p-adic Hodge parameters in the crystabelline representations of GL_n

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Abstract

Let K be a finite extension of $\mathbb{Q}_p$, and ρ be an n-dimensional (non-critical generic) crystabelline representation of the absolute Galois group of K of regular Hodge-Tate weights. We associate to ρ an explicit locally $\mathbb{Q}_p$-analytic representation $\pi_1(\rho)$ of $\mathrm{GL}_n(K)$, which encodes some p-adic Hodge parameters of ρ. When $K = \mathbb{Q}_p$, it encodes the full information hence reciprocally determines ρ. When ρ is associated to p-adic automorphic representations, we show under mild hypotheses that $\pi_1(\rho)$ is a subrepresentation of the $\mathrm{GL}_n(K)$-representation globally associated to ρ.

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1 Introduction

The locally analytic p-adic Langlands program for $\mathrm{GL}_n(\mathbb{Q}_p)$ aims at building a correspondence between n-dimensional p-adic continuous representations of the abosulte Galois group $\mathrm{Gal}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ and certain locally analytic representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. In particular, it is expected to match the parameters on both sides via the conjectural correspondence.

On the Galois side, the p-adic $\mathrm{Gal}_{\mathbb{Q}_p}$-representations are central objects in the p-adic Hodge theory, and are classified by Fontaine's theory. Among these representations, the de Rham ones are particularly important, as they include those arising from geometry ([29]). The p-adic Langlands program for de Rham representations is expected to be compatible with the classical local Langlands correspondence (e.g. see [16]). More precisely, by Fontaine's theory, for a de Rham representation ρ over a p-adic field E, one can associate an n-dimensional Weil-Deligne representation $\mathtt{r}$, which furthermore corresponds, via the classical local Langlands correspondence, to an irreducible smooth representation $\pi_{\mathrm{sm}}(\mathtt{r})$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E. If ρ has regular Hodge-Tate weights $\mathbf{h} = (h_1, \cdots, h_n)$, then the locally algebraic representation

$$\pi_{\mathrm{alg}}(\mathtt{r}, \mathbf{h}) := \pi_{\mathrm{sm}}(\mathtt{r}) \otimes_E L(\mathbf{h} - \theta)$$

is expected to be the locally algebraic subrepresentation of the conjectural locally analytic representation $\pi^?(\rho)$ associated to ρ, where $\theta = (n-1, \cdots, 0)$ and $L(\mathbf{h}-\theta)$ is the algebraic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ of highest weight $\mathbf{h}-\theta$. One can clearly recover $\mathtt{r}$ (up to F-semi-simplification) and $\mathbf{h}$ from the representation of $\pi_{\mathrm{alg}}(\mathtt{r}, \mathbf{h})$. However, passing from ρ to $(\mathtt{r}, \mathbf{h})$, one loses the information of Hodge filtration of ρ. A fundamental question in the p-adic Langlands program is to find the missing information on Hodge filtration on the automorphic side, say, in the conjectural locally analytic representation $\pi^?(\rho)$. After the pioneer work of Breuil ([4][6]), the question was settled for $\mathrm{GL}_2(\mathbb{Q}_p)$ by Colmez, establishing the p-adic Langlands correspondence ([21]). It remains quite mysterious for general $\mathrm{GL}_n(\mathbb{Q}_p)$. In this paper, we address the question for (non-critical generic) crystabelline $\mathrm{Gal}_{\mathbb{Q}_p}$-representations ρ, those that become crystalline when restricted to the absolute Galois group of a certain abelian extension of $\mathbb{Q}_p$.

For simplicity, we assume in the introduction that ρ itself is crystalline. Then by Fontaine's theory, ρ is equivalent to the associated filtered φ-module $D_{\mathrm{cris}}(\rho)$. We assume the φ-action is *generic* (and we simply call such ρ generic), which means the φ-eigenvalues $\underline{\alpha} = (\alpha_i)$ on $D_{\mathrm{cris}}(\rho)$ are distinct, and $\alpha_i\alpha_j^{-1} \neq p$ for $i \neq j$. In this case, $\mathtt{r} \cong \oplus_{i=1}^n \mathrm{unr}(\alpha_i)$ and we denote $\mathtt{r}$ by $\underline{\alpha}$. The classical local Langalnds correspondence in this case is simply given by

$$\pi_{\mathrm{sm}}(\underline{\alpha}) \cong (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \mathrm{unr}(\underline{\alpha})\eta)^{\infty}$$

where $\mathrm{unr}(\underline{\alpha}) = \mathrm{unr}(\alpha_1)\boxtimes \cdots \mathrm{unr}(\alpha_n)$, $\eta = |\cdot|^{1-n}\boxtimes|\cdot|^{2-n}\boxtimes\cdots\boxtimes 1$ are unramified characters of $T(\mathbb{Q}_p)$, and B^- is the Borel subgroup of lower triangular matrices. Let $\mathrm{Fil}_H^{\bullet}$ denote the Hodge filtration, which is a complete flag in $D_{\mathrm{cris}}(\rho)$ as $\mathbf{h}$ is regular. Let $e_i \in D_{\mathrm{cris}}(\rho)$ be an eigenvector for α_i. Under the basis $\{e_i\}$, $\mathrm{Fil}_H^{\bullet}$ is parametrized by an element in $T\backslash \mathrm{GL}_n / B$, which we call the *p-adic Hodge parameter* of ρ. Recall that ρ is called *non-critical* if $\mathrm{Fil}_H^{\bullet}$ is in a relative general position with respect to all the $n!$ φ-stable (complete) flags. When $n=2$, $T\backslash \mathrm{GL}_2 / B$ is a finite set of cardinality

3. So there are at most 3 isomorphism classes[1] of ρ, distinguished by the relative position of $\mathrm{Fil}_H^\bullet$ with the two φ-stable flags. The information is reflected by the extra socle phenomenon on the $\mathrm{GL}_2(\mathbb{Q}_p)$-side. In this context, Breuil formulated a conjecture concerning the locally analytic socle of GL_n, which characterizes the relative positions of $\mathrm{Fil}_H^\bullet$ with the φ-stable flags. The conjecture was subsequently proved (under Taylor-Wiles hypotheses) by Breuil-Hellmann-Schraen ([14]). However, a significant difference between the cases $n = 2$ and $n \geq 3$ lies in the extra parameters for non-critical ρ (with fixed $(\underline{\alpha}, \mathbf{h})$): when $n = 2$, the non-critical ρ is unique, whereas for $n \geq 3$, there are additional (new) parameters for non-critical ρ (as $T\backslash \mathrm{GL}_n /B$ is now an infinite set). We refer to Example 2.10 for a concrete example of $n = 3$.

In the paper, we reveal these p-adic Hodge parameters on the $\mathrm{GL}_n(\mathbb{Q}_p)$-side. It turns out it is convenient to work with (φ, Γ)-modules over the Robba ring instead of Galois representations. Denote by $\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$ the set of isomorphism classes of non-critical crystalline (φ, Γ)-modules overlying $\underline{\alpha}$ of regular Hodge-Tate weights $\mathbf{h}$. Under the basis of φ-eigenvectors $\{e_i\}$ in the precedent paragraph (noting that $D_{\mathrm{cris}}(D) \cong \oplus_{i=1}^n E e_i$, as φ-module, for all $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$), the set $\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$ can be identified with a Zariski open subset of $T\backslash \mathrm{GL}_n /B$. For each $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$, we associate an explicit locally analytic $\mathrm{GL}_n(\mathbb{Q}_p)$-representation $\pi_1(D)$ (see Theorem 1.3 below for the construction). We have:

Theorem 1.1. *(1) (Local correspondence) For $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \boldsymbol{h})$, $\mathrm{soc}_{\mathrm{GL}_n(\mathbb{Q}_p)} \pi_1(D) \cong \pi_{\mathrm{alg}}(\underline{\alpha}, \boldsymbol{h})$, and $\pi_1(D) \twoheadrightarrow \pi_{\mathrm{alg}}(\underline{\alpha}, \boldsymbol{h})^{\oplus(2^n - \frac{n(n+1)}{2} - 1)}$. Moreover, for $D' \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \boldsymbol{h})$, $\pi_1(D) \cong \pi_1(D')$ if and only if $D' \cong D$.*

(2) (Local-global compatibility) Suppose ρ is automorphic for the setting of [18] (or the setting in § 4.2.2), and let $\widehat{\pi}(\rho)$ be the unitary Banach representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ (globally) associated to ρ. Assume $D_{\mathrm{rig}}(\rho) \in \Phi\Gamma_{\mathrm{nc}}(\phi, \boldsymbol{h})$. Then for $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \boldsymbol{h})$,

$$\pi_1(D) \hookrightarrow \widehat{\pi}(\rho)^{\mathrm{an}} \text{ if and only if } D \cong D_{\mathrm{rig}}(\rho).$$

In particular, $\widehat{\pi}(\rho)^{\mathrm{an}}$ determines ρ.

The quotient $\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})^{\oplus(2^n - \frac{n(n+1)}{2} - 1)}$ of $\pi_1(D)$ appears in the "third" layer in its socle filtration. Let $\pi_{\min}(D)$ be the minimal subrepresentation of $\pi_1(D)$ such that the composition $\pi_{\min}(D) \hookrightarrow \pi_1(D) \twoheadrightarrow \pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})^{\oplus(2^n - \frac{n(n+1)}{2} - 1)}$ is surjective. The representation $\pi_{\min}(D)$ has a much cleaner structure, for example, its socle filtration has only three grades (see § 3.2). Note that one can replace everywhere $\pi_1(D)$ in the statements by $\pi_{\min}(D)$. The extra locally algebraic constituents in the cosocle of $\pi_1(D)$ were unexpected, not to mention their huge multiplicity. It is one of the reasons why it took a long time to find the Hodge parameters. In fact, the work grows out from the finding of such extra constituents in [26] while trying to exclude such constituents for GL_2. We remark that the existence of the extra locally algebraic constituent was first proved by Hellmann-Hernandez-Schraen in the split case for $\mathrm{GL}_3(\mathbb{Q}_p)$ ([31]). The multiplicity $2^n - \frac{n(n+1)}{2} - 1$ can be explained as follows: for each choice of 3 distinct φ-eigenvalues, the corresponding 3-dimensional filtered φ-submodule of $D_{\mathrm{cris}}(D)$ carries one parameter. With these parameters fixed, the associated 4-dimensional filtered φ-submodule for each choice of 4 distinct φ-eigenvalues adds an additional parameter... Continuing in this way, the total count of parameters amounts to $(\binom{n}{3} + \binom{n}{4} + \cdots + \binom{n}{n}) = 2^n - \frac{n(n+1)}{2} - 1$.

For a finite extension K of $\mathbb{Q}_p$, we also construct a locally $\mathbb{Q}_p$-analytic representation $\pi_1(D)$ with $\mathrm{soc}_{\mathrm{GL}_n(K)} \pi_1(D) \cong \pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})$ and $\pi_1(D) \twoheadrightarrow \pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})^{\oplus(2^n - \frac{n(n+1)}{2} - 1)[K:\mathbb{Q}_p]}$. The local-global

[1] The étaleness of ρ will imply that some of these classes may not occur. In most general cases, there is typically a unique isomorphism class. But note if we relax the étaleness condition, and consider crystalline (φ, Γ)-modules instead of ρ, all these classes can appear.

compatibility result still holds. But a major difference is that when $K \neq \mathbb{Q}_p$, $\pi_1(D)$ just determines the filtered φ^f-module $D_{\mathrm{cris}}(D)_\sigma$ (where f is the unramified degree of K over $\mathbb{Q}_p$) for each embedding σ rather than D itself. For example, when $n=2$, $\pi_1(D)$ are all isomorphic (for different $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$) but there are still extra parameters, see for example [5, § 3].

We make a few additional remarks on Theorem 1.1.

Remark 1.2. *(1) Very little was known about such a local correspondence when $n \geq 3$. We highlight some related results. When $n=3$, in [10], we showed how to recover the Hodge parameters in the semi-stable non-crystalline case (given by the Fontaine-Mazur $\mathcal{L}$-invariants) in the locally analytic $\mathrm{GL}_3(\mathbb{Q}_p)$-representations and proved a local-global compatibility result in the ordinary case. When the Weil-Deligne representation $\mathtt{r}$ associated to ρ is indecomposable, the (largely open) conjecture on Ext^1 in [9] (see also [11]) suggests a way to recover the p-adic Hodge parameters on the automorphic side. In contrast, the (non-critical) crystalline case was somewhat more mysterious, as such parameters are entirely new for $n \geq 3$. We finally mention that the results for $\mathrm{GL}_3(\mathbb{Q}_p)$ were presented in the note [25] (not intended for publication), which may help readers quickly understand the story.*

(2) The phenomenon where the Hodge parameters lie in the extension group of certain locally algebraic representation by certain locally analytic representation traces back to Breuil's initializing work in [4].

(3) Similar results are also obtained in the patched setting. Let Π_∞ be the patched Banach representation over the patched Galois deformation ring R_∞ of [18]. We show that if there is a maximal ideal $\mathfrak{m}_\rho$ of $R_\infty[1/p]$ associated to ρ such that $\Pi_\infty[\mathfrak{m}_\rho]^{\mathrm{lalg}} \neq 0$, then for $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \boldsymbol{h})$,

$$\pi_1(D) \hookrightarrow \Pi_\infty[\mathfrak{m}_\rho] \text{ if and only if } D \cong D_{\mathrm{rig}}(\rho).$$

(4) We finally remark the representation $\pi_1(D)$ should still be far from the final complete locally analytic $\mathrm{GL}_n(\mathbb{Q}_p)$-representation associated to D (so we choose not to use the notation $\pi(D)$).

