$ (recalling we have fixed $I_{\mathscr{F}_i} \hookrightarrow \pi(D_1)$)

$$\widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_i} := I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big) \oplus_{I_{\mathscr{F}_i}} \pi(D_1) \tag{23}$$

is no other than the universal extension of $\mathrm{Ext}^1_{\mathscr{F}_i}(\pi_{\mathrm{alg}}(D_1), \pi(D_1)) \otimes_E \pi_{\mathrm{alg}}(D_1)$ by $\pi(D_1)$ (using the fixed isomorphism $\pi_{\mathrm{alg}}(D_1) \cong I_{\mathscr{F}_i,0}$). And the $R_{D_1,\mathscr{F}_i}$-action on it (via (23)) coincides with the one given in a similar way as in (22) using rec_i.

Let $\widehat{\mathcal{O}}^{g'}_{\widehat{T},\mathfrak{m}_{\delta_{\mathscr{F}_i}}}$ be the completion at $\mathfrak{m}_{\delta_{\mathscr{F}_i}}$ of the fibre of the composition

$$\widehat{T} \longrightarrow \mathbb{A}^2 \xrightarrow{(x-y,\mathrm{id})} \mathbb{A}^1$$

at $h_1 - h_2$, where the first map is the weight map. Denote by $R_{D_1,g'}$ the universal deformation ring of those deformations which are de Rham up to twist by characters (over artinian E-algebras) with $\mathfrak{m}_0$ its maximal ideal. We have a natural quotient map $R_{D_1,\mathscr{F}_i} \twoheadrightarrow R_{D_1,g'}$, which induces an isomorphism $\widehat{\mathcal{O}}^{g'}_{\widehat{T},\mathfrak{m}_{\delta_{\mathscr{F}_i}}} \xrightarrow{\sim} R_{D_1,g'}$. Denote by $\widetilde{\delta}^{\mathrm{univ}}_{\mathscr{F}_i,g'}$ the universal deformation of $\delta_{\mathscr{F}_i}$ over $\widehat{\mathcal{O}}^{g'}_{\widehat{T},\mathfrak{m}_{\delta_{\mathscr{F}_i}}}/\mathfrak{m}^2_{\delta_{\mathscr{F}_i}} \cong R_{D_1,g'}/\mathfrak{m}_0^2$. Similarly as above, its dual $\widetilde{\delta}^{\mathrm{univ},\vee}_{\mathscr{F}_i,g'}$ is the universal extension of $\mathrm{Ext}^1_{g'}(\delta_{\mathscr{F}_i}, \delta_{\mathscr{F}_i}) \otimes_E \delta_{\mathscr{F}_i}$ by $\delta_{\mathscr{F}_i}$, with the $\mathfrak{m}_0/\mathfrak{m}_0^2$-action given in a similar way as in (22).

Lemma 4.3. *(1) $I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i,g'}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big)$ is the universal extension of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(I_{\mathscr{F}_i,0}, I_{\mathscr{F}_i,0}) \otimes_E I_{\mathscr{F}_i,0}$ by $I_{\mathscr{F}_i,0}$. Moreover the action of $x \in \mathfrak{m}_0/\mathfrak{m}_0^2$ is given by*

$$I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i,g'}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(I_{\mathscr{F}_i,0}, I_{\mathscr{F}_i,0}) \otimes_E I_{\mathscr{F}_i,0} \xrightarrow{x \circ \mathrm{rec}_i} I_{\mathscr{F}_i,0} \hookrightarrow I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i,g'}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big).$$

(2) The natural injection $I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i,g'}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big) \hookrightarrow I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_i}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big)$ is $R_{D_1,\mathscr{F}_i} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant.

(3) We have a $\mathrm{GL}_2(\mathbb{Q}_p) \times R_{D_1,g'}/\mathfrak{m}_0^2$-equivariant isomorphism

$$I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_1,g'}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big) \cong I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}_{\mathscr{F}_2,g'}^{\mathrm{univ},\vee}(\varepsilon^{-1}\boxtimes 1)\big).$$

Proof. (1) follows by similar arguments as in the proof of Lemma 4.2. (2) is clear. (3) follows from (1), (15) and Lemma 3.6. $\square$

We have a $\mathrm{GL}_2(\mathbb{Q}_p)\times R_{D_1}$-equivariant injection (where the amalgamated sum is for the $\mathrm{GL}_2(\mathbb{Q}_p)\times R_{D_1,g'}/\mathfrak{m}_0^2$-action, noting the fixed injection $I_{\mathscr{F}_i,0}\hookrightarrow \pi(D_1)$ is obviously R_{D_1}-equivariant)

$$\widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_i,g'} := I_{B^-}^{\mathrm{GL}_2}\big(\widetilde{\delta}^{\mathrm{univ},\vee}_{\mathscr{F}_1,g'}(\varepsilon^{-1}\boxtimes 1)\big)\oplus_{I_{\mathscr{F}_i,0}}\pi(D_1)\longrightarrow \widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_i}.$$

By Lemma 4.3, we have a $\mathrm{GL}_2(\mathbb{Q}_p)\times R_{D_1}$-equivariant isomorphism $\widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_1,g'}\cong \widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_2,g'}=: \widetilde{\pi}(D_1)^{\mathrm{univ}}_{g'}$. Comparing the R_{D_1}-action and using the construction of rec (21), we obtain

Proposition 4.4. *We have a $\mathrm{GL}_2(\mathbb{Q}_p)\times R_{D_1}/\mathfrak{m}^2$-equivariant isomorphism*

$$\widetilde{\pi}(D_1)^{\mathrm{univ}}\cong \widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_1}\oplus_{\widetilde{\pi}(D_1)^{\mathrm{univ}}_{g'}}\widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_2}.$$

4.2 Local-global compatibility on the parabolic Jacquet-Emerton modules

We prove a local-global compatibility result on the parabolic Jacquet-Emerton module of the patched locally analytic representation. In this section, let T_1 be the torus subgroup of GL_2, $B_1\supset T_1$ be the subgroup of upper triangular matrices of GL_2.