We now give the construction of $\pi_1(D)$. We first look at the Galois side. For each $w \in S_n$, let $\mathrm{Ext}^1_w(D,D)$ be the extension group of trianguline deformations of D with respect to the refinement $w(\underline{\alpha})$ (see the discussion above (2.5)). Recall there is a natural (weight) map

$$\kappa_w : \mathrm{Ext}^1_w(D,D) \longrightarrow \mathrm{Hom}(T(\mathbb{Q}_p), E)$$

sending $\widetilde{D}$ to ψ such that $\widetilde{D}$ is trianguline with parameter $\mathrm{unr}(w(\underline{\alpha}))z^{\mathbf{h}}(1+\psi\epsilon)$ (that is a character of $T(\mathbb{Q}_p)$ over $E[\epsilon]/\epsilon^2$). The map κ_w is surjective (e.g. see [2, Prop. 2.3.10]). One can show that $\mathrm{Ker}\,\kappa_w$, as a subspace of $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$, is independent of the choice of w, denoted by $\mathrm{Ext}^1_0(D,D)$ (cf. Lemma 2.12). For a subspace $\mathrm{Ext}^1_?(D,D) \subset \mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$ containing $\mathrm{Ext}^1_0(D,D)$, set $\overline{\mathrm{Ext}}^1_?(D,D) := \mathrm{Ext}^1_?(D,D)/\mathrm{Ext}^1_0(D,D)$. We have hence a bijection

$$\kappa_w : \overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow{\sim} \mathrm{Hom}(T(\mathbb{Q}_p), E).$$

By [19], the following "amalgamating" map is surjective (see also [36] [30])

$$\oplus_{w \in S_n} \overline{\mathrm{Ext}}^1_w(D,D) \twoheadrightarrow \overline{\mathrm{Ext}}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(D,D). \tag{1.1}$$

Now we look at the $\mathrm{GL}_n(\mathbb{Q}_p)$-side. For each w, consider the locally analytic principal series (ε denoting the cyclotomic character)

$$\mathrm{PS}(w, \underline{\alpha}, \mathbf{h}) := (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \mathrm{unr}(w(\underline{\alpha}))z^{\mathbf{h}}\varepsilon^{-1} \circ \theta)^{\mathrm{an}}.$$

The explicit structure of $\mathrm{PS}(w, \underline{\alpha}, \mathbf{h})$ is well-understood by Orlik-Strauch ([42]). For example, we have $\mathrm{soc}_{\mathrm{GL}_n(\mathbb{Q}_p)} \mathrm{PS}(w, \underline{\alpha}, \mathbf{h}) \cong \pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})$, which has multiplicity one as irreducible constituent of $\mathrm{PS}(w, \underline{\alpha}, \mathbf{h})$. For $w \in S_n$, consider the composition

$$\zeta_w : \mathrm{Hom}(T(\mathbb{Q}_p), E) \to \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\mathrm{PS}(w, \underline{\alpha}, \mathbf{h}), \mathrm{PS}(w, \underline{\alpha}, \mathbf{h})) \to \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \mathrm{PS}(w, \underline{\alpha}, \mathbf{h})),$$

where the first map sends ψ to $(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \mathrm{unr}(w(\underline{\alpha}))z^{\mathbf{h}}(\varepsilon^{-1} \circ \theta)(1 + \psi\epsilon))^{\mathrm{an}}$, and the second map is the natural pull-back map. Using Schraen's spectral sequence ([45, (4.38)]), one can show that ζ_w is in fact bijective. Now we amalgamate these principal series: let $\pi(\underline{\alpha}, \mathbf{h})$ be the unique quotient of the amalgamation $\oplus^{w\in S_n}_{\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)} \mathrm{PS}(w, \underline{\alpha}, \mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\underline{\alpha}, \lambda)$ (which was introduced and denoted by $\pi(D)^{\mathrm{fs}}$ in [15]). For each $w \in S_n$, there is a natural injection $\mathrm{PS}(w, \underline{\alpha}, \mathbf{h}) \hookrightarrow \pi(\underline{\alpha}, \mathbf{h})$ which induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \mathrm{PS}(w, \underline{\alpha}, \mathbf{h})) \longhookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \pi(\underline{\alpha}, \mathbf{h})).$$

We denote by $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \pi(\underline{\alpha}, \mathbf{h}))$ its image. The following "amalgamating" map is also surjective (see Proposition 3.7 (2) and compare with (1.1)):

$$\oplus_{w\in S_n} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \pi(\underline{\alpha}, \mathbf{h})) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \pi(\underline{\alpha}, \mathbf{h})). \tag{1.2}$$

The construction of $\pi_1(D)$ follows from the following theorem.

Theorem 1.3 (cf. Theorem 3.21)**.** *For $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \boldsymbol{h})$, there is a unique (surjective) map*

$$t_D : \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \boldsymbol{h}), \pi(\underline{\alpha}, \boldsymbol{h})) \twoheadrightarrow \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D, D)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \oplus_{w\in S_n} \overline{\mathrm{Ext}}^1_w(D, D) & \xrightarrow[\sim]{(\zeta_w\circ\kappa_w)} & \oplus_w \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha}, \boldsymbol{h}), \pi(\underline{\alpha}, \boldsymbol{h})) \\ {\scriptstyle (1.1)}\downarrow & & \downarrow{\scriptstyle (1.2)} \\ \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D, D) & \xleftarrow{t_D} & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \boldsymbol{h}), \pi(\underline{\alpha}, \boldsymbol{h})). \end{array}$$

Moreover, $\dim_E \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D, D) = \frac{n(n+1)}{2} + n$, $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\alpha, \boldsymbol{h}), \pi(\alpha, \boldsymbol{h})) = 2^n + n - 1$ *hence* $\dim_E \mathrm{Ker}(t_D) = 2^n - \frac{n(n+1)}{2} - 1$.

The representation $\pi_1(D)$ is then defined to be the (tautological) extension of $\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}) \otimes_E \mathrm{Ker}(t_D) \cong \pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})^{\oplus(2^n - \frac{n(n+1)}{2} - 1)}$ by $\pi(\underline{\alpha}, \mathbf{h})$. More precisely, choosing a basis $\{v_i\}$ of $\mathrm{Ker}(t_D)$ with $\mathscr{E}(v_i)$ the associated extension of $\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})$ by $\pi(\underline{\alpha}, \mathbf{h})$, $\pi_1(D)$ is the amalgamated sum of these $\mathscr{E}(v_i)$ along $\pi(\underline{\alpha}, \mathbf{h})$, which is clearly independent of the choice of $\{v_i\}$. The structure of $\pi(\underline{\alpha}, \mathbf{h})$ is complicated (see for example [15]). However, the theorem actually holds with $\pi(\underline{\alpha}, \mathbf{h})$ replaced by its subrepresentation given by the first two layers in its socle filtration, which has a much easier and cleaner structure, see Theorem 3.21 and § 3.1.2. The extension of $\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}) \otimes_E \mathrm{Ker}(t_D)$ by this subrepresentation actually gives $\pi_{\min}(D)$ in the discussion below Theorem 1.1.

One can deduce from Theorem 1.3:

Corollary 1.4 (cf. Corollary 3.24)**.** *The map t_D induces a bijection*

$$t_D : \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}, \boldsymbol{h}), \pi_1(D)) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D, D).$$

It is not so clear from the construction that $\pi_1(D)$ determines D, and we will discuss this point a bit later.

We first explain the proof of the local-global compatibility (Theorem 1.1 (2)). For this, we will use an alternative formulation of Theorem 1.3 given as follows. Let π^{univ} (resp. π_w^{univ}) be the (universal) extension of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_E \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))$ $\big($resp. of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_E \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))\big)$ by $\pi(\underline{\alpha},\mathbf{h})$ (defined in a similar way as in the discussion below Theorem 1.3). By (1.2), π^{univ} is generated by all the subrepresentations π_w^{univ} for $w\in S_n$. On the Galois side, letting R_D be the universal deformation ring of deformations of D over Artinian local E-algebras and $\mathfrak{m}$ be its maximal ideal, the quotient $\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D,D)$ corresponds to a local Artinian E-subalgebra A_D of $R_D/\mathfrak{m}^2$, and $\overline{\mathrm{Ext}}^1_w(D,D)$ corresponds to a quotient $A_{D,w}$ of A_D. Using the isomorphism $\zeta_w\circ\kappa_w$, there exists a natural action of $A_{D,w}$ on π_w^{univ} such that $x\in \mathfrak{m}_{A_{D,w}}/\mathfrak{m}^2_{A_{D,w}}\cong \overline{\mathrm{Ext}}^1_w(D,D)^\vee\cong \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))^\vee$ acts via

$$\pi_w^{\mathrm{univ}}\twoheadrightarrow \pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_E \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))\xrightarrow{x}\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\hookrightarrow \pi_w^{\mathrm{univ}}.$$

The following corollary gives a reformulation of Theorem 1.3.

Corollary 1.5 (cf. Theorem 3.35, Corollary 3.36)**.** *There exists a unique action of A_D on π^{univ} such that for each $w\in S_n$, the A_D-action on its subrepresentation π_w^{univ} factors through the natural $A_{D,w}$-action. Moreover, we have $\pi_1(D)\cong\pi^{\mathrm{univ}}[\mathfrak{m}_{A_D}]$.*

Suppose we are in the patched setting as in Remark 1.2 (4), and let $D=D_{\mathrm{rig}}(\rho)$. Working with the patched eigenvariety of [13], and using the global triangulation theory ([34][38]) and Emerton's adjunction formula [27], we can obtain $A_D\times\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant injections $\pi_w^{\mathrm{univ}}\hookrightarrow\Pi_\infty[\mathfrak{m}_\rho^2]$ for all $w\in S_n$, where the A_D-action on the right hand side comes from the R_∞-action (noting R_D is isomorphic to the universal Galois deformation ring of ρ). These injections "amalgamate" to an $A_D\times\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant injection

$$\pi^{\mathrm{univ}}\hookrightarrow\Pi_\infty[\mathfrak{m}_\rho^2], \tag{1.3}$$

hence by Corollary 1.5 an injection $\iota:\pi_1(D)\cong\pi^{\mathrm{univ}}[\mathfrak{m}_{A_D}]\hookrightarrow\Pi_\infty[\mathfrak{m}_\rho^2+\mathfrak{m}_{A_D}]$. With some extra arguments (where we refer to § 4.1 for details), one can show ι has image contained in $\Pi_\infty[\mathfrak{m}_\rho]$. Now for $D'\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$, if $\pi_1(D')\hookrightarrow\Pi_\infty[\mathfrak{m}_\rho]$, one can prove (cf. the proof of Corollary 4.6) that it factors through the injection (1.3), i.e. we have

$$\pi_1(D')\hookrightarrow\pi^{\mathrm{univ}}\hookrightarrow\Pi_\infty[\mathfrak{m}_\rho^2].$$

As $A_D(\hookrightarrow R_D/\mathfrak{m}^2)$ acts on $\Pi_\infty[\mathfrak{m}_\rho]$ hence on its sub $\pi_1(D')$ via $A_D/\mathfrak{m}_{A_D}$ and (1.3) is A_D-equivariant, $\pi_1(D')\hookrightarrow\pi^{\mathrm{univ}}$ has image contained in $\pi^{\mathrm{univ}}[\mathfrak{m}_{A_D}]\cong\pi_1(D)$. As $\pi_1(D')$ and $\pi_1(D)$ have the same irreducible constituents with the same multiplicities, this implies $\pi_1(D')\xrightarrow{\sim}\pi_1(D)$.

We now explain why $\pi_1(D)$ determines D. We use an induction argument. We first discuss how to see the Hodge parameters inductively on the Galois side. Let $\underline{\alpha}^1:=(\alpha_1,\cdots,\alpha_{n-1})$, $\mathbf{h}^1:=(h_1>h_2>\cdots>h_{n-1})$ and $\mathbf{h}^2:=(h_2>h_3>\cdots>h_n)$. Then D admits a unique saturated (φ,Γ)-submodule D_1 (resp. a unique quotient C_1) such that $D_1\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}^1,\mathbf{h}^1)$ (resp. $C_1\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}^1,\mathbf{h}^2)$). In fact, we have two filtrations on D:

$$\mathscr{F}:\quad 0\longrightarrow D_1\longrightarrow D\longrightarrow\mathcal{R}_E(\mathrm{unr}(\alpha_n)z^{h_n})\longrightarrow 0,$$
$$\mathscr{G}:\quad 0\longrightarrow\mathcal{R}_E(\mathrm{unr}(\alpha_n)z^{h_1})\longrightarrow D\longrightarrow C_1\longrightarrow 0.$$

Let $\iota_D\in\mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$ be the composition

$$\iota_D: D_1\hookrightarrow D\twoheadrightarrow C_1.$$

Proposition 1.6 (cf. Proposition 2.3)**.** *For $n\geq 3$, we have* $\dim_E \mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)=2$, *and D is determined by D_1, C_1, α_n and (the E-line generated by) ι_D.*

Set $\Phi\Gamma_{\mathrm{nc}}(D_1,C_1,\alpha_n)\subset \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$ to be the subset of isomorphism classes of (φ,Γ)-modules D satisfying moreover $\mathrm{Hom}_{(\varphi,\Gamma)}(D_1,D)\cong \mathrm{Hom}_{(\varphi,\Gamma)}(D,C_1)\cong E$. By Proposition 1.6, $\Phi\Gamma_{\mathrm{nc}}(D_1,C_1,\alpha_n)$ can be identified with a subset of the projective space of $\mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$. We will show, under an induction hypothesis, that ι_D can be revealed in $\pi_1(D)$. For this, we first show that ι_D can be detected by paraboline deformations of D with respect to the two filtrations $\mathscr{F}$ and $\mathscr{G}$. Similarly as in (1.1) by considering the paraboline deformations with respect to $\mathscr{F}$ and $\mathscr{G}$, we have a natural map

$$\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D_1,D_1)\oplus \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(C_1,C_1)\longrightarrow \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D,D), \tag{1.4}$$

sending $\widetilde{D}_1$ (resp. $\widetilde{C}_1$) to a (or any) deformation $\widetilde{D}$ of D of the form (whose image in $\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D,D)$ does not depend on the choice):

$$0\longrightarrow \widetilde{D}_1\longrightarrow \widetilde{D}\longrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\mathrm{unr}(\alpha_n)z^{h_n})\longrightarrow 0,$$

$$(\text{resp. } 0\longrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\mathrm{unr}(\alpha_n)z^{h_1})\longrightarrow \widetilde{D}\longrightarrow \widetilde{C}_1\longrightarrow 0).$$

We determine the kernel of (1.4). For any $\iota\in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$, consider the natural maps

$$\begin{aligned}\iota^- &: \mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1,D_1)\longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1,D_1)\longrightarrow \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D_1,D_1),\\ \iota^+ &: \mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1,D_1)\longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1,C_1)\longrightarrow \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(C_1,C_1).\end{aligned}$$

We set V_ι to be the image of the following map

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}(C_1,D_1)\xrightarrow{(\iota^-,\iota^+)}\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D_1,D_1)\oplus \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(C_1,C_1).$$

Theorem 1.7 (cf. Corollary 2.34)**.** *(1) For ι, $\iota'\in \mathrm{Hom}_{(\varphi,\Gamma)}(D_1,C_1)$, $V_\iota=V_{\iota'}$ if and only $E\iota=E\iota'$.*

(2) We have $\mathrm{Ker}(1.4)=V_{\iota_D}$.

We move to the automorphic side. Applying Theorem 1.3 to D_1 and C_1 respectively, we obtain locally analytic representations $\pi_1(D_1)$ and $\pi_1(C_1)$ of $\mathrm{GL}_{n-1}(\mathbb{Q}_p)$. We have by Corollary 1.4:

$$\begin{aligned}t_{D_1}: \mathrm{Ext}^1_{\mathrm{GL}_{n-1}(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^1),\pi_1(D_1)) &\xrightarrow{\sim} \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D_1,D_1),\\ t_{C_1}: \mathrm{Ext}^1_{\mathrm{GL}_{n-1}(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^2),\pi_1(C_1)) &\xrightarrow{\sim} \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(C_1,C_1).\end{aligned}$$

The representation $\pi_1(D)$ is in fact compatible with parabolic inductions and paraboline deformations (see Proposition 3.27). In particular, we have natural injections

$$\begin{aligned}j_{D_1}&: I_{P^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\big((\pi_1(D_1)\varepsilon^{-1})\boxtimes \mathrm{unr}(\alpha_n)z^{h_n}\big)\lhook\joinrel\longrightarrow \pi_1(D)\\ j_{C_1}&: I_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\big((\mathrm{unr}(\alpha_n)z^{h_1}\varepsilon^{1-n})\boxtimes \pi_1(C_1)\big)\lhook\joinrel\longrightarrow \pi_1(D),\end{aligned}$$

where $I_{P^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(V)$ denotes Emerton's parabolic induction ([27]), which roughly speaking is the minimal closed subrepresentation of the standard locally analytic parabolic induction containing V, and where $P=\begin{pmatrix}\mathrm{GL}_{n-1} & * \\ 0 & \mathrm{GL}_1\end{pmatrix}$, and $Q=\begin{pmatrix}\mathrm{GL}_1 & * \\ 0 & \mathrm{GL}_{n-1}\end{pmatrix}$. As $\pi_1(D_1)$ (resp. $\pi_1(C_1)$) has exactly $(2^{n-1}-\frac{n(n-1)}{2}-1)$-copies of $\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^1)$ (resp. $\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^2)$) in the cosocle, it is not difficult to see there are exactly $(2^{n-1}-\frac{n(n-1)}{2}-1)$-copies of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})$ in the cosocle of both $\mathrm{Im}(j_{D_1})$ and $\mathrm{Im}(j_{C_1})$. One can show furthermore these locally algebraic constituents are "disjoint" (e.g. using Proposition 3.13 and Lemma 3.23): letting $\pi_1(D_1,C_1,\alpha_n)$ be the subrepresentation of $\pi_1(D)$ generated by the image of j_{D_1} and j_{C_1}, and $\pi(\underline{\alpha},\mathbf{h})$, then $\pi_1(D_1,C_1,\alpha_n)$ is isomorphic to an extension of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})^{\oplus 2(2^{n-1}-\frac{n(n-1)}{2}-1)}$ by $\pi(\underline{\alpha},\mathbf{h})$. We have compositions

$$j^-:\overline{\mathrm{Ext}}^1(D_1,D_1)\xrightarrow[\sim]{t_{D_1}^{-1}}\mathrm{Ext}^1_{\mathrm{GL}_{n-1}}\big(\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^1),\pi_1(D_1)\big)\to\mathrm{Ext}^1_{\mathrm{GL}_n}\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi_1(D_1,C_1,\alpha_n)\big),$$

$$j^+:\overline{\mathrm{Ext}}^1(C_1,C_1)\xrightarrow[\sim]{t_{C_1}^{-1}}\mathrm{Ext}^1_{\mathrm{GL}_{n-1}}\big(\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^2),\pi_1(C_1)\big)\to\mathrm{Ext}^1_{\mathrm{GL}_n}\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi_1(D_1,C_1,\alpha_n)\big),$$

where the second map in each composition is induced by taking the corresponding parabolic induction (with P^- for j^- and Q^- for j^+, see § 3.2.1 for details). By the compatibility of the map t_D in Theorem 1.3 with parabolic inductions (cf. Proposition 3.27), one can show that t_D factors as (where the first map is the natural push-forward map via $\pi(\underline{\alpha},\mathbf{h})\hookrightarrow\pi_1(D_1,C_1,\alpha_n)$, and one can show it is surjective)

$$\mathrm{Ext}^1_{\mathrm{GL}_n}(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))\longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_n}(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi_1(D_1,C_1,\alpha_n))\xrightarrow{\overline{t_D}}\!\!\!\!\rightarrow\overline{\mathrm{Ext}}^1(D,D)$$

and the following diagram commutes:

$$\begin{array}{ccc}\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D_1,D_1)\oplus\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(C_1,C_1) & & \\ \downarrow{\scriptstyle(1.4)} & \searrow^{j=(j^-,j^+)} & \\ \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D,D) & \xleftarrow{\overline{t_D}} & \mathrm{Ext}^1(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi_1(D_1,C_1,\alpha_n))\end{array}$$

By Theorem 1.7 (2), j sends V_{ι_D} to $\mathrm{Ker}(\overline{t_D})$. We have in fact (see the proof of Theorem 3.33):

Proposition 1.8. *We have* $\mathrm{Ker}(\overline{t_D})=j(V_{\iota_D})$ $\big($*of dimension* $\frac{n(n-3)}{2}+1\big)$*, and the representation* $\pi_1(D)$ *is isomorphic to the tautological extension of* $\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h})\otimes_E j(V_{\iota_D})$ *by* $\pi_1(D_1,C_1,\alpha_n)$*.*

We can finally inductively prove Theorem 1.1 (1). By induction hypothesis, $\pi_1(D_1)$ and $\pi_1(C_1)$ determine D_1 and C_1 respectively. From which, it is not difficult to deduce that $\pi_1(D_1,C_1,\alpha_n)$ determines D_1, C_1 and α_n. On the other hand, using Theorem 1.7 (and some representation theory), one sees $j(V_{\iota_D})$ still determines ι_D. So $\pi_1(D)$ determines D_1, C_1, α_n, ι_D hence determines D by Proposition 1.6. We refer to Theorem 3.33 for details. Finally, note that, under the isomorphism $(\zeta_w\circ\kappa_w)_{w\in S_n}$, there is an exact sequence

$$0\longrightarrow\mathrm{Ker}(1.2)\longrightarrow\mathrm{Ker}(1.1)\longrightarrow\mathrm{Ker}(t_D)\longrightarrow 0.$$

As $\mathrm{Ker}(t_D)$ determines D, we deduce $\mathrm{Ker}(1.1)$ hence the kernel of the composition

$$\oplus_{w\in S_n}\mathrm{Hom}(T(\mathbb{Q}_p),E)\xrightarrow[\sim]{(\kappa_w^{-1})}\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(D,D)\xrightarrow{(1.1)}\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D,D).$$

also determines D. This fact (purely on Galois side) is of interest on its own right.

We refer to the the context for the more precise and detailed statements. One main difference from what's discussed in the introduction is that we mainly work with $\pi_{\min}(D)$ instead of $\pi_1(D)$ in the introduction, which has a cleaner structure but requires a bit more on Orlik-Strauch representations.

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2 Hodge filtration and higher intertwining

2.1 Notation and preliminaries

Let K be a finite extension of $\mathbb{Q}_p$, E be a finite extension of $\mathbb{Q}_p$ containing all the embeddings of K in $\overline{\mathbb{Q}_p}$. Let $\Sigma_K := \{\sigma : K \hookrightarrow E\}$, and $d_K := [K : \mathbb{Q}_p]$. For $\mathbf{k} = (k_\sigma)_{\sigma\in\Sigma_K} \in \mathbb{Z}^{\Sigma_K}$, denote by $z^{\mathbf{k}} := \prod_{\sigma\in\Sigma_K} \sigma(z)^{k_\sigma}$ the ($\mathbb{Q}_p$-)algebraic character of $K^\times$ of weight $\mathbf{k}$. Let $|\cdot|_K : K^\times \to E^\times$ be the unramified character such that $|\varpi_K| = p^{-[K_0:\mathbb{Q}_p]}$ for a uniformizer ϖ_K of K, where K_0 is the maximal unramified subextension of K over $\mathbb{Q}_p$.

Let $\mathcal{R}_{K,E}$ be the E-coefficient Robba ring for K. For a continuous character $\chi : K^\times \to E^\times$, denote by $\mathcal{R}_{K,E}(\chi)$ the associated rank one (φ,Γ)-module over $\mathcal{R}_{K,E}$ (see for example [34, § 6.2]). We write Ext^i (and $\mathrm{Hom} = \mathrm{Ext}^0$) without "$(\varphi,\Gamma)$" in the subscript for the i-th extension group of (φ,Γ)-modules (cf. [37]). For de Rham (φ,Γ)-modules M and N, denote by $\mathrm{Ext}^1_g(M,N) \subset \mathrm{Ext}^1(M,N)$ the subspace of de Rham extensions. For a (φ,Γ)-module M, we identify *elements* in $\mathrm{Ext}^1(M,M)$ with deformations of M over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$. Indeed, the $E[\epsilon]/\epsilon^2$-structure on $\widetilde{M} \in \mathrm{Ext}^1(M,M)$ is given by letting ϵ act via

$$\epsilon : \widetilde{M} \twoheadrightarrow M \xrightarrow{\mathrm{id}} M \hookrightarrow \widetilde{M}.$$

We denote by $W^+_{\mathrm{dR}}(M)$ the (semi-linear) Gal_K-representation over $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ associated to M (cf. [3, Prop. 2.2.6 (2)]). There is a natural decomposition $W^+_{\mathrm{dR}}(M) \cong \oplus_{\sigma\in\Sigma_K} W^+_{\mathrm{dR},\sigma}(M)$ with respect to $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E \cong \oplus_{\sigma\in\Sigma_K} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E$.

Let M be a crystabelline (φ,Γ)-module of rank d over $\mathcal{R}_{K,E}$. We can associate to M a filtered Deligne-Fontaine module $(D_{\mathrm{pst}}(M), D_{\mathrm{dR}}(M))$ such that

- $D_{\mathrm{pst}}(M) = (W_e(M) \otimes_{B_e} B_{\mathrm{cris}})^{\mathrm{Gal}_{K'}}$ which is free of rank d over $K'_0 \otimes_{\mathbb{Q}_p} E$ equipped with a commuting K'_0-semi-linear action of φ and $\mathrm{Gal}(K'/K)$, K' is an *abelian* extension of K, and K'_0 is the maximal unramified extension of K' (over $\mathbb{Q}_p$), and where $W_e(M)$ is the $B_e = B_{\mathrm{cris}}^{\varphi=1}$-representation associated to M ([3, Prop. 2.2.6 (1)]),
- $D_{\mathrm{dR}}(M) \cong (D_{\mathrm{pst}}(M) \otimes_{K'_0} K')^{\mathrm{Gal}(K'/K)}$ is free of rank d over $K \otimes_{\mathbb{Q}_p} E$, equipped with a Hodge filtration Fil_H of $K \otimes_{\mathbb{Q}_p} E$-submodules (not necessarily free).

By [16, Prop. 4.1], to $D_{\mathrm{pst}}(M)$, one can associate a Weil-Deligne representation $\mathtt{r}(M)$ over E. We call M is *generic* if $\mathtt{r}(M)$ is generic, which means $\mathtt{r}(M)$ is semi-simple and isomorphic to $\oplus_{i=1}^d \phi_i$ with $\phi_i\phi_j^{-1} \neq 1, |\cdot|_K$ for $i \neq j$. In fact, M being generic crystabelline is equivalent to that there exist smooth characters ϕ_i for $i = 1, \cdots, d$ such that $M[1/t] \cong \oplus_{i=1}^d \mathcal{R}_{K,E}(\phi_i)[1/t]$, and $\phi_i\phi_j^{-1} \neq 1, |\cdot|_K$ for $i \neq j$. An ordering of $(\phi_1, \cdots, \phi_d)$ is refereed to as a *refinement* of M. Indeed, an ordering $w(\phi) = (\phi_{w^{-1}(1)}, \cdots, \phi_{w^{-1}(d)})$ for $w \in S_d$, corresponds uniquely to a filtration $\mathscr{F}_w$ (of saturated (φ,Γ)-submodules) on M such that $(\mathrm{gr}^i_{\mathscr{F}_w} M)[1/t] \cong \mathcal{R}_{K,E}(\phi_{w^{-1}(i)})[1/t]$. We frequently view $w(\phi)$

as a (smooth) character of $T(K)$ (the torus subgroup of $\mathrm{GL}_d(K)$) for any $w \in S_d$. We also call these characters of $T(K)$ refinements of M.

Let $\mathbf{h} := (\mathbf{h}_i)_{i=1,\cdots,d} = (\mathbf{h}_\sigma)_{\sigma\in\Sigma_K} = (h_{\sigma,1} \geq \cdots \geq h_{\sigma,d})_{\sigma\in\Sigma_K}$ be the Hodge-Tate-Sen weights of M (normalized such that the weight of the cyclotomic character is 1). Let $w \in S_d$, we call the refinement $w(\phi)$ (or $\mathscr{T}_w$) *non-critical* if the Hodge-Tate-Sen weights of $\mathrm{gr}^i_{\mathscr{T}_w} M$ are exactly $\mathbf{h}_i$ (which are hence decreasing with growth of i). We call M is non-critical, if all the refinements of M are non-critical. We denote by $\Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ the set of isomorphism classes of non-critical crystabelline (φ,Γ)-modules of refinement ϕ and of Hodge-Tate-Sen weights $\mathbf{h}$. Finally, we call M has *regular* Hodge-Tate-Sen weights if $\mathbf{h}$ is strictly dominant, i.e. $h_{i,\sigma} > h_{i+1,\sigma}$ for all $\sigma \in \Sigma_K$.

Suppose M is generic crystabelline with refinement ϕ. For a subset $\mathbf{r} = \{r_1, \cdots, r_k\} \subset \{1, \cdots, d\}$, denote by $M_{\mathbf{r}}$ (resp. $M^{\mathbf{r}}$) the saturated (φ,Γ)-submodule of M (resp. the quotient of M) which has a refinement given by $(\phi_{r_1}, \cdots, \phi_{r_k})$. Assuming M is non-critical, $M_{\mathbf{r}}$ and $M^{\mathbf{r}}$ are non-critical as well for any $\mathbf{r}$ (noting any triangulation of $M_{\mathbf{r}}$ or of $M^{\mathbf{r}}$ extends to a triangulation of M). In this case, the Hodge-Tate-Sen weights of $M_{\mathbf{r}}$ (resp. $M^{\mathbf{r}}$) are $(\mathbf{h}_1, \cdots, \mathbf{h}_k)$ (resp. $(\mathbf{h}_{d-k+1}, \cdots, \mathbf{h}_d)$).