Let Π_∞ be the patched Banach representation in [11] (for $\mathrm{GL}_n(F)=\mathrm{GL}_3(\mathbb{Q}_p)$), which is equipped with an action of the patched Galois deformation ring $R_\infty\cong R_{\overline{\rho}}^{\square}\widehat{\otimes}_{\mathcal{O}_E}R_\infty^{\wp}$ (where $\wp$ is "$\widetilde{\mathfrak{p}}$" and $\overline{\rho}$ is the local Galois representation $\overline{r}$ of *loc. cit.*). We refer to *loc. cit.* for details. Let

$$\mathcal{E}\longrightarrow(\mathrm{Spf}\,R_{\overline{\rho}}^{\square})^{\mathrm{rig}}\times\widehat{T}\times(\mathrm{Spf}\,R_\infty^{\wp})^{\mathrm{rig}} \tag{24}$$

be the associated patched eigenvariety (see [14, § 4.1.2], that is an easy variation of those introduced in [8]), $\mathcal{M}$ be the natural coherent sheaf on $\mathcal{E}$ such that there is an $T(\mathbb{Q}_p)\times R_\infty$-equivariant isomorphism (see [8, § 3.1] for "$R_\infty-\mathrm{an}$")

$$\Gamma(\mathcal{E},\mathcal{M})^\vee\cong J_B(\Pi_\infty^{R_\infty-\mathrm{an}}).$$

Let $X_{\mathrm{tri}}^{\square}(\overline{\rho})$ be the trianguline variety [8, § 2.2]. Recall (24) factors through an embedding $\mathcal{E}\hookrightarrow X_{\mathrm{tri}}^{\square}(\overline{\rho})\times(\mathrm{Spf}\,R_\infty^{\wp})^{\mathrm{rig}}$ (cf. [8, Thm. 1.1], see also [21][24]). Finally, let $n_1:=\dim(\mathrm{Spf}\,R_\infty^{\wp})^{\mathrm{rig}}$, then $\dim\mathcal{E}=(3^2+\frac{3(3+1)}{2})+n_1$.

Let $\rho:\mathrm{Gal}_{\mathbb{Q}_p}\to\mathrm{GL}_3(\mathbb{Q}_p)$ such that ρ has a modulo p reduction equal to $\overline{\rho}$ and $D_{\mathrm{rig}}(\rho)=:D$ is given as in § 2. We use the notation there, in particular, D is determined by $\underline{\phi}$, h, and $\iota_D:D_1\hookrightarrow C_1$. Let $\mathfrak{m}_\rho\subset R_{\overline{\rho}}^{\square}[1/p]$ be the maximal ideal associated to ρ, and suppose there exists a maximal ideal $\mathfrak{m}^{\wp}$ of $R_\infty^{\wp}[1/p]$ such that $\Pi_\infty[\mathfrak{m}]^{\mathrm{lalg}}\neq 0$ for $\mathfrak{m}=(\mathfrak{m}_\rho,\mathfrak{m}^{\wp})$, the corresponding maximal ideal of $R_\infty[1/p]$. By [11, § 4], we have $\Pi_\infty[\mathfrak{m}]^{\mathrm{lalg}}\cong\pi_{\mathrm{alg}}(\underline{\phi},\lambda)$. This implies that for any refinement $w(\underline{\phi})$, $x_w:=(x_{w,\wp},\mathfrak{m}^{\wp})=(\rho,\jmath(w(\underline{\phi}))\delta_B z^\lambda,\mathfrak{m}^{\wp})\in\mathcal{E}$. By Hypothesis 2.1, all the points are non-critical. In particular, $X_{\mathrm{tri}}^{\square}(\overline{\rho})$ is smooth at the points $x_{w,\wp}$ (cf. [8, Thm. 2.6 (iii)]). As $(\mathrm{Spf}\,R_\infty^{\wp})^{\mathrm{rig}}$ is also smooth at $\mathfrak{m}^{\wp}$, $\mathcal{E}$ is smooth at all x_w. By [7, Lem. 3.8] and the multiplicity one property in the construction in [11], we see $\mathcal{M}$ is locally free of rank one at all x_w. Let $\mathcal{I}:=(\mathfrak{m}_\rho^2,\mathfrak{m}^{\wp})\subset R_\infty[1/p]$.

Let κ_3 denote the composition $\mathcal{E}\to\widehat{T}\xrightarrow{\mathrm{pr}_3}\widehat{\mathbb{Q}_p^\times}$, where pr_3 denotes the projection induced by the restriction map $\chi\mapsto\chi|_{\mathrm{diag}(1,1,\mathbb{Q}_p^\times)}$. Let $\overline{\mathcal{E}}$ be the fibre of $\mathcal{E}$ at $\phi_3 z^{h_3}|\cdot|^{-2}$ via κ_3. Thus $\overline{\mathcal{E}}$ consists of points of the form $(\rho',\delta_1'\boxtimes\delta_2'\boxtimes\phi_3 z^{h_3}|\cdot|^{-2},\mathfrak{m}'^{\wp})$. We first show a local-global compatibility result for the tangent space of $\overline{\mathcal{E}}$ at the point $x:=x_1$ (which clearly generalizes to x_w for $w\in S_3$). One may obtain similar results for the tangent space of $\mathcal{E}$ at x, but the results for $\overline{\mathcal{E}}$ are enough for our application.