2.2 Hodge parameters

In this section, we give a reinterpretation of (some) p-adic Hodge parameters of a generic non-critical crystabelline (φ,Γ)-module.

Let $\phi = (\phi_i)_{i=1,\cdots,n}$ be generic, and $\mathbf{h} = (\mathbf{h}_\sigma)_{\sigma\in\Sigma_K} = (\mathbf{h}_i)_{i=1,\cdots,n} = (h_{\sigma,1} > h_{\sigma,2} > \cdots > h_{\sigma,n})$. Let $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. Let $D_1 := D_{\{1,\cdots,n-1\}}$ and $C_1 := D^{\{1,\cdots,n-1\}}$, we have two exact sequences:

$$0 \longrightarrow D_1 \longrightarrow D \longrightarrow \mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}) \longrightarrow D \longrightarrow C_1 \longrightarrow 0.$$

Denote by ι_D the composition $D_1 \hookrightarrow D \twoheadrightarrow C_1$, which is injective as $\mathrm{Hom}(D_1, \mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1})) = 0$.

Proposition 2.1. *We have* $\dim_E \mathrm{Hom}(D_1, C_1) = \begin{cases} 1 & n = 2 \\ 2 & n \geq 3. \end{cases}$

Proof. The case for $n < 3$ is trivial, and we assume $n \geq 3$. Let $\mathbf{r} := \{1, \cdots, n-3\}$, and consider $(C_1)_{\mathbf{r}}$, the saturated submodule of C_1 of rank $n-3$ over $\mathcal{R}_{K,E}$ with a refinement $(\phi_1, \cdots, \phi_{n-3})$. As C_1 is non-critical of Hodge-Tate weights $(\mathbf{h}_2, \cdots \mathbf{h}_n)$, $(C_1)_{\mathbf{r}}$ is non-critical of Hodge-Tate weights $(\mathbf{h}_2, \cdots, \mathbf{h}_{n-2})$. Thus $(C_1)_{\mathbf{r}}$ is isomorphic to a (non-split) successive extension of $\mathcal{R}_{K,E}(\phi_i z^{\mathbf{h}_{i+1}})$ for $i = 1, \cdots, n-3$. Consider

$$0 \longrightarrow \mathrm{Hom}(D_1, (C_1)_{\mathbf{r}}) \longrightarrow \mathrm{Hom}(D_1, C_1) \longrightarrow \mathrm{Hom}(D_1, C_1/(C_1)_{\mathbf{r}}).$$

Any map in $\mathrm{Hom}(D_1, (C_1)_{\mathbf{r}})$ clearly factors through $(D_1)^{\mathbf{r}}$, the latter being isomorphic to a (non-split) successive extension of $\mathcal{R}_{K,E}(\phi_i z^{\mathbf{h}_{i+2}})$ for $i = 1, \cdots, n-3$. By an easy dévissage and using the fact $h_{\sigma,i+2} < h_{\sigma,i+1}$, we deduce $\mathrm{Hom}((D_1)^{\mathbf{r}}, (C_1)_{\mathbf{r}}) = 0$ hence $\mathrm{Hom}(D_1, (C_1)_{\mathbf{r}}) = 0$. Again by an easy dévissage, we have $\dim_E \mathrm{Hom}(D_1, C_1/(C_1)_{\mathbf{r}}) = \dim_E \mathrm{Hom}(D_1, (C_1)^{\{n-2,n-3\}}) \leq 2$. Hence $\dim_E \mathrm{Hom}(D_1, C_1) \leq 2$.

Now let $\mathbf{r} = \{2, \cdots, n-1\}$. Consider $(D_1)^{\mathbf{r}}$ and $(C_1)_{\mathbf{r}}$, which are both non-critical (φ,Γ)-modules of refinement $(\phi_2, \cdots, \phi_{n-1})$ and of Hodge-Tate weights $(\mathbf{h}_2, \cdots, \mathbf{h}_{n-1})$. And D has the following two forms $[\mathcal{R}_{K,E}(\phi_1 z^{\mathbf{h}_1}) \text{—} (D_1)^{\mathbf{r}} \text{—} \mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n})]$ and $[\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}) \text{—} (C_1)_{\mathbf{r}} \text{—} \mathcal{R}_{K,E}(\phi_1 z^{\mathbf{h}_n})]$.

Claim. $(D_1)^{\mathbf{r}} \cong (C_1)_{\mathbf{r}}$.

It suffices to show $(D_1)^{\mathbf{r}}$ is a submodule of $D/\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1})$. Let $\mathbf{r}' := \{3, \cdots, n\}$, then $D^{\mathbf{r}'}$ has Hodge-Tate weights $(\mathbf{h}_3, \cdots, \mathbf{h}_n)$. The composition $\iota_1 : (D_1)^{\mathbf{r}} \hookrightarrow D/\mathcal{R}_{K,E}(\phi_1 z^{\mathbf{h}_1}) \twoheadrightarrow D^{\mathbf{r}'}$ is clearly injective. Let C be the pull-back of D via ι_1, which is isomorphic to a de Rham extension of $(D_1)^{\mathbf{r}}$ by $D_{\{1,n\}}$. However $\mathrm{Ext}^1_g((D_1)^{\mathbf{r}}, D_{\{1,n\}}) = 0$ (for example by [23, Cor. A.4], noting $D_{\{1,n\}}$ has Hodge-Tate weights $(\mathbf{h}_1, \mathbf{h}_2)$). Hence

$$C/\mathcal{R}_{K,E}(\phi_1 z^{\mathbf{h}_1}) \cong (D_1)^{\mathbf{r}} \oplus \mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_2}) \text{ and } C/\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}) \cong (D_1)^{\mathbf{r}} \oplus \mathcal{R}_{K,E}(\phi_1 z^{\mathbf{h}_2}).$$

In particular, the pull-back of $D/\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1})$ (that is an extension of $D^{\mathbf{r}'}$ by $\mathcal{R}_{K,E}(\phi_1 z^{\mathbf{h}_2})$) via ι_1 is split, which implies $(D_1)^{\mathbf{r}} \hookrightarrow D/\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1})$. The claim follows.

Let α_1 be the following composition (which is *not* injective)

$$\alpha_1 : D_1 \twoheadrightarrow (D_1)^{\mathbf{r}} \cong (C_1)_{\mathbf{r}} \hookrightarrow C_1.$$

It is clear that α_1 and ι_D are linearly independent in $\mathrm{Hom}(D_1, C_1)$, hence $\dim_E \mathrm{Hom}(D_1, C_1) \geq 2$. This finishes the proof. □

Remark 2.2. *Let $i \in \{1, \cdots, n-1\}$, and $\boldsymbol{r} := \{1, \cdots, n-1\} \setminus \{i\}$. By the same argument as in the proof of the proposition, we have $(D_1)^{\boldsymbol{r}} \cong (C_1)_{\boldsymbol{r}}$. We denote by α_i the following composition*

$$\alpha_i : D_1 \twoheadrightarrow (D_1)^{\boldsymbol{r}} \cong (C_1)_{\boldsymbol{r}} \hookrightarrow C_1. \tag{2.1}$$

It is clear that α_i are pair-wisely linearly independent in $\mathrm{Hom}(D_1, C_1)$.

Consider the cup-product

$$\mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), D_1) \times \mathrm{Hom}(D_1, C_1) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), C_1). \tag{2.2}$$

Proposition 2.3. *Under the cup-product, $E[D] \subset [\iota_D]^{\perp}$, and we have an equality if $K = \mathbb{Q}_p$. In particular, when $K = \mathbb{Q}_p$, D is determined by D_1, C_1, $\phi_n z^{\boldsymbol{h}_n}$ and ι_D.*

Proof. As ι_D factors through D, the map induced by the pairing $\langle -, \iota_D \rangle$ (in (2.2) is equal to the following composition

$$\mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), D_1) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), C_1).$$

The first map sends $[D]$ to zero, hence $\langle D, \iota_D \rangle = 0$. In fact, by dévissage, the kernel of the composition is $\mathrm{Hom}(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), C_1/D_1)$, which, by (an easy generalization to K of) [11, Lem. 5.1.1], has dimension d_K. In particular, when $K = \mathbb{Q}_p$, it is exactly generated by $[D]$. This finishes the proof. □

In the rest of the section, we discuss what information of D can be detected by ι_D for general K. The reader who is mainly interested in the $\mathbb{Q}_p$-case can skip to the next section. Fix $\sigma \in \Sigma_K$, and define $\mathfrak{T}_\sigma(\mathbf{h})$ to be the weight such that $\mathfrak{T}_\sigma(\mathbf{h})_{\tau,i} = \begin{cases} h_{\tau,i} & \tau = \sigma \\ h_{\tau,n} & \tau \neq \sigma \end{cases}$ which is in particular constant for $\tau \neq \sigma$. The following proposition is a direct consequence of Fontaine's classification of B^+_{dR}-representations (e.g. using similar arguments as in [26, Lem. 2.1]).

Proposition 2.4. *Let $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \boldsymbol{h})$, and let $\sigma \in \Sigma_K$. There exists a unique (φ, Γ)-module (up to isomorphism) D_σ over $\mathcal{R}_{K,E}$ such that $D_\sigma[1/t] \cong D[1/t]$, $D \subset D_\sigma$, and the Hodge-Tate weights of D_σ are $\mathfrak{T}_\sigma(\boldsymbol{h})$.*

Remark 2.5. *We have an isomorphism of Deligne-Fontaine modules $D_{\mathrm{pst}}(D) \xrightarrow{\sim} D_{\mathrm{pst}}(D_\sigma)$, such that the induced map $D_{\mathrm{dR}}(D) \to D_{\mathrm{dR}}(D_\sigma)$ is a morphism of filtered $K \otimes_{\mathbb{Q}_p} E$-modules, satisfying $D_{\mathrm{dR}}(D)_\sigma \xrightarrow{\sim} D_{\mathrm{dR}}(D_\sigma)$ (as filtered E-vector space).*

Lemma 2.6. *Let D, D_σ be as in Proposition 2.4. For each $w \in S_n$, $w(\underline{\phi})z^{\mathfrak{T}_\sigma(\boldsymbol{h})}$ is a trianguline parameter of D_σ.*

Proof. Consider the composition $\mathcal{R}_{K,E}(\phi_{w^{-1}(1)}z^{\mathbf{h}_1}) \hookrightarrow D \hookrightarrow D_\sigma$. It is not difficult to see the saturation of the image in D_σ is just $\mathcal{R}_{K,E}(\phi_{w^{-1}(1)}z^{\mathfrak{T}_\sigma(\mathbf{h})_1})$, and

$$D/\mathcal{R}_{K,E}(\phi_{w^{-1}(1)}z^{\mathbf{h}_1}) \longrightarrow D_\sigma/\mathcal{R}_{K,E}(\phi_{w^{-1}(1)}z^{\mathfrak{T}_\sigma(\mathbf{h})_1}).$$

Continuing with the argument, the lemma follows. □

We have hence a (surjective) map

$$\mathfrak{T}_\sigma : \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h}) \longrightarrow \Phi\Gamma_{\mathrm{nc}}(\phi, \mathfrak{T}_\sigma(\mathbf{h})),\ D \mapsto D_\sigma. \tag{2.3}$$

Lemma 2.7. *We have $\dim_E \mathrm{Hom}(D, D_\sigma) = 1$.*

Proof. By dévissage, we reduce to show $\mathrm{Hom}(\mathcal{R}_{K,E}(\phi_i z^{\mathbf{h}_i}), D_\sigma) = 0$ for $i > 1$. But this follows easily from the above lemma. Indeed, if $\mathcal{R}_{K,E}(\phi_i z^{\mathbf{h}_i}) \hookrightarrow D_\sigma$ for some $i > 1$, then the morphism $\mathcal{R}_{K,E}(\phi_i z^{\mathfrak{T}_\sigma(\mathbf{h})_1}) \hookrightarrow D_\sigma$ (induced by $\mathcal{R}_{K,E}(\phi_i z^{\mathbf{h}_1}) \hookrightarrow D$) can not be saturated, a contradiction. □

Let $D_{1,\sigma} := (D_\sigma)_{\{1,\cdots,n-1\}}$ and $C_{1,\sigma} := (D_\sigma)^{\{1,\cdots,n-1\}}$. By Lemma 2.6, it is not difficult to see $D_{1,\sigma}$ (resp. $C_{1,\sigma}$) has Hodge-Tate weights $(\mathfrak{T}_\sigma(\mathbf{h})_1, \cdots, \mathfrak{T}_\sigma(\mathbf{h})_{n-1})$ $\big($resp. $(\mathfrak{T}_\sigma(\mathbf{h})_2, \cdots, \mathfrak{T}_\sigma(\mathbf{h})_n)\big)$. In fact, we have $D_{1,\sigma} = \mathfrak{T}_\sigma(D_1)$ and $C_{1,\sigma} = \mathfrak{T}_\sigma(C_1)$ (where $\mathfrak{T}_\sigma$ is defined in a similar way as (2.3)). Similarly as in Lemma 2.7, we have

$$\dim_E \mathrm{Hom}(D_1, D_{1,\sigma}) = \dim_E \mathrm{Hom}(C_1, C_{1,\sigma}) = 1.$$

We fix embeddings $D_1 \hookrightarrow D_{1,\sigma}$ and $C_1 \hookrightarrow C_{1,\sigma}$.

Lemma 2.8. *For any $\iota \in \mathrm{Hom}(D_1, C_1)$, the composition $D_1 \xrightarrow{\iota} C_1 \hookrightarrow C_{1,\sigma}$ factors through a unique morphism $\iota_\sigma : D_{1,\sigma} \to C_{1,\sigma}$. Moreover, the map $\iota \mapsto \iota_\sigma$ gives a bijection*

$$\mathrm{Hom}(D_1, C_1) \xrightarrow{\sim} \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma}). \tag{2.4}$$

Proof. The first part follows by comparing the weights. Let $i \in \{1, \cdots, n-1\}$, and $\mathbf{r} = \{1, \cdots, n-1\} \setminus \{i\}$. By same argument in the proof of Proposition 2.1, $(D_{1,\sigma})^{\mathbf{r}} \cong (C_{1,\sigma})_{\mathbf{r}}$. It is also clear that $(C_1)_{\mathbf{r}} \hookrightarrow C_{1,\sigma}$ factors through $(C_{1,\sigma})_{\mathbf{r}}$. So the composition of α_i (2.1) with $C_1 \hookrightarrow C_{1,\sigma}$ factors through $D_1 \hookrightarrow D_{1,\sigma} \twoheadrightarrow (D_{1,\sigma})^{\mathbf{r}}$. As $\mathrm{Hom}(D_1, C_1)$ is spanned by α_i, the first part follows (noting the uniqueness is clear). By the same argument in the proof of Proposition 2.1, $\dim_E \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma}) \leq 2$. It suffices to show (2.4) is injective, but it is clear. □

Let $\iota_{D_\sigma} \in \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma})$ be the composition $D_{1,\sigma} \hookrightarrow D_\sigma \twoheadrightarrow C_{1,\sigma}$, which is equal to the image of ι_D via (2.4).

Proposition 2.9. *For the cup-product*

$$\mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\boldsymbol{h}_n}), D_{1,\sigma}) \times \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma}) \to \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\boldsymbol{h}_n}), C_{1,\sigma}),$$

we have $[\iota_{D_\sigma}]^\perp = E[D_\sigma]$. In particular, D_σ is determined by $D_{1,\sigma}$, $C_{1,\sigma}$, $\phi_n z^{\boldsymbol{h}_n}$ and ι_{D_σ}.

Proof. Taking the cup-product with ι_{D_σ} is equal to the following composition

$$\mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), D_{1,\sigma}) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), D_\sigma) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}), C_{1,\sigma}),$$

which is the push-forward map via ι_{D_σ}. We see $\langle D_\sigma, \iota_{D_\sigma}\rangle = 0$. Using [11, Lem. 5.1.1],one calculates that the kernel is now one dimensional. The proposition follows. □

Example 2.10. *We give an example to illustrate how ι_{D_σ} determines D_σ (or equivalently the Hodge σ-filtration of D). Suppose $n=3$, K unramified and D is crystalline (generic non-critical) of regular Hodge-Tate-Sen weights $\boldsymbol{h}$. In this case we have $D_{\mathrm{cris}}(D) \cong D_{\mathrm{dR}}(D) \cong \oplus_{\tau\in\Sigma_K} D_{\mathrm{cris}}(D)_\tau$, where each $D_{\mathrm{cris}}(D)_\tau$ is a filtered φ^{d_K}-module. Fix $\sigma \in \Sigma_K$. Note that we have an isomorphism of filtered φ^{d_K}-module $D_{\mathrm{cris}}(D_\sigma)_\sigma \cong D_{\mathrm{cris}}(D)_\sigma$.*

Let α_1, α_2, α_3 be the three distinct eigenvalues of φ^{d_K} on $D_{\mathrm{cris}}(D_\sigma)_\tau$ (for any τ). Let $e_{i,\sigma}$ be an α_i-eigenvector in $D_{\mathrm{cris}}(D_\sigma)_\sigma$, hence $D_{\mathrm{cris}}(D_\sigma)_\sigma \cong Ee_{1,\sigma} \oplus Ee_{2,\sigma} \oplus Ee_{3,\sigma}$. For $j=0,\cdots,d_K-1$, we have $D_{\mathrm{cris}}(D_\sigma)_{\sigma\circ\mathrm{Frob}^{-j}} \cong E\varphi^j(e_{1,\sigma}) \oplus E\varphi^j(e_{2,\sigma}) \oplus E\varphi^j(e_{3,\sigma})$ (where Frob *denotes the absolute Frobenius), and $D_{\mathrm{cris}}(D_{1,\sigma})_{\sigma\circ\mathrm{Frob}^{-j}} \cong E\varphi^j(e_{1,\sigma}) \oplus E\varphi^j(e_{2,\sigma})$ for $j=0,\cdots,d_K-1$, which is equipped with the induced Hodge filtration. As $D_{1,\sigma}$ is non-critical, multiplying $e_{1,\sigma}$, $e_{2,\sigma}$ by non-zero scalars, we can and do assume $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_{1,\sigma})_\sigma = \mathrm{Fil}^j D_{\mathrm{cris}}(D_{1,\sigma})_\sigma$, $-h_{1,\sigma} < j \leq -h_{2,\sigma}$, is generated by $e_{1,\sigma}+e_{2,\sigma}$. As D_σ is non-critical for all the refinements, multiplying $e_{3,\sigma}$ by a non-zero scalar, we can and do assume $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_\sigma)_\sigma = \mathrm{Fil}^j D_{\mathrm{cris}}(D_\sigma)_\sigma$, $-h_{2,\sigma} < j \leq -h_{3,\sigma}$, is generated by $e_1 + a_{D_\sigma} e_2 + e_3$. The filtered φ^{d_K}-module $D_{\mathrm{cris}}(D_\sigma)_\sigma$ is in fact parametrized (and determined) by $a_{D_\sigma} \in E \setminus \{0,1\}$: we have*

$$\mathrm{Fil}^j D_{\mathrm{cris}}(D_\sigma)_\sigma = \begin{cases} D_{\mathrm{cris}}(D_\sigma)_\sigma & j \leq -h_{1,\sigma} \\ E(e_{1,\sigma}+e_{2,\sigma}) \oplus E(e_{1,\sigma}+a_{D_\sigma}e_{2,\sigma}+e_{3,\sigma}) & -h_{1,\sigma} < j \leq -h_{2,\sigma} \\ E(e_{1,\sigma}+a_{D_\sigma}e_{2,\sigma}+e_{3,\sigma}) & -h_{2,\sigma} < j \leq -h_{3,\sigma} \\ 0 & j > -h_{3,\sigma} \end{cases}$$

For $\tau \neq \sigma$, we have $\mathrm{Fil}^j D_{\mathrm{cris}}(D_\sigma)_\tau = \begin{cases} D_{\mathrm{cris}}(D_\sigma)_\tau & j \leq -h_{n,\tau} \\ 0 & j > -h_{n,\tau} \end{cases}$. So D_σ is indeed determined by the single parameter a_{D_σ} (in contrast, D itself has many more parameters). Note that

$$\mathrm{Fil}^{\max} D_{\mathrm{cris}}(C_{1,\sigma})_\sigma = \mathrm{Fil}^j D_{\mathrm{cris}}(C_{1,\sigma})_\sigma,\ -h_{2,\sigma} < j \leq -h_{3,\sigma},$$

is generated by $e_{1,\sigma}+a_{D_\sigma}e_{2,\sigma}$ (as it is equipped with the quotient filtration). The map ι_{D_σ} uniquely corresponds to a morphism of filtered φ^{d_K}-modules $\iota_{D_\sigma} : D_{\mathrm{cris}}(D_{1,\sigma})_\sigma \to D_{\mathrm{cris}}(C_{1,\sigma})_\sigma$ sending $e_{i,\sigma}$ to $e_{i,\sigma}$ for $i=1,2$. We see a_{D_σ} can be read out from the relative position of the two lines $\iota_{D_\sigma}\big(\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_{1,\sigma})_\sigma\big)$ and $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(C_{1,\sigma})_\sigma$ in $D_{\mathrm{cris}}(C_{1,\sigma})_\sigma$. Thus a_{D_σ} (hence D_σ) is determined by ι_{D_σ}.

2.3 Deformations of crystabelline (φ,Γ)-modules

Let $D \in \Phi_{\mathrm{nc}}\Gamma(\phi,\mathbf{h})$. In this section, we collect some facts on certain deformations of D.

2.3.1 Trianguline and paraboline deformations, I

We first consider trianguline deformations.

For $w \in S_n$, denote by $\mathrm{Ext}^1_w(D,D) \subset \mathrm{Ext}^1(D,D)$ the subspace of trianguline deformations with respect to the refinement $w(\phi)$. More precisely, for $\widetilde{D} \in \mathrm{Ext}^1(D,D)$ (viewed as a (φ,Γ)-module over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2})$), $\widetilde{D} \in \mathrm{Ext}^1_w(D,D)$ if and only if $\widetilde{D}$ is isomorphic to a successive extension of $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w^{-1}(i)} z^{\mathbf{h}_i}\psi_i)$ for $\psi_i \in \mathrm{Hom}(K^\times, E)$. In this case, we call the character $w(\phi)z^{\mathbf{h}}(1+\psi\epsilon)$ (with $\psi := (\psi_1, \cdots, \psi_n)$) of $T(K)$ over $E[\epsilon]/\epsilon^2$ the *trianguline parameter* of $\widetilde{D}$ with respect to $w(\phi)$. Let κ_w be the following composition:

$$\kappa_w : \mathrm{Ext}^1_w(D,D) \longrightarrow \mathrm{Ext}^1_{T(K)}(w(\phi)z^{\mathbf{h}}, w(\phi)z^{\mathbf{h}}) \xrightarrow{\sim} \mathrm{Hom}(T(K), E), \tag{2.5}$$

where the first map sends $\widetilde{D}$ to its trianguline parameter with respect to $w(\phi)$, and the inverse of the second map sends ψ to $w(\phi)z^{\mathbf{h}}(1+\psi\epsilon)$. We also denote $\mathrm{Ext}^1_w(D,D)$ by $\mathrm{Ext}^1_{w(\phi)}(D,D)$ or $\mathrm{Ext}^1_{\mathscr{T}_w}(D,D)$ where $\mathscr{T}_w$ is the filtration on D associated to $w(\phi)$ whenever it is convenient for the context. The following proposition is well-known, see for example [2, § 2] [39, § 2].

Proposition 2.11. *(1)* $\dim_E \mathrm{Ext}^1(D,D) = 1 + n^2 d_K$, $\dim_E \mathrm{Ext}^1_g(D,D) = 1 + \frac{n(n-1)}{2} d_K$ *and* $\dim_E \mathrm{Ext}^1_w(D,D) = 1 + \frac{n(n+1)}{2} d_K$ *for all* $w \in S_n$.

(2) For $w \in S_n$, κ_w *is surjective.*

(3) For $w \in S_n$, $\mathrm{Ext}^1_g(D,D) \subset \mathrm{Ext}^1_w(D,D)$ *and is equal to the preimage of the subspace* $\mathrm{Hom}_{\mathrm{sm}}(T(K), E) \subset \mathrm{Hom}(T(K), E)$ *of smooth characters via* κ_w.

Recall there is a right action of S_n on $T(K)$: $w(a_1, \cdots, a_n) = (a_{w(1)}, \cdots, a_{w(n)})$ for $w \in S_n$. It induces a left action of S_n on $\mathrm{Hom}(T(K), E)$: $(w\psi)(a_1, \cdots, a_n) = \psi(a_{w(1)}, \cdots, a_{w(n)})$. It is clear that $\mathrm{Hom}_{\mathrm{sm}}(T(K), E)$ is stabilized by the action.

Lemma 2.12. *Let* $w_1, w_2 \in S_n$, *the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Ext}^1_g(D,D) & \xrightarrow{\kappa_{w_1}} & \mathrm{Hom}_{\mathrm{sm}}(T(K), E) \\ \Big\| & & {\scriptstyle w_2 w_1^{-1}} \Big\downarrow \wr \\ \mathrm{Ext}^1_g(D,D) & \xrightarrow{\kappa_{w_2}} & \mathrm{Hom}_{\mathrm{sm}}(T(K), E). \end{array} \tag{2.6}$$

Proof. The lemma is well-known, but we include a proof for the convenience of the reader. It suffices to prove the statement for the case where $w_2 w_1^{-1}$ is a simple reflection, say, s_k. Let $\widetilde{D} \in \mathrm{Ext}^1_g(D,D)$ and suppose $\kappa_{w_i}(\widetilde{D}) = (\psi_{i,1}, \cdots, \psi_{i,n})$. By definition, $\widetilde{D}$ admits triangulations:

$$\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w_i^{-1}(1)} z^{\mathbf{h}_1}(1+\psi_{i,1}\epsilon)) \text{—} \cdots \text{—} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w_i^{-1}(n)} z^{\mathbf{h}_1}(1+\psi_{i,n}\epsilon)).$$

Note by assumption $w_1^{-1}(j) = w_2^{-1}(j)$ for $j \neq k, k+1$. Consequently, for $j < k$ or $j > k+1$, we have $\mathrm{Fil}^j_{\mathscr{T}_{w_1}} \widetilde{D} \cong \mathrm{Fil}^j_{\mathscr{T}_{w_2}} \widetilde{D}$, since $\mathrm{Hom}\big(\mathrm{Fil}^j_{\mathscr{T}_{w_1}} \widetilde{D}, \widetilde{D}/\mathrm{Fil}^j_{\mathscr{T}_{w_2}} \widetilde{D}\big) = 0$.

As $\mathrm{Hom}\Big(\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w_1^{-1}(1)} z^{\mathbf{h}_1}(1+\psi_{1,1}\epsilon)), \widetilde{D}\Big) \cong E[\epsilon]/\epsilon^2$, using dévissage for $\mathscr{T}_{w_2}$, we easily deduce that if $k > 1$,

$$\mathrm{Hom}\Big(\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w_1^{-1}(1)} z^{\mathbf{h}_1}(1+\psi_{1,1}\epsilon)), \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w_2^{-1}(1)} z^{\mathbf{h}_1}(1+\psi_{2,1}\epsilon))\Big) \cong E[\epsilon]/\epsilon^2,$$

hence $H^0_{(\varphi,\Gamma)}\big(\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1+(\psi_{1,1}-\psi_{2,1})\epsilon)\big) \cong E[\epsilon]/\epsilon^2$ (noting $w_1^{-1}(1) = w_2^{-1}(1)$). So $\psi_{1,1} = \psi_{2,1}$. We can then consider the $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$-module $\widetilde{D}/\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w_1^{-1}(1)} z^{\mathbf{h}_1}(1+\psi_{1,1}\epsilon))$ equipped with the filtrations induced by $\mathscr{T}_{w_1}$ and $\mathscr{T}_{w_2}$. Continuing with the above argument, we have $\psi_{1,j} = \psi_{2,j}$ for $j < k$.