The embedding $\mathcal{E} \hookrightarrow (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T} \times (\mathrm{Spf}\, R_{\infty}^{\wp})^{\mathrm{rig}}$ induces $\overline{\mathcal{E}} \hookrightarrow (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T}_1 \times (\mathrm{Spf}\, R_{\infty}^{\wp})^{\mathrm{rig}}$. Let $\mathcal{U}$ be an affinoid smooth neighbourhood of x in $\overline{\mathcal{E}}$ of the form $\mathcal{U}_{\wp} \times \mathcal{U}^{\wp}$ where $\mathcal{U}_{\wp} \subset (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T}_1$ and $\mathcal{U}^{\wp} \subset (\mathrm{Spf}\, R_{\infty}^{\wp})^{\mathrm{rig}}$. Let $\mathfrak{m}_{x,\wp}$ be the maximal ideal of $\mathcal{O}(\mathcal{U}_{\wp})$ associated to $x_{\wp}$. Let $\mathcal{I}_1 := (\mathfrak{m}_{x,\wp}^2, \mathfrak{m}^{\wp})$ (which is also the closed ideal generated by $\mathcal{I}$ via $R_{\infty}[1/p] \to \mathcal{O}(\mathcal{U})$). Denote by $\mathcal{M}_{\tilde{x}}$ the fibre of $\mathcal{M}$ at $\mathcal{O}(\mathcal{U})/\mathcal{I}_1$ (which does not depend the choice of $\mathcal{U}$), which is equipped with a natural action of $R_{\infty} \times T(\mathbb{Q}_p)$ (where $\mathrm{diag}(1,1,\mathbb{Q}_p^{\times})$ acts by $\phi_3 z^{h_3} |\cdot|^{-2}$). We have natural $R_{\infty} \times T(\mathbb{Q}_p)$-equivariant injections

$$\mathcal{M}_x^{\vee} \longrightarrow \mathcal{M}_{\tilde{x}}^{\vee} \longrightarrow J_B(\Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathcal{I}]). \tag{25}$$

Lemma 4.5. *The map $\mathcal{M}_{\tilde{x}}^{\vee} \hookrightarrow J_B(\Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathcal{I}])$ is balanced.*

Proof. It follows by the same argument as in [14, Lem. 4.11]. □

By [19, Thm. 0.13], the maps in (25) induce $R_{\infty} \times \mathrm{GL}_3(\mathbb{Q}_p)$-equivariant injections:

$$I_{B^-}^{\mathrm{GL}_3}(\mathcal{M}_x^{\vee} \delta_B^{-1}) \longrightarrow I_{B^-}^{\mathrm{GL}_3}(\mathcal{M}_{\tilde{x}}^{\vee} \delta_B^{-1}) \longrightarrow \Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathcal{I}]. \tag{26}$$

Lemma 4.6. *The map (25) induces balanced $R_{\infty} \times L_{P_1}(\mathbb{Q}_p)$-equivariant injections*

$$(I_{B_1^-}^{\mathrm{GL}_2} \mathcal{M}_x^{\vee} \delta_{B_1}^{-1}) \boxtimes \phi_3 z^{h_3} |\cdot|^{-2} \longrightarrow (I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x}}^{\vee} \delta_{B_1}^{-1})) \boxtimes \phi_3 z^{h_3} |\cdot|^{-2} \longrightarrow J_{P_1}(\Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathcal{I}]). \tag{27}$$

Proof. For an admissible locally analytic representation W of $T(\mathbb{Q}_p)$, using

$$(\mathrm{Ind}_{B^-}^{\mathrm{GL}_3} W)^{\mathrm{an}} \cong \big(\mathrm{Ind}_{P_1^-}^{\mathrm{GL}_3} (\mathrm{Ind}_{B^- \cap L_{P_1}}^{L_{P_1}} W)^{\mathrm{an}}\big)^{\mathrm{an}},$$

we have by definition $I_{P_1^-}^{\mathrm{GL}_3}(I_{B^- \cap L_{P_1}}^{L_{P_1}} W) \xrightarrow{\sim} I_{B^-}^{\mathrm{GL}_3} W$. The lemma then follows from (26) and [19, (0.10)]. □

We discuss the R_{∞}-action. Let $R_{\rho}^{\square}$ be the universal framed deformation ring of ρ that is a formally smooth completed local E-algebra. Let $R_{\rho,\mathscr{F}}^{\square}$ be the universal framed $\mathscr{F}$-deformation ring, and $R_{\rho,\mathscr{F}_1}^{\square}$ be the universal framed trianguline deformation ring of ρ with respect to the refinement $\underline{\phi}$ (that we also denote by $\mathscr{F}_1$). Denote by $\overline{R}_{\rho,\mathscr{F}}^{\square}$ the universal framed deformation ring of $\mathscr{F}$-deformations of the form $[\widetilde{D}_{1,A}$—$\mathcal{R}_A(\phi_3 z^{h_3})]$, and $\overline{R}_{\rho,\mathscr{F}_1}^{\square}$ the universal framed deformation ring of trianguline deformations with respect to $\mathscr{F}_1$ of the form $[\widetilde{D}_{1,A}$—$\mathcal{R}_A(\phi_3 z^{h_3})]$. We have a commutative diagram of formally smooth completed local E-algebras:

$$\begin{array}{ccccc} R_{\rho}^{\square} & \longrightarrow & \overline{R}_{\rho,\mathscr{F}}^{\square} & \longrightarrow & \overline{R}_{\rho,\mathscr{F}_1}^{\square} \\ & & \uparrow & & \uparrow \\ & & R_{D_1} & \longrightarrow & R_{D_1,\mathscr{F}_1} \end{array}$$

where all the horizontal maps are surjective. Recall that the completion of $R_{\overline{\rho}}^{\square}[1/p]$ at ρ is naturally isomorphic to $R_{\rho}^{\square}$, and consequently the completion of $X_{\mathrm{tri}}^{\square}(\overline{\rho})$ at $x_{\wp}$ is naturally isomorphic to $R_{\rho,\mathscr{F}_1}^{\square}$ (e.g. see the proof of [8, Thm. 2.6]). We see the $R_{\overline{\rho}}^{\square}$-action on $\mathcal{M}_{\tilde{x}}$ factors through $\overline{R}_{\rho,\mathscr{F}_1}^{\square}$ hence also factors through $\overline{R}_{\rho,\mathscr{F}}^{\square}$. Moreover, $\mathcal{M}_{\tilde{x}}$ is free of rank one over $\overline{R}_{\rho,\mathscr{F}_1}^{\square}/\mathfrak{m}_{\rho}^2$, which is thus isomorphic to $\mathcal{N}_0 \otimes_{R_{D_1,\mathscr{F}_1}/\mathfrak{m}_{D_1}^2} \overline{R}_{\rho,\mathscr{F}_1}^{\square}/\mathfrak{m}_{\rho}^2$ for a free $R_{D_1,\mathscr{F}_1}/\mathfrak{m}_{D_1}^2$-module $\mathcal{N}_0$. We equip $\mathcal{N}_0$ with the natural

$T_1(\mathbb{Q}_p)$-action (induced by Spf $R_{D,\mathscr{F}_1} \to \widehat{T}_1$), then $\mathcal{N}_0^\vee$ as $R_{D_1,\mathscr{F}_1} \times T_1(\mathbb{Q}_p)$-module is no other than $\widetilde{\delta}_{\mathscr{F}_1}^{\mathrm{univ},\vee}$ in the precedent section. We have thus a $T_1(\mathbb{Q}_p)$-equivariant isomorphism of $\overline{R}_{\rho,\mathscr{F}_1}^{\square}$-module:

$$(\mathcal{N}_0 \otimes_{R_{D_1,\mathscr{F}_1}/\mathfrak{m}_{D_1}^2} \overline{R}_{\rho,\mathscr{F}_1}^{\square}/\mathfrak{m}_\rho^2) \otimes_E (z^{-2} \boxtimes \varepsilon^{-1}) \xrightarrow{\sim} \mathcal{M}_{\tilde{x}}.$$

Consider the weight map

$$\kappa : \overline{\mathcal{E}} \longrightarrow \widehat{T}_1 \longrightarrow \mathbb{A}^2 \xrightarrow{(a,b)\mapsto a-b} \mathbb{A}^1,$$

and let $\mathcal{M}_{\tilde{x},g'}$ be the fibre of $\mathcal{M}_{\tilde{x}}$ at $h_1 - h_2 - 1$. The action of $\overline{R}_{\rho,\mathscr{F}_1}^{\square}$ on $\mathcal{M}_{\tilde{x},g'}$ factors through $\overline{R}_{\rho,g'}^{\square}$, the universal deformation ring of deformations of the form $[\widetilde{D}_{1,A} \text{—} \mathcal{R}_A(\phi_3 z^{h_3})]$ with $\widetilde{D}_{1,A}$ de Rham up to twist by a certain character over A. We have $\overline{R}_{\rho,g'}^{\square} \cong \overline{R}_{\rho,\mathscr{F}_i}^{\square} \otimes_{R_{D_1,\mathscr{F}_i}} R_{D_1,g'}$ and $\mathcal{M}_{\tilde{x},g'}$ is a free $\overline{R}_{\rho,g'}^{\square}/\mathfrak{m}_\rho^2$-module. Letting $\mathcal{N}_{0,g'} := \mathcal{N}_0 \otimes_{R_{D_1,\mathscr{F}_1}} R_{D_1,g'}$, we have an isomorphism $(\mathcal{N}_{0,g'} \otimes_{R_{D_1,g'}/\mathfrak{m}_{D_1}^2} \overline{R}_{\rho,g'}^{\square}/\mathfrak{m}_\rho^2) \otimes_E (z^{-2} \boxtimes \varepsilon^{-1}) \xrightarrow{\sim} \mathcal{M}_{\tilde{x},g'}$, and $\mathcal{N}_{0,g'}^\vee \cong \widetilde{\delta}_{\mathscr{F}_1,g'}^{\mathrm{univ},\vee}$. In summary, we have $R_{D_1} \times T_1(\mathbb{Q}_p)$-equivariant maps:

$$\begin{array}{ccccc} \mathcal{N}_0 \otimes_E (z^{-2} \boxtimes \varepsilon^{-1}) & \longrightarrow & \mathcal{M}_{\tilde{x}} & & \\ \downarrow & & \downarrow & & \\ \mathcal{N}_{0,g'} \otimes_E (z^{-2} \boxtimes \varepsilon^{-1}) & \longrightarrow & \mathcal{M}_{\tilde{x},g'} & \longrightarrow & \mathcal{M}_x \end{array}$$

where the bottom composition is surjective. We then deduce:

Lemma 4.7. *We have an $R_{D_1} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant commutative diagram*

$$\begin{array}{ccccc} I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_x^\vee \delta_{B_1}^{-1}) & \longrightarrow & I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x},g'}^\vee \delta_{B_1}^{-1}) & \longrightarrow & I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x}}^\vee \delta_{B_1}^{-1}) \\ \sim\downarrow & & \downarrow & & \downarrow \\ (I_{\mathscr{F}_1,0}) \otimes_E z^{-1} \circ \det & \longrightarrow & I_{B_1^-}^{\mathrm{GL}_2}(\widetilde{\delta}_{\mathscr{F}_1,g'}^{\mathrm{univ}}(\varepsilon^{-1} z^{-1} \boxtimes z^{-1})) & \longrightarrow & I_{B_1^-}^{\mathrm{GL}_2}(\widetilde{\delta}_{\mathscr{F}_1}^{\mathrm{univ}}(\varepsilon^{-1} z^{-1} \boxtimes z^{-1})). \end{array}$$

Denote by $J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])_0$ the maximal subrepresentation of $J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])$ on which the group $Z_3 := \mathrm{diag}(1, 1, \mathbb{Q}_p^\times)$ acts by $\phi_3 z^{h_3} |\cdot|^{-2}$ and which, as $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, is isomorphic to an extension of a certain copy of $\pi_{\mathrm{alg}}(D_1) \otimes_E z^{-1} \circ \det$ by a certain copy of $\pi_{\mathrm{alg}}(D_1) \otimes_E z^{-1} \circ \det$. The following proposition follows from the fact $\mathcal{M}$ is locally free of rank 1 at x.

Proposition 4.8. *The morphism $I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x},g'}^\vee \delta_{B_1}^{-1}) \boxtimes \phi_3 z^{h_3} |\cdot|^{-2} \to J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])$ factors through an $R_\infty \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism*

$$I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x},g'}^\vee \delta_{B_1}^{-1}) \xrightarrow{\sim} J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])_0.$$

It is clear that $I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x}}^\vee \delta_{B_1}^{-1})[\mathfrak{m}_\rho] \cong I_{\mathscr{F}_i} \otimes_E z^{-1} \circ \det$, and we put

$$\begin{aligned} \pi_{\tilde{x}} &:= I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x}}^\vee \delta_{B_1}^{-1}) \oplus_{I_{\mathscr{F}_i} \otimes_E z^{-1} \circ \det} \big(\pi(D_1) \otimes_E z^{-1} \circ \det\big), \\ \pi_{\tilde{x},g'} &:= I_{B_1^-}^{\mathrm{GL}_2}(\mathcal{M}_{\tilde{x},g'}^\vee \delta_{B_1}^{-1}) \oplus_{\pi_{\mathrm{alg}}(D_1) \otimes_E z^{-1} \circ \det} \big(\pi(D_1) \otimes_E z^{-1} \circ \det\big), \end{aligned}$$

where the amalgamated sums are both for the action of R_∞ and $\mathrm{GL}_2(\mathbb{Q}_p)$. Let $\mathscr{E}_{g',P_1}$ be the $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation of $J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])[Z_3 = \phi_3 z^{h_3} |\cdot|^{-2}]$ generated by $\pi(D_1) \otimes_E z^{-1} \circ \det \hookrightarrow J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}_\rho])[Z_3 = \phi_3 z^{h_3} |\cdot|^{-2}]$ and $J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])_0$. By Lemma 4.7 and Proposition 4.8, we have

Proposition 4.9. *We have an $R_{D_1} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant commutative diagram*

$$\begin{array}{ccccccc}
\pi(D_1) \otimes z^{-1} & \hookrightarrow & \pi_{\tilde{x},g'} & \hookrightarrow & \pi_{\tilde{x}} & \hookrightarrow & J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])[Z_3 = \phi_3 z^{h_3}|\cdot|^{-2}] \\
\Vert & & \downarrow & & \downarrow & & \\
\pi(D_1) \otimes z^{-1} & \hookrightarrow & \widetilde{\pi}(D_1)^{\mathrm{univ}}_{g'} \otimes z^{-1} & \hookrightarrow & \widetilde{\pi}(D_1)^{\mathrm{univ}}_{\mathscr{F}_1} \otimes z^{-1} & &
\end{array}$$

Moreover, the composition $\pi_{\tilde{x},g'} \to \pi_{\tilde{x}} \to J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])$ factors through an $R_\infty \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism $\pi_{\tilde{x},g'} \xrightarrow{\sim} \mathscr{E}_{g',P_1}$.

Consider the point x_{s_1}. By the same argument, the statement in the above proposition holds with x replaced by x_{s_1}, and $\mathscr{F}_1$ replaced by $\mathscr{F}_2$. By taking amalgamated sum and using Proposition 4.4, we finally obtain

Corollary 4.10. *We have an $R_{D_1} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant commutative diagram*

$$\begin{array}{ccccccc}
\pi(D_1) \otimes z^{-1} & \hookrightarrow & \mathscr{E}_{g',P_1} & \hookrightarrow & \pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}} & \hookrightarrow & J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])[Z_3 = \phi_3 z^{h_3}|\cdot|^{-2}] \\
\Vert & & \downarrow & & \downarrow & & \\
\pi(D_1) \otimes z^{-1} & \hookrightarrow & \widetilde{\pi}(D_1)^{\mathrm{univ}}_{g'} \otimes z^{-1} & \hookrightarrow & \widetilde{\pi}(D_1)^{\mathrm{univ}} \otimes z^{-1} & &
\end{array}$$

Moreover, the map $j : (\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}}) \boxtimes \phi_3 z^{h_3}|\cdot|^{-2} \to J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])$ is balanced.