For $j = k$, we have (noting $\mathrm{Fil}^{k-1}_{\mathscr{T}_{w_1}} \widetilde{D} = \mathrm{Fil}^{k-1}_{\mathscr{T}_{w_2}} \widetilde{D}$)

$$\mathrm{Hom}\Big(\mathcal{R}_{K,E[\epsilon]/\epsilon^2}\big(\phi_{w_1^{-1}(k)} z^{\mathbf{h}_k}(1+\psi_{1,k\epsilon})\big), \widetilde{D}/\mathrm{Fil}^{k-1}_{\mathscr{T}_{w_2}} \widetilde{D}\Big) \cong E[\epsilon]/\epsilon^2.$$

Using dévissage for $\mathscr{T}_{w_2}$ (and the fact $w_2 w_1^{-1} = s_k$), we get

$$\mathrm{Hom}\Big(\mathcal{R}_{K,E[\epsilon]/\epsilon^2}\big(\phi_{w_1^{-1}(k)} z^{\mathbf{h}_k}(1+\psi_{1,k}\epsilon)\big), \mathcal{R}_{K,E[\epsilon]/\epsilon^2}\big(\phi_{w_2^{-1}(k+1)} z^{\mathbf{h}_{k+1}}(1+\psi_{2,k+1}\epsilon)\big)\Big) \cong E[\epsilon]/\epsilon^2,$$

hence $\psi_{1,k} = \psi_{2,k+1}$. Exchanging $\mathscr{T}_{w_1}$ and $\mathscr{T}_{w_2}$, we get $\psi_{2,k} = \psi_{1,k+1}$.

For $j > k+1$, using the same argument as in the case of $j < k$ with $\widetilde{D}$ replaced by $\widetilde{D}/\mathrm{Fil}^{k+1}_{\mathscr{T}_{w_1}} \widetilde{D}$, we see $\psi_{1,j} = \psi_{2,j}$. This concludes the proof. $\square$

Let $\mathrm{Ext}^1_0(D,D) := \mathrm{Ker}\,\kappa_w$ (for some $w \in S_n$ *a priori*). By Proposition 2.11 (3), $\mathrm{Ext}^1_0(D,D) \subset \mathrm{Ext}^1_g(D,D)$. Using Lemma 2.12, we see $\mathrm{Ext}^1_0(D,D) = \mathrm{Ker}\,\kappa_w$ for all $w \in S_n$. Moreover, by Proposition 2.11 (1) (2), we have

$$\dim_E \mathrm{Ext}^1_0(D,D) = \frac{n(n-1)}{2} d_K + 1 - n.$$

For $\mathrm{Ext}^1_*(D,D) \subset \mathrm{Ext}^1(D,D)$ (with $* = g, w, \ldots$), if $\mathrm{Ext}^1_*(D,D) \supset \mathrm{Ext}^1_0(D,D)$, we set

$$\overline{\mathrm{Ext}}^1_*(D,D) := \mathrm{Ext}^1_*(D,D)/\mathrm{Ext}^1_0(D,D).$$

We have hence isomorphisms

$$\overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow[\sim]{\kappa_w} \mathrm{Hom}(T(K),E), \quad \overline{\mathrm{Ext}}^1_g(D,D) \xrightarrow[\sim]{\kappa_w} \mathrm{Hom}_{\mathrm{sm}}(T(K),E).$$

Let $\mathrm{Ext}^1_{g'}(D,D) \subset \mathrm{Ext}^1(D,D)$ be the subspace of de Rham deformations up to twist by characters of $K^\times$ over $(E[\epsilon]/\epsilon^2)^\times$. Similarly, put $\mathrm{Hom}_{g'}(T(K),E)$ to be the subspace of characters ψ such that there exists a character ψ_0 of $K^\times$ satisfying $\psi - \psi_0 \circ \det \in \mathrm{Hom}_{\mathrm{sm}}(T(K),E)$. One easily deduces from Proposition 2.11 (3) that for all $w \in S_n$, $\mathrm{Ext}^1_{g'}(D,D) \subset \mathrm{Ext}^1_w(D,D)$ and is equal to the preimage of $\mathrm{Hom}_{g'}(T(K),E)$ under κ_w. Thus

$$\dim_E \mathrm{Ext}^1_{g'}(D,D) = 1 + (\frac{n(n-1)}{2} + 1) d_K.$$

Moreover, (2.6) holds with "g" and "sm" replaced by "g'". By [19] (see also [36] for the $n=2$-case, noting the proposition also follows from Corollary 2.33 below and an easy induction argument), we have

Proposition 2.13. *The natural map $\oplus_{w \in S_n} \mathrm{Ext}^1_w(D,D) \to \mathrm{Ext}^1(D,D)$ is surjective and induces a surjective map*

$$\oplus_{w \in S_n} \overline{\mathrm{Ext}}^1_w(D,D) \longrightarrow\!\!\!\!\rightarrow \overline{\mathrm{Ext}}^1(D,D).$$

Now we consider general paraboline deformations of D. Let $P \supset B$ be a standard parabolic subgroup of GL_n with the standard Levi subgroup $L_P \supset T$ equal to $\mathrm{diag}(\mathrm{GL}_{n_1}, \cdots, \mathrm{GL}_{n_r})$. A filtration $\mathscr{F}_P : 0 = \mathrm{Fil}^0_{\mathscr{F}_P} D \subsetneq \mathrm{Fil}^1_{\mathscr{F}_P} D \subsetneq \cdots \subsetneq \mathrm{Fil}^r_{\mathscr{F}_P} D = D$ of saturated (φ,Γ)-submodules of D

is called a P-*filtration* if $M_i := \operatorname{rank} \operatorname{gr}^i_{\mathscr{F}_P} D = n_i$. A deformation $\widetilde{D}$ of D over $E[\epsilon/\epsilon^2]$ is called an $\mathscr{F}_P$-*deformation*, if $\widetilde{D}$ admits a filtration $\operatorname{Fil}^i_{\mathscr{F}_P} \widetilde{D}$ of saturated (φ,Γ)-submodules of D over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$ (which means $\operatorname{Fil}^i_{\mathscr{F}_P} \widetilde{D}$ is free over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$) such that $\operatorname{gr}^i_{\mathscr{F}_P} \widetilde{D}$ is a deformation of M_i over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$. Denote by $\operatorname{Ext}^1_{\mathscr{F}_P}(D,D) \subset \operatorname{Ext}^1(D,D)$ the subspace of $\mathscr{F}_P$-deformations. By [19, § 3.3] (which is for $K=\mathbb{Q}_p$, but all the arguments generalize easily to general K), we have

Proposition 2.14. $\dim_E \operatorname{Ext}^1_{\mathscr{F}_P}(D,D) = 1 + d_K \dim P = 1 + d_K \sum_{1\leq i\leq j\leq r} n_i n_j$. *The natural map*

$$\kappa_{\mathscr{F}_P} : \operatorname{Ext}^1_{\mathscr{F}_P}(D,D) \longrightarrow \prod_{i=1}^r \operatorname{Ext}^1(M_i,M_i), \tag{2.7}$$

sending $\widetilde{D}$ *to* $(\operatorname{gr}^i_{\mathscr{F}_P} \widetilde{D})_{i=1,\cdots,r}$, *is surjective.*

Fro $w \in S_n$, we call the B-filtration $\mathscr{T}_w$ (associated to $w(\phi)$) is compatible with $\mathscr{F}_P$, if $\mathscr{T}_w$ induces a complete flag on $\operatorname{Fil}^i_{\mathscr{F}_P} D$ for all i. In this case, we have $\operatorname{Ext}^1_{\mathscr{T}_w}(D,D) \subset \operatorname{Ext}^1_{\mathscr{F}_P}(D,D)$. For $i=1,\cdots,r$, we let $\mathscr{T}_{w,i}$ be the induced filtration on M_i $(=\operatorname{gr}^i_{\mathscr{F}_P} D)$.

Corollary 2.15. *Keep the above situation.*

(1) $\operatorname{Ext}^1_w(D,D)$ *is the preimage of* $\prod_{i=1}^r \operatorname{Ext}^1_{\mathscr{T}_{w,i}}(M_i,M_i)$ *via* $\kappa_{\mathscr{F}_P}$. *In particular, the map* $\kappa_{\mathscr{F}_P}$ *induces a surjective map*

$$\kappa_{\mathscr{F}_P} : \operatorname{Ext}^1_w(D,D) \twoheadrightarrow \prod_{i=1}^r \operatorname{Ext}^1_{\mathscr{T}_{w,i}}(M_i,M_i).$$

(2) The map $\kappa_{\mathscr{F}_P}$ *sends* $\operatorname{Ext}^1_0(D,D)$ *to* $\prod_{i=1}^r \operatorname{Ext}^1_0(M_i,M_i)$ *and induces an isomorphism*

$$\kappa_{\mathscr{F}_P} : \overline{\operatorname{Ext}}^1_{\mathscr{F}_P}(D,D) \xrightarrow{\sim} \prod_{i=1}^r \overline{\operatorname{Ext}}^1(M_i,M_i). \tag{2.8}$$

Proof. The first part of (1) is by definition, and the second part follows from Proposition 2.14. It is clear that the following diagram commutes

$$\begin{array}{ccc} \operatorname{Ext}^1_w(D,D) & \longrightarrow & \prod_{i=1}^r \operatorname{Ext}^1_{\mathscr{T}_{w,i}}(M_i,M_i) \\ {\scriptstyle\kappa_w}\downarrow & & \downarrow{\scriptstyle(\kappa_{\mathscr{T}_{w,i}})} \\ \operatorname{Hom}(T(K),E) & \xrightarrow{\sim} & \prod_{i=1}^r \operatorname{Hom}(T_i(K),E) \end{array} \tag{2.9}$$

where T_i is the torus subgroup of GL_{n_i}. The first part of (2) follows. By (1) and Proposition 2.13, (2.8) is surjective. However, it is straightforward to see

$$\dim_E \overline{\operatorname{Ext}}^1_{\mathscr{F}_P}(D,D) = \sum_{i=1}^r \dim_E \overline{\operatorname{Ext}}^1(M_i,M_i) (= d_K \dim(B\cap L_P) - n).$$

Hence (2.8) is bijective. □

We denote by $\operatorname{Ext}^1_{\mathscr{F}_P,g'}(D,D)$ the preimage of $\prod_{i=1}^r \operatorname{Ext}^1_{g'}(M_i,M_i)$ via (2.7). Denote by

$$\operatorname{Hom}_{P,g'}(T(K),E) \subset \operatorname{Hom}(T(K),E)$$

the subspace of characters ψ such that there is a character ψ_P of $Z_{L_P}(K)$ satisfying that $\psi - \psi_P \circ \det_{L_P}$ is a smooth character of $T(K)$ (with $\det_{L_P} : T(K) \to Z_{L_P}(K)$ the determinant map). It is straightforward to see $\dim_E \operatorname{Hom}_{P,g'}(T(K),E) = n + rd_K$. We finally discuss some intertwining property of trianguline deformations which generalizes (2.6).

Corollary 2.16. *(1) Let $w \in S_n$ such that $\mathscr{T}_w$ is compatible with $\mathscr{F}_P$, then* $\mathrm{Ext}^1_{\mathscr{F}_P,g'}(D,D) \subset \mathrm{Ext}^1_w(D,D)$.

(2) Let $w_1, w_2 \in S_n$ such that $\mathscr{T}_{w_1}$, $\mathscr{T}_{w_2}$ are compatible with $\mathscr{F}_P$ (so $w_2 w_1^{-1} \in \mathscr{W}_P$), we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{Ext}}^1_{\mathscr{F}_P,g'}(D,D) & \xrightarrow[\sim]{\kappa_{w_1}} & \mathrm{Hom}_{P,g'}(T(K),E) \\ \Vert & & \quad\downarrow{\scriptstyle w_2 w_1^{-1}}\,\wr \\ \overline{\mathrm{Ext}}^1_{\mathscr{F}_P,g'}(D,D) & \xrightarrow[\sim]{\kappa_{w_2}} & \mathrm{Hom}_{P,g'}(T(K),E). \end{array}$$

.

Proof. (1) follows from Corollary 2.15 (1) and the fact $\mathrm{Ext}^1_{g'}(M_i,M_i) \subset \mathrm{Ext}^1_{\mathscr{T}_{w,i}}(M_i,M_i)$. By Corollary 2.15 (2), we have $\overline{\mathrm{Ext}}^1_{\mathscr{F}_P,g'}(D,D) \xrightarrow{\sim} \prod_{i=1}^r \overline{\mathrm{Ext}}^1_{g'}(M_i,M_i)$. (2) then follows from the commutative diagram (2.9) and Lemma 2.12 (applied to each M_i, and with "g", "sm" replaced by "g'"). ☐

2.3.2 Trianguline and paraboline deformations, II

Let D be as in the precedent section. We consider some partially de Rham deformations of D. The reader who is mainly interested in the $\mathbb{Q}_p$-case can skip this section. Throughout the section, we fix $\sigma \in \Sigma_K$. For an extension group $\mathrm{Ext}^1_?(D,D)$, we denote by $\mathrm{Ext}^1_{\sigma,?}(D,D) \subset \mathrm{Ext}^1_?(D,D)$ the subspace consisting of those that are $\Sigma_K \setminus \{\sigma\}$-de Rham. If $\mathrm{Ext}^1_?(D,D) \supset \mathrm{Ext}^1_0(D,D)$, then it is clear that $\mathrm{Ext}^1_{\sigma,?}(D,D) \supset \mathrm{Ext}^1_0(D,D)$ and we set

$$\overline{\mathrm{Ext}}^1_{\sigma,?}(D,D) := \mathrm{Ext}^1_{\sigma,?}(D,D)/\,\mathrm{Ext}^1_0(D,D)$$

which is a subspace of $\overline{\mathrm{Ext}}^1_?(D,D)$.

Lemma 2.17. *We have* $\dim_E \mathrm{Ext}^1_\sigma(D,D) = 1 + \frac{n(n-1)}{2}(d_K-1) + n^2$.

Proof. The lemma follows from [23, Cor. A.4]. ☐

Let P be a standard parabolic subgroup, and $\mathscr{F}_P$ be a P-filtration on D with $\mathrm{gr}^i_{\mathscr{F}_P} D =: M_i$. The surjection $\kappa_{\mathscr{F}_P}$ (2.7) induces a surjection (using the fact that the partial de Rhamness is inherited by taking subquotients)

$$\kappa_{\mathscr{F}_P} : \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D) \twoheadrightarrow \prod_{i=1}^r \mathrm{Ext}^1_\sigma(M_i,M_i). \tag{2.10}$$

Proposition 2.18. *(1) We have* $\dim_E \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D) = 1 + (d_K-1)\frac{n(n-1)}{2} + \dim P$.

(2) The map (2.10) is surjective and induces an isomorphism

$$\overline{\mathrm{Ext}}_{\sigma,\mathscr{F}_P}(D,D) \xrightarrow{\sim} \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(M_i,M_i).$$

Proof. For any $\widetilde{D} \in \mathrm{Ker}(\kappa_{\mathscr{F}_P})$, it is easy to see $\widetilde{D}$ is de Rham. Hence $\mathrm{Ker}\,\kappa_{\mathscr{F}_P} \subset \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D)$, and $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D)$ is exactly the preimage of $\prod_{i=1}^r \mathrm{Ext}^1_\sigma(M_i,M_i)$. (1) follows from Lemma 2.17 and Proposition 2.14. (2) follows by comparing dimensions. ☐

Corollary 2.19. *For $w \in S_n$, κ_w (2.5) induces an isomorphism*

$$\overline{\mathrm{Ext}}_{\sigma,w}(D,D) \xrightarrow{\sim} \mathrm{Hom}_{\sigma}(T(K),E).$$

We will show later (in Corollary 2.40 below) the induced map

$$\oplus_{w\in S_n} \mathrm{Ext}^1_{\sigma,w}(D,D) \longrightarrow \mathrm{Ext}^1_{\sigma}(D,D) \tag{2.11}$$

is surjective (and the same holds with Ext^1 replaced by $\overline{\mathrm{Ext}}^1$) . Consider now certain extension groups of $D_\sigma := \mathfrak{T}_\sigma(D)$ (cf. (2.3)).

Proposition 2.20. *(1) We have $\dim_E \mathrm{Ext}^1(D_\sigma, D_\sigma) = 1 + n^2 d_K$.*

(2) We have $\dim_E \mathrm{Ext}^1_g(D_\sigma, D_\sigma) = 1 + \frac{n(n-1)}{2}$.

(3) Let P be a standard parabolic subgroup of GL_n, and $\mathscr{F}_P$ be a P-filtration of D_σ with $\mathrm{gr}_i \mathscr{F}_P \cong M_{i,\sigma}$. We have $\dim_E \mathrm{Ext}^1_{\mathscr{F}_P}(D_\sigma, D_\sigma) = 1 + d_K \dim P$ and $\mathrm{Ext}^1_g(D_\sigma, D_\sigma) \subset \mathrm{Ext}^1_{\mathscr{F}_P}(D_\sigma, D_\sigma)$. Moreover, the natural map

$$\mathrm{Ext}^1_{\mathscr{F}_P}(D_\sigma, D_\sigma) \longrightarrow \prod_{i=1}^{r} \mathrm{Ext}^1(M_{i,\sigma}, M_{i,\sigma})$$

is surjective.

Proof. (1) is standard. (2) follows from [23, Cor. A.4]. The statements in (3) except $\mathrm{Ext}^1_g(D_\sigma, D_\sigma) \subset \mathrm{Ext}^1_{\mathscr{F}_P}(D_\sigma, D_\sigma)$ follow by the same argument as in [19, § 3.3]. However, the inclusion follows from the fact $\mathrm{Ext}^1_g\big(\mathrm{Fil}^i_{\mathscr{F}_P} D_\sigma, D_\sigma/\mathrm{Fil}^i_{\mathscr{F}_P} D_\sigma\big) = 0$ for $i = 1, \cdots, r-1$ (e.g. using [23, Cor. A.4]). ☐

Remark 2.21. *Recall for each $w \in S_n$, $w(\phi)$ is also a refinement of D_σ and we still use $\mathscr{T}_w$ to denote the associated B-filtration on D_σ. Applying Proposition 2.20 (3) to $\mathscr{T}_w$, we obtain a natural surjection*

$$\kappa_w : \mathrm{Ext}^1_w(D_\sigma, D_\sigma) \twoheadrightarrow \mathrm{Hom}(T(K), E). \tag{2.12}$$

Note when $K \neq \mathbb{Q}_p$, $\mathrm{Ext}^1_g(D_\sigma, D_\sigma)$ is however properly contained in the preimage of $\mathrm{Hom}_{\mathrm{sm}}(T(K), E)$.

For $\Sigma_K \setminus \{\sigma\}$-de Rham deformations of D_σ, we have:

Proposition 2.22. *(1) We have $\dim_E \mathrm{Ext}^1_\sigma(D_\sigma, D_\sigma) = 1 + n^2$.*

(2) Let P be a standard parabolic subgroup of GL_n, and $\mathscr{F}_P$ be a P-filtration of D_σ with $\mathrm{gr}^i_{\mathscr{F}_P} D_\sigma \cong M_{i,\sigma}$. Then $\dim_E \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D_\sigma, D_\sigma) = 1 + \dim P$.

Proof. All the statements follows by [23, Cor. A.4]. ☐

Now we consider the relation between deformations of D and those of D_σ. Similarly as in Proposition 2.4, we have

Proposition 2.23. *There is a natural map*

$$\mathfrak{T}_\sigma : \mathrm{Ext}^1(D,D) \longrightarrow \mathrm{Ext}^1(D_\sigma, D_\sigma), \tag{2.13}$$

sending an deformation $\widetilde{D}$ to $\widetilde{D}_\sigma$, where $\widetilde{D}_\sigma$ is the unique (φ,Γ)-module over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$ such that $\widetilde{D} \subset \widetilde{D}_\sigma$, $\widetilde{D}[1/t] \cong \widetilde{D}_\sigma[1/t]$, and the Sen σ-weights of $\widetilde{D}_\sigma$ are equal to those of $\widetilde{D}$, and the Sen τ-weights (over E) of $\widetilde{D}_\sigma$ are constantly $h_{\tau,n}$ for $\tau \neq \sigma$.

It is clear that $\mathfrak{T}_\sigma$ restricts to $\mathrm{Ext}^1_\sigma(D,D)\to \mathrm{Ext}^1_\sigma(D_\sigma,D_\sigma)$. A P-filtration $\mathscr{F}_P$ on D with $\mathrm{gr}^i_{\mathscr{F}_P} D = M_i$ induces a P-filtration, still denoted by $\mathscr{F}_P$, on D_σ such that $\mathrm{gr}^i_{\mathscr{F}_P} D_\sigma \cong M_{i,\sigma}$, where $M_{i,\sigma}$ is the (φ,Γ)-module associated to M_i as in Proposition 2.4. It is also clear $\mathfrak{T}_\sigma$ restricts to a map $\mathfrak{T}_\sigma : \mathrm{Ext}^1_{\mathscr{F}_P}(D,D)\to \mathrm{Ext}^1_{\mathscr{F}_P}(D_\sigma,D_\sigma)$.

Proposition 2.24. *(1) The induced map $\mathfrak{T}_\sigma : \mathrm{Ext}^1_g(D,D)\to \mathrm{Ext}^1_g(D_\sigma,D_\sigma)$ is surjective.*

(2) The induced map $\mathfrak{T}_\sigma : \mathrm{Ext}^1_\sigma(D,D)\to \mathrm{Ext}^1_\sigma(D_\sigma,D_\sigma)$ is surjective.

(3) The induced map $\mathfrak{T}_\sigma : \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D)\to \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D_\sigma,D_\sigma)$ is surjective, and the following diagram commutes

$$\begin{array}{ccc} \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D) & \longrightarrow & \prod_{i=1}^r \mathrm{Ext}^1_\sigma(M_i,M_i) \\ {\scriptstyle \mathfrak{T}_\sigma}\downarrow & & {\scriptstyle \mathfrak{T}_\sigma}\downarrow \\ \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D_\sigma,D_\sigma) & \longrightarrow & \prod_{i=1}^r \mathrm{Ext}^1_\sigma(M_{i,\sigma},M_{i,\sigma}). \end{array} \tag{2.14}$$

Consequently, the map $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D_\sigma,D_\sigma)\to \prod_{i=1}^r \mathrm{Ext}^1_\sigma(M_{i,\sigma},M_{i,\sigma})$ is surjective.

Proof. (1): first the kernel has dimension bigger than $\dim_E \mathrm{Ext}^1_g(D,D) - \dim_E \mathrm{Ext}^1_g(D_\sigma,D_\sigma) = (d_K-1)\frac{n(n-1)}{2}$. On the other hand, the composition

$$\mathrm{Ext}^1(D,D)\longrightarrow \mathrm{Ext}^1(D_\sigma,D_\sigma)\longrightarrow \mathrm{Ext}^1(D,D_\sigma)$$

coincides with the natural push-forward map. However, by [11, Lem. 5.1.1], one can prove the kernel of $\mathrm{Ext}^1(D,D)\to \mathrm{Ext}^1(D,D_\sigma)$ has dimension equal to $(d_K-1)\frac{n(n-1)}{2}$. (1) follows. We also deduce the kernel of (2.13) (which is clearly contained in $\mathrm{Ext}^1_g(D,D)$) has dimension $(d_K-1)\frac{n(n-1)}{2}$. By comparing dimensions, (2) and the first part of (3) follow. The commutativity of (2.14) is clear from the definition of $\mathfrak{T}_\sigma$. The last part of (3) is then a consequence of (2) and Proposition 2.18 (2). ☐

Remark 2.25. *As* Ker(2.13) *is contained in $\mathrm{Ext}^1_g(D,D)$ hence in $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D)$, $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D,D)$ is in fact the preimage of $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D_\sigma,D_\sigma)$ under $\mathfrak{T}_\sigma$.*

Corollary 2.26. *Let $w_1,w_2\in S_n$, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Ext}^1_g(D_\sigma,D_\sigma) & \xrightarrow{\kappa_{w_1}} & \mathrm{Hom}_{\mathrm{sm}}(T(K),E) \\ \| & & {\scriptstyle w_2w_1^{-1}}\downarrow \\ \mathrm{Ext}^1_g(D_\sigma,D_\sigma) & \xrightarrow{\kappa_{w_2}} & \mathrm{Hom}_{\mathrm{sm}}(T(K),E), \end{array}$$

and the horizontal maps are surjective.

Proof. The commutativity follows by the same argument as in Lemma 2.12. For $w\in S_n$, by Proposition 2.24, we have

$$\begin{array}{ccccc} \mathrm{Ext}^1_g(D,D) & \longrightarrow & \mathrm{Ext}^1_w(D,D) & \longrightarrow & \mathrm{Hom}(T(K),E) \\ {\scriptstyle \mathfrak{T}_\sigma}\downarrow & & {\scriptstyle \mathfrak{T}_\sigma}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ \mathrm{Ext}^1_g(D_\sigma,D_\sigma) & \longrightarrow & \mathrm{Ext}^1_w(D_\sigma,D_\sigma) & \longrightarrow & \mathrm{Hom}(T(K),E). \end{array}$$

The surjectivity of κ_{w_i} follows from Proposition 2.11 (3). ☐

Let $\mathrm{Ext}^1_0(D_\sigma, D_\sigma) \subset \mathrm{Ext}^1_g(D_\sigma, D_\sigma)$ be the kernel of $\kappa_w|_{\mathrm{Ext}^1_g(D_\sigma, D_\sigma)}$ (for one or equivalently any $w \in S_n$, by Corollary 2.26). Note it is however not equal to the kernel of (2.12) when $K \neq \mathbb{Q}_p$. For $\mathrm{Ext}^1_?(D_\sigma, D_\sigma) \supset \mathrm{Ext}^1_0(D_\sigma, D_\sigma)$, denote by $\overline{\mathrm{Ext}}^1_?(D_\sigma, D_\sigma) := \mathrm{Ext}^1_?(D_\sigma, D_\sigma)/\mathrm{Ext}^1_0(D_\sigma, D_\sigma)$. The following corollary is a direct consequence of Proposition 2.24 (2) (3) by comparing dimensions.

Corollary 2.27. *(1) The (surjective) map $\mathrm{Ext}^1_\sigma(D, D) \to \mathrm{Ext}^1_\sigma(D_\sigma, D_\sigma)$ induces an isomorphism*

$$\overline{\mathrm{Ext}}^1_\sigma(D, D) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1_\sigma(D_\sigma, D_\sigma).$$

(2)The (surjective) map $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D, D) \to \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(D_\sigma, D_\sigma)$ induces an isomorphism

$$\overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_P}(D, D) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_P}(D_\sigma, D_\sigma).$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_P}(D, D) & \xrightarrow{\sim} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(M_i, M_i) \\ {\scriptstyle \mathfrak{T}_\sigma}\downarrow{\scriptstyle \sim} & & {\scriptstyle \mathfrak{T}_\sigma}\downarrow{\scriptstyle \sim} \\ \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_P}(D_\sigma, D_\sigma) & \xrightarrow{\sim} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(M_{i,\sigma}, M_{i,\sigma}). \end{array}$$

2.4 Hodge filtration and higher intertwining

Let D be as in the previous section, we show that there are more higher intertwining relations in paraboline deformations of D than those considered in Corollary 2.13. Notably, such intertwining relations turn out to carry some information on the Hodge filtration of D.

Let $\mathscr{F}$ be the filtration $D_1 \subset D$, and $\mathscr{G}$ be the filtration $\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}) \subset D$. By Proposition 2.14, we have

$$\dim_E \mathrm{Ext}^1_{\mathscr{F}}(D, D) = \dim_E \mathrm{Ext}^1_{\mathscr{G}}(D, D) = 1 + (n^2 - n + 1)d_K.$$

And there are natural surjections (identifying $\mathrm{Ext}^1_{K^\times}(\delta, \delta)$ with $\mathrm{Hom}(K^\times, E)$):

$$\kappa_{\mathscr{F}} = (\kappa_{\mathscr{F},1}, \kappa_{\mathscr{F},2}) : \mathrm{Ext}^1_{\mathscr{F}}(D, D) \twoheadrightarrow \mathrm{Ext}^1(D_1, D_1) \times \mathrm{Hom}(K^\times, E),$$

$$\kappa_{\mathscr{G}} = (\kappa_{\mathscr{G},1}, \kappa_{\mathscr{G},2}) : \mathrm{Ext}^1_{\mathscr{G}}(D, D) \twoheadrightarrow \mathrm{Ext}^1(C_1, C_1) \times \mathrm{Hom}(K^\times, E).$$

We introduce certain subspaces of $\mathrm{Ext}^1(D_1, D_1)$ and $\mathrm{Ext}^1(C_1, C_1)$. For $\iota \in \mathrm{Hom}(D_1, C_1)$. Consider the pull-back and push-forward maps:

$$\iota^- : \mathrm{Ext}^1(C_1, D_1) \longrightarrow \mathrm{Ext}^1(D_1, D_1), \ \iota^+ : \mathrm{Ext}^1(C_1, D_1) \longrightarrow \mathrm{Ext}^1(C_1, C_1). \tag{2.15}$$

Set

$$\mathrm{Ext}^1_\iota(D_1, D_1) := \iota^-(\mathrm{Ext}^1(C_1, D_1)), \ \mathrm{Ext}^1_\iota(C_1, C_1) := \iota^+(\mathrm{Ext}^1(C_1, D_1)).$$

Lemma 2.28. *For $i \in \{1, \cdots, n-1\}$, we have $\dim_E \mathrm{Ext}^1_{\alpha_i}(D_1, D_1) = (n-1)(n-2)d_K$ (see (2.1) for α_i). Moreover for $j \in \{1, \cdots, n-1\}$, $j \neq i$,*

$$\dim_E \big(\mathrm{Ext}^1_{\alpha_i}(D_1, D_1) \cap \mathrm{Ext}^1_{\alpha_j}(D_1, D_1)\big) = (n-1)(n-3)d_K.$$

The same statement holds with D_1 replaced by C_1.