Proof. It rests to show the map is balanced. It is clear that

$$(\pi(D_1) \otimes z^{-1}) \boxtimes \phi_3 z^{h_3}|\cdot|^{-2} \longrightarrow J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}])$$

comes from an injection $I_{P_1^-}^{\mathrm{GL}_3}\big(((\pi(D_1)\otimes z^{-1})\boxtimes \phi_3 z^{h_3}|\cdot|^{-2})\delta_{P_1}^{-1}\big) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$. Together with (26), we see $\pi_{\tilde{x}} \boxtimes \phi_3 z^{h_3}|\cdot|^{-2} \hookrightarrow J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])$ comes from the corresponding map $I_{P_1^-}^{\mathrm{GL}_3}((\pi_{\tilde{x}} \boxtimes \phi_3 z^{h_3}|\cdot|^{-2})\delta_{P_1}^{-1}) \to \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$. The same holds with $\pi_{\tilde{x}}$ replaced by $\pi_{\tilde{x}_{s_1}}$. It is then not difficult to see j corresponds to $I_{P_1^-}^{\mathrm{GL}_3}\big(((\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}}) \boxtimes \phi_3 z^{h_3}|\cdot|^{-2})\delta_{P_1}^{-1}\big) \to \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$. So j is balanced by [19, (0.10)]. □

Remark 4.11. *Note that any extension of $\pi_{\mathrm{alg}}(D_1) \otimes_E z^{-1} \circ \det$ by $\pi(D_1) \otimes_E z^{-1} \circ \det \hookrightarrow J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}_\rho])[Z_3 = \phi_3 z^{h_3}|\cdot|^{-2}]$, which is contained in $J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}_\rho])[Z_3 = \phi_3 z^{h_3}|\cdot|^{-2}]$, lies in $\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}}$. Indeed, let V be such an extension. As $\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}}$ contains the universal extension of $\pi_{\mathrm{alg}}(D_1) \otimes_E z^{-1} \circ \det$ by $\pi(D_1) \otimes_E z^{-1} \circ \det$ (by the above commutative diagram), amalgamating V with certain subrepresentation of $\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}}$, we reduce to the case where V is split. But in this case, V is already contained in $\mathscr{E}_{g',P_1}$. Consequently, for such an extension, the induced $R_\rho^\square/\mathfrak{m}_\rho^2$-action factors through $\overline{R}_{\rho,\mathscr{F}}^\square$.*

Replacing the points $\{x, x_{s_1}\}$ by $\{x_{s_1 s_2}, x_{s_2 s_1 s_2}\}$, P_1 by P_2, Z_3 by $Z_1 := \mathrm{diag}(\mathbb{Q}_p^\times, 1, 1)$ and using the same arguments, we get

Corollary 4.12. *We have an $R_{C_1} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant commutative diagram*

$$\begin{array}{ccccccc}
\pi(C_1) \otimes |\cdot|^{-1} & \hookrightarrow & \mathscr{E}_{g',P_2} & \hookrightarrow & \pi_{\tilde{x}_{s_1 s_2}} \oplus_{\mathscr{E}_{g',P_2}} \pi_{\tilde{x}_{s_2 s_1 s_2}} & \hookrightarrow & J_{P_2}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])[Z_1 = \phi_3 z^{h_1-2}] \\
\| & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
\pi(C_1) \otimes |\cdot|^{-1} & \hookrightarrow & \widetilde{\pi}(C_1)^{\mathrm{univ}}_{g'} \otimes |\cdot|^{-1} & \hookrightarrow & \widetilde{\pi}(C_1)^{\mathrm{univ}} \otimes |\cdot|^{-1} & &
\end{array}$$

Moreover, the map $\phi_3 z^{h_1-2} \boxtimes \left(\pi_{\tilde{x}_{s_1 s_2}} \oplus_{\mathscr{E}_{g',P_2}} \pi_{\tilde{x}_{s_2 s_1 s_2}}\right) \to J_{P_2}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}])$ is balanced.

4.3 Surplus locally algebraic constituents and local-global compatibility

We prove our main local-global compatibility result. We keep the notation of the precedent section.

Lemma 4.13. *For any extension $\pi \in \mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$ or $\pi \in \mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$, π is not a subrepresentation of $\Pi_\infty[\mathfrak{m}]$.*

Proof. We only prove the case for $\mathscr{F}$, the case for $\mathscr{G}$ being the same. Suppose $\pi \hookrightarrow \Pi_\infty[\mathfrak{m}]$. By [7, Lem. 4.16], for all $w \in S_3$, we have ("$\{-\}$" denoting the corresponding generalized eigenspace)

$$\dim_E J_B(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}])[T(\mathbb{Q}_p) = \jmath(w(\underline{\phi}))\delta_B z^\lambda] = \dim_E J_B(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}])\{T(\mathbb{Q}_p) = \jmath(w(\underline{\phi}))\delta_B z^\lambda\} = 1.$$

Together with Proposition 3.3 (3) and Remark 3.4, we deduce $\pi \notin \mathrm{Ext}^1_{\mathscr{F}_i}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$. It is clear (by (16), Proposition 3.5)

$$\mathrm{Ext}^1_1(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda)) + \mathrm{Ext}^1_{s_1}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda)) = \mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda)).$$

There exists hence $\pi' \in \mathrm{Ext}^1_1(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$ such that

$$[\pi''] := [\pi] - [\pi'] \in \mathrm{Ext}^1_{s_1}(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda)).$$

By Remark 3.4, there exists a deformation $\widetilde{\chi}$ of $\jmath(\underline{\phi}) z^\lambda \delta_B$ (resp. $\widetilde{\chi}'$ of $\jmath(s_1(\underline{\phi})) z^\lambda \delta_B$) such that

$$\widetilde{\chi} \longrightarrow J_B(\pi') \text{ (resp. } \widetilde{\chi}' \longrightarrow J_B(\pi'')).$$

Let v be a non-zero element in the tangent space of $\mathcal{E}$ at x such that the associated character of $T(\mathbb{Q}_p)$ is $\widetilde{\chi}$, and $\mathcal{I}_v$ be its associated ideal of $R_\infty[1/p]$. Let $\widetilde{D}_v$ be the associated deformation of D. Then $\widetilde{\chi}^\sharp := \widetilde{\chi}\delta_B^{-1}(\varepsilon^2 \boxtimes \varepsilon \boxtimes 1)$ is a trianguline parameter of $\widetilde{D}_v$. Note $\widetilde{\chi}^\sharp$ has the form $\phi_1 z^{h_1}(1+\psi_1\epsilon) \boxtimes \phi_2 z^{h_2}(1+\psi_2\epsilon) \boxtimes \phi_3 z^{h_3}(1+(\psi_3)\epsilon)$. Using an easy variation of (26) for $\mathcal{E}$ instead of $\overline{\mathcal{E}}$ (see also the proof of [15, Prop. C.5]), we have $\pi' \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$. As $\pi \in \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$, we deduce $\pi'' \in \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$, hence $\widetilde{\chi}' \hookrightarrow J_B(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v])$. By construction of $\mathcal{E}$, we see $(\mathcal{I}_v, \widetilde{\chi}') \hookrightarrow \mathcal{E}$. But this implies $(\widetilde{\chi}')^\sharp$ is a trianguline parameter of $\widetilde{D}'_v$ where $\widetilde{D}'_v \cong \widetilde{D}_v$ as self-extension of D. Note $(\widetilde{\chi}')^\sharp$ has the form $\phi_2 z^{h_1}(1+\psi'_2\epsilon) \boxtimes \phi_1 z^{h_2}(1+\psi'_1\epsilon) \boxtimes \phi_3 z^{h_3}(1+\psi'_3\epsilon)$. Using Proposition 2.4 (2), it is not difficult to see there exists $a \in E^\times$ such that $\psi'_3 = a\psi_3$, and $\psi_1 - \psi_2, \psi'_1 - \psi'_2 \in \mathrm{Hom}_{\mathrm{sm}}(\mathbb{Q}_p^\times, E)$. By Remark 3.4, both π' and π'' lie in $\mathrm{Ext}^1_1(\pi_{\mathrm{alg}}(\underline{\phi}, \lambda), \pi_1(\underline{\phi}, \lambda))$, hence so does π, a contradiction. □

Theorem 4.14. *For an injection $\iota \in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1, C_1)$, $\pi(\underline{\phi}, \lambda, \iota) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$ if and only if $\iota = \iota_D$.*

Proof. Let $(\widetilde{D}_1, \widetilde{C}_1) \in \mathscr{I}_D^0$ and let $\widetilde{D}$ be an associated deformation of D with $\kappa_{\mathscr{F}}(\widetilde{D}) = (\widetilde{D}_1, 0)$ and $\kappa_{\mathscr{G}}(\widetilde{D}) = (\widetilde{C}_1, 0)$ (as in Theorem 2.13). Let v be a non-zero element in the tangent space of $R_{\rho,\mathscr{F}}^{\square}$ associated to $\widetilde{D}$, $\mathcal{I}_{v,\wp} \subset R_{\overline{\rho}}^{\square}[1/p]$ be the associated ideal, and $\mathcal{I}_v := \mathcal{I}_{v,\wp} + \mathfrak{m}^{\wp} \subset R_\infty[1/p]$ (which corresponds to the element $(v, 0)$ in the tangent space of $(\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times (\mathrm{Spf}\, R_\infty^{\wp})^{\mathrm{rig}}$ at $\mathfrak{m}$). We have a natural exact sequence (associated to the non-zero v)

$$0 \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}] \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v] \xrightarrow{\kappa} \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}].$$

Claim: $(\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}})[\mathcal{I}_v] \cong \mathrm{rec}(\widetilde{D}_1) \otimes_E z^{-1} \circ \det$.

Proof of the claim. Let $\mathcal{J}_v \subset R_{D_1}$ be the kernel of the natural map $R_{D_1} \to R_{\overline{\rho}}^{\square}[1/p]/\mathcal{I}_{v,\wp}$. The right vertical map in Corollary 4.10 induces $f : (\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}})[\mathcal{I}_v] \to \widetilde{\pi}(D_1)^{\mathrm{univ}}[\mathcal{J}_v] \otimes_E z^{-1} \circ \det \cong \mathrm{rec}(\widetilde{D}_1) \otimes_E z^{-1} \circ \det$. Note the kernel of the map is a finite copy of $\pi_{\mathrm{alg}}(D_1) \otimes_E z^{-1} \circ \det$. If f is not an isomorphism, $(\pi_{\tilde{x}} \oplus_{\mathscr{E}_{g',P_1}} \pi_{\tilde{x}_{s_1}})[\mathcal{I}_v]$ has to contain $(\pi_{\mathrm{alg}}(D_1) \oplus \pi(D_1)) \otimes_E z^{-1} \circ \det$. Applying the Jacquet-Emerton functor $J_{B \cap L_{P_1}}(-)$, and using [7, Lem. 4.16] and similar arguments as in the proof of Lemma 4.13, this implies that $\widetilde{D}$ is trianguline, a contradiction. The claim follows.