Proof. We only prove it for D_1, C_1 being similar. Let $\mathbf{r} := \{1, \cdots, n-1\} \setminus \{i\}$. The map α_i^- factors through

$$\mathrm{Ext}^1(C_1, D_1) \twoheadrightarrow \mathrm{Ext}^1((D_1)^{\mathbf{r}}, D_1) \hookrightarrow \mathrm{Ext}^1(D_1, D_1)$$

where the corresponding surjectivity and injectivity follow easily by dévissage. The first part follows from the fact $\dim_E \mathrm{Ext}^1((D_1)^{\mathbf{r}}, D_1) = (n-1)(n-2)d_K$. We also see $\mathrm{Ext}^1_{\alpha_i}(D_1, D_1)$ is just the kernel of the natural pull-back map $\mathrm{Ext}^1(D_1, D_1) \to \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_i z^{\mathbf{h}_1}), D_1)$. Hence $\mathrm{Ext}^1_{\alpha_i}(D_1, D_1) \cap \mathrm{Ext}^1_{\alpha_j}(D_1, D_1)$ is equal to the kernel of the pull-back map $\mathrm{Ext}^1(D_1, D_1) \twoheadrightarrow \mathrm{Ext}^1((D_1)_{\{i,j\}}, D_1)$. The second part follows by a direct calculation. □

Proposition 2.29. *Let $\iota \in \mathrm{Hom}(D_1, C_1)$ be an injection.*

(1) $\dim_E \mathrm{Ext}^1_\iota(D_1, D_1) = \dim_E \mathrm{Ext}^1_\iota(C_1, C_1) = 1 + (n-1)(n-2)d_K$.

(2) $\mathrm{Ext}^1_g(D_1, D_1) \subset \mathrm{Ext}^1_\iota(D_1, D_1)$ and $\mathrm{Ext}^1_g(C_1, C_1) \subset \mathrm{Ext}^1_\iota(C_1, C_1)$.

(3) For $\iota' \in \mathrm{Hom}(D_1, C_1)$, $\mathrm{Ext}^1_{\iota'}(D_1, D_1) = \mathrm{Ext}^1_\iota(D_1, D_1)$ if and only if $\mathrm{Ext}^1_{\iota'}(C_1, C_1) = \mathrm{Ext}^1_\iota(C_1, C_1)$ if and only if $\iota' = a\iota$ for some $a \in E^\times$.

Proof. We only prove it for D_1 with C_1 being similar. For any $i \in \{1, \cdots, n-1\}$ and $\mathbf{i} := \{1, \cdots, i\}$, the saturation of $(D_1)_{\mathbf{i}}$ in C_1 via ι is just $(C_1)_{\mathbf{i}}$. We then deduce $W^+_{\mathrm{dR}}(D_1^\vee)/\iota^\vee\big(W^+_{\mathrm{dR}}(C_1^\vee)\big)$ is a successive extension of $\oplus_{\sigma\in\Sigma_K} t^{-h_{j,\sigma}} B^+_{\mathrm{dR},\sigma}/t^{-h_{j+1,\sigma}} B^+_{\mathrm{dR},\sigma}$ with j decreasing. Together with the fact $W^+_{\mathrm{dR}}(D_1) \cong \oplus_{\substack{k=1,\cdots,n-1\\ \sigma\in\Sigma_K}} t^{h_{k,\sigma}} B^+_{\mathrm{dR},\sigma}$, we have an isomorphism of B^+_{dR}-representation of Gal_K:

$$W^+_{\mathrm{dR}}(D_1^\vee \otimes_{\mathcal{R}_{K,E}} D_1)/\iota^\vee\big(W^+_{\mathrm{dR}}(C_1^\vee \otimes_{\mathcal{R}_{K,E}} D_1)\big) \cong \bigoplus_{\substack{k=1,\cdots,n-1\\ \sigma\in\Sigma_K}} M_{k,\sigma},$$

where $M_{k,\sigma}$ is isomorphic to a successive extension of $t^{h_{k,\sigma}-h_{j,\sigma}} B^+_{\mathrm{dR},\sigma}/t^{h_{k,\sigma}-h_{j+1,\sigma}} B^+_{\mathrm{dR},\sigma}$. It is not difficult to see $H^0(\mathrm{Gal}_K, M_{k,\sigma}) \cong E$. By dévissage and using [11, Lem. 5.1.1], this implies $\dim_E \mathrm{Ker}\, \iota^- = (n-1)d_K - 1$ hence

$$\dim_E \mathrm{Im}\, \iota^- = (n-1)^2 d_K - (n-1)d_K + 1 = 1 + (n-1)(n-2)d_K.$$

Recall $\dim_E \mathrm{Ext}^1_g(D_1, D_1) = 1 + \frac{(n-1)(n-2)}{2} d_K$. By definition, we have an exact sequence

$$H^1_g(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee) \hookrightarrow H^1(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee) \longrightarrow H^1(\mathrm{Gal}_K, W^+_{\mathrm{dR}}(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee)).$$

It is not difficult to see $\dim_E H^1(\mathrm{Gal}_K, W^+_{\mathrm{dR}}(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee)) = \frac{(n-1)(n-2)}{2} d_K$, from which we deduce $\dim_E \mathrm{Ext}^1_g(C_1, D_1) = \dim_E H^1_g(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee) \geq (n-1)^2 d_K - \frac{(n-1)(n-2)}{2} d_K = \frac{n(n-1)}{2} d_K$. The map ι^- obviously induces

$$\iota_g^- : \mathrm{Ext}^1_g(C_1, D_1) \longrightarrow \mathrm{Ext}^1_g(D_1, D_1).$$

It is clear that $\mathrm{Ker}\, \iota^- \subset \mathrm{Ext}^1_g(C_1, D_1)$ hence is equal to $\mathrm{Ker}\, \iota_g^-$. We have $\dim_E \mathrm{Ext}^1(C_1, D_1) = (n-1)^2 d_K$, so $\dim_E \mathrm{Ker}\, \iota^- = \dim_E \mathrm{Ker}\, \iota_g^- = (n-1)d_K - 1$. By comparing the dimensions, we see ι_g^- is surjective (and $\dim_E \mathrm{Ext}^1_g(C_1, D_1) = \frac{n(n-1)}{2} d_K$). Finally, consider the cup-product

$$\mathrm{Ext}^1(C_1, D_1) \times \mathrm{Hom}(D_1, C_1) \longrightarrow \mathrm{Ext}^1(D_1, D_1).$$

Suppose $\iota' \notin E[\iota]$, then ι' and ι form a basis of $\mathrm{Hom}(D_1, C_1)$. If $\mathrm{Ext}^1_{\iota'}(D_1, D_1) = \mathrm{Ext}^1_\iota(D_1, D_1)$, we then easily deduce $\mathrm{Ext}^1_{\alpha_i}(D_1, D_1) \subset \mathrm{Ext}^1_\iota(D_1, D_1)$ for all $i = \{1, \cdots, n-1\}$. However, for $i \neq j$, by Lemma 2.28, $\dim_E(\mathrm{Ext}^1_{\alpha_i}(D_1, D_1) + \mathrm{Ext}^1_{\alpha_j}(D_1, D_1)) = (n-1)^2 d_K$, a contradiction. □

Let T_1 be the torus subgroup of GL_{n-1}, and $\phi^1 := \phi_1 \boxtimes \cdots \boxtimes \phi_{n-1}$, which is a refinement of both D_1 and C_1. Let $\mathbf{h}^1 := (\mathbf{h}_1, \cdots, \mathbf{h}_{n-1})$, and $\mathbf{h}^2 := (\mathbf{h}_2, \cdots, \mathbf{h}_n)$. For the refinement ϕ^1, we have maps

$$\mathrm{Ext}^1_g(D_1, D_1) \xrightarrow{\kappa_{\phi^1}} \mathrm{Hom}_{\mathrm{sm}}(T_1(K), E),\ \mathrm{Ext}^1_g(C_1, C_1) \xrightarrow{\kappa_{\phi^1}} \mathrm{Hom}_{\mathrm{sm}}(T_1(K), E).$$

Lemma 2.30. *For $M \in \mathrm{Ext}^1_g(C_1, D_1)$, $\kappa_{\phi^1} \circ \iota_g^-(M) = \kappa_{\phi^1} \circ \iota_g^+(M)$.*

Proof. By definition, there is a natural injection $\tilde{\iota} : \iota_g^-(M) \hookrightarrow \iota_g^+(M)$ which sits in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1 & \longrightarrow & \iota_g^-(M) & \longrightarrow & D_1 & \longrightarrow & 0 \\ & & \iota\downarrow & & \tilde{\iota}\downarrow & & \iota\downarrow & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & \iota_g^+(M) & \longrightarrow & C_1 & \longrightarrow & 0. \end{array}$$

It is easy to see $\tilde{\iota}$ is moreover $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$-linear if $\iota_g^-(M)$ and $\iota_g^+(M)$ are equipped with the natural $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$-action. Suppose $\kappa_{\phi^1} \circ \iota_g^-(M) = (\psi_1, \cdots, \psi_{n-1})$ and $\kappa_{\phi^1} \circ \iota_g^+(M) = (\psi_1', \cdots, \psi_{n-1}')$. Then $\iota_g^-(M)$ (resp. $\iota_g^+(M)$) is isomorphic, as (φ, Γ)-module over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$, to a successive extension of $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_i}(1+\psi_i\epsilon))$ (resp. $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_{i+1}}(1+\psi_i'\epsilon))$) for $i = 1, \cdots, n-1$. It is not difficult to see $\tilde{\iota}$ induces injections $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_i}(1+\psi_i\epsilon)) \hookrightarrow \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_{i+1}}(1+\psi_i'\epsilon))$ of (φ, Γ)-modules over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$. Hence $\psi_i = \psi_i'$ for all i. ∎

Denote by $\Phi\Gamma_{\mathrm{nc}}(D_1, C_1, \phi_n) \subset \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ the subset of isomorphism classes of (φ, Γ)-modules D such that $\mathrm{Hom}(D_1, D) = \mathrm{Hom}(D, C_1) \cong E$. For an injection $\iota \in \mathrm{Hom}(D_1, C_1)$, we set $\mathscr{I}_\iota$ to be the following set

$$\{(\widetilde{D}_1, \widetilde{C}_1) \in \mathrm{Ext}^1_\iota(D_1, D_1) \times \mathrm{Ext}^1_\iota(C_1, C_1) \mid \exists M \in \mathrm{Ext}^1(C_1, D_1) \text{ s.t. } \iota^-(M) = \widetilde{D}_1, \iota^+(M) = \widetilde{C}_1\}.$$

If $\iota = \iota_D$ for some $D \in \Phi\Gamma_{\mathrm{nc}}(D_1, C_1, \phi_n)$, we write $\mathscr{I}_D := \mathscr{I}_{\iota_D}$. The following corollary is a direct consequence of Proposition 2.29 (3) and Proposition 2.3.

Corollary 2.31. *We have $\mathscr{I}_\iota = \mathscr{I}_{\iota'}$ if and only if $\iota' = a\iota$ for some $a \in E^\times$. In particular, for D, $D' \in \Phi\Gamma_{\mathrm{nc}}(D_1, C_1, \phi_n)$ we have $\mathscr{I}_D = \mathscr{I}_{D'}$ if and only if $\iota_D = a\iota_{D'}$ for $a \in E^\times$. When $K = \mathbb{Q}_p$, this is equivalent to $D \cong D'$.*

Theorem 2.32 (Higher intertwining). *Let $D \in \Phi\Gamma_{\mathrm{nc}}(D_1, C_1, \phi_n)$ and $\widetilde{D} \in \mathrm{Ext}^1_{\mathscr{F}}(D, D)$ with $\kappa_{\mathscr{F}}(\widetilde{D}) = (\widetilde{D}_1, \psi)$. The followings are equivalent:*

1. *$\widetilde{D} \in \mathrm{Ext}^1_{\mathscr{F}}(D, D) \cap \mathrm{Ext}^1_{\mathscr{G}}(D, D)$.*
2. *$\widetilde{D}_1 \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(1 - \psi\epsilon) \in \mathrm{Ext}^1_{\iota_D}(D_1, D_1)$.*

Moreover, if the equivalent conditions hold, then $\kappa_{\mathscr{G},2}(\widetilde{D}) = \psi$ and there exists $M \in \mathrm{Ext}^1(C_1, D_1)$ such that $\widetilde{D}_1 = \iota_D^-(M) \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1+\psi\epsilon)$ and $\kappa_{\mathscr{G},1}(\widetilde{D}) = \iota_D^+(M) \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1+\psi\epsilon)$.

Proof. Twisting $\widetilde{D}$ by $1 - \psi\epsilon$, we can and do assume $\kappa_{\mathscr{F},2}(\widetilde{D}) = 0$. By definition, $\widetilde{D} \in \mathrm{Ext}^1_{\mathscr{G}}(D, D)$ if and only if it lies in the kernel of the composition

$$\mathrm{Ext}^1(D, D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}), D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}), C_1). \tag{2.16}$$

Similarly, $\mathrm{Ext}^1_{\mathscr{F}}(D,D)$ is equal to the kernel of the composition

$$\mathrm{Ext}^1(D,D) \longrightarrow \mathrm{Ext}^1(D,\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n})) \longrightarrow \mathrm{Ext}^1(D_1,\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n})).$$

By dévissage, it is not difficult to deduce an exact sequence

$$0 \to \mathrm{Ext}^1(D,D_1) \longrightarrow \mathrm{Ext}^1_{\mathscr{F}}(D,D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}),\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n})) \longrightarrow 0.$$

As $\kappa_{\mathscr{F},2}(\widetilde{D})=0$, $\widetilde{D}$ lies in the image of $\mathrm{Ext}^1(D,D_1)\to\mathrm{Ext