We obtain hence an injection $(\mathrm{rec}(\widetilde{D}_1) \otimes_E z^{-1} \circ \det) \boxtimes (\phi_3 z^{h_3} |\cdot|^{-2}) \hookrightarrow J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v])$, which is balanced by Corollary 4.10. By [19, Thm. 0.13] (noting similarly as in the discussion below Proposition 3.2, j^- can also be obtained by applying Emerton's $I_{P_1^-}^{\mathrm{GL}_3}(-)$), it induces $i^-(\widetilde{D}_1) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$. Note that the composition

$$i^-(\widetilde{D}_1) \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v] \xrightarrow{\kappa} \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$$

has image exactly equal to $\pi_{\mathrm{alg}}(\underline{\phi}, \lambda)$. Indeed, the inclusion is clear. However, the composition can not have zero image by Lemma 4.13. Similarly, we have $i^+(\widetilde{C}_1) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$ whose composition with κ has image equal to $\pi_{\mathrm{alg}}(\underline{\phi}, \lambda)$ as well (noting v can also be viewed as an element in the tangent space of $R_{\rho,\mathscr{G}}^{\square}$ by Theorem 2.13). So there is an extension $\pi \subset i^-(\widetilde{D}_1) \oplus_{\pi_1(\underline{\phi},\lambda)} i^+(\widetilde{C}_1)$ of $\pi_{\mathrm{alg}}(\underline{\phi}, \lambda)$ by $\pi_1(\underline{\phi}, \lambda)$ such that $\pi \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$.

We show π has to be isomorphic to $\pi(\underline{\phi}, \lambda, \iota_D)$. As in the proof of Proposition 3.10, let $(x_0, y_0) \in \mathscr{I}_D$ be non-zero de Rham, and $([\widetilde{D}_1'] = [\widetilde{D}_1] + x_0, [\widetilde{C}_1'] = [\widetilde{C}_1] + y_0) \in \mathscr{I}_D^0$. Let v', $\widetilde{D}'$, $\mathcal{I}_{v'}$ be the similar objects associated to the pair. By the same argument, $i^-(\widetilde{D}_1')$, $i^+(\widetilde{C}_1')$ are both subrepresentations of $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_{v'}]$. We also know $\pi \subset \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}] \subset \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_{v'}]$. If π is not isomorphic to $\pi(\underline{\phi}, \lambda, \iota_D)$, $[\pi]$ has the form $a i^-([\widetilde{D}_1]) - b i^+([\widetilde{C}_1])$ with $a \neq b$. Moreover, by Proposition 3.10 (and the proof), π is not contained in $i^-(\widetilde{D}_1') \oplus_{\pi_1(\underline{\phi},\lambda)} i^+(\widetilde{C}_1')$, we have hence an injection

$$\pi \oplus_{\pi_1(\underline{\phi},\lambda)} i^-(\widetilde{D}_1') \oplus_{\pi_1(\underline{\phi},\lambda)} i^+(\widetilde{C}_1') \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_{v'}].$$

Using $i^-([x_0]) = i^+([y_0])$ (for example see the proof of Lemma 3.9), it is not difficult to deduce (the representation corresponding to) $i^-([x_0])$ injects into $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_{v'}]$. However, applying $J_B(-)$ and (again) using [7, Lem. 4.16] and similar arguments in the proof of Lemma 4.13, this will imply that $\widetilde{D}'$ is trianguline, a contradiction.

We prove the "only if" part of the theorem. Suppose there exists an injection $\iota \notin E[\iota_D]$ such that $\pi(\underline{\phi}, \lambda, \iota) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$. Recall the extension class $[\pi(\underline{\phi}, \lambda, \iota)]$ has the form $i^-([\widetilde{D}_1]) - i^+([\widetilde{C}_1])$ with $(\widetilde{D}_1, \widetilde{C}_1) \in \mathscr{I}_\iota^0$. By Proposition 2.10, $\widetilde{D}_1 \notin \mathrm{Ext}^1_{\iota_D}(D_1, D_1)$ and $\widetilde{C}_1 \notin \mathrm{Ext}^1_{\iota_D}(C_1, C_1)$. Let $\widetilde{D}$ be a deformation of D such that $\kappa_{\mathscr{G}}(\widetilde{D}) = (\widetilde{C}_1, 0)$ (cf. Remark 2.8), and v, $\mathcal{I}_v$ be associated to $\widetilde{D}$ similarly as above. By the same argument (using the claim with D_1 replaced by C_1), we have $i^+(\widetilde{C}_1) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$. As $\pi(\underline{\phi}, \lambda, \iota) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}] \subset \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$, we deduce $i^-(\widetilde{D}_1) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v]$ hence $\mathrm{rec}(\widetilde{D}_1) \otimes_E z^{-1} \circ \det \hookrightarrow J_{P_1}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}_v])[Z_3 = \phi_3 z^{h_3} |\cdot|^{-2}]$. By Remark 4.11, the $R_\rho^\square$-action on $i^-(\widetilde{D}_1)$ factors through $\overline{R}_{\rho,\mathscr{F}}^\square$, hence so does its action on $i^+(\widetilde{C}_1)$ (whose extension class is equal to $[\pi(\underline{\phi}, \lambda, \iota)] + i^-([\widetilde{D}_1])$, noting the $R_\rho^\square$-action on $\pi(\underline{\phi}, \lambda, \iota)$ factors through $R_\rho^\square/\mathfrak{m}_\rho$). However, as $\widetilde{C}_1 \notin \mathrm{Ext}^1_{\iota_D}(C_1, C_1)$ hence $\widetilde{D} \notin \mathrm{Ext}^1_{\mathscr{F}}(D, D)$ (Theorem 2.13), we see the image of $\mathcal{I}_{v,\wp}$ in $\overline{R}_{\rho,\mathscr{F}}^\square$ is $\mathfrak{m}$. Thus $i^+(\widetilde{C}_1)$ ($\subset \Pi_\infty^{R_\infty}[\mathcal{I}_v]$) has to be annihilated by $\mathfrak{m}$, contradicting Lemma 4.13. $\square$

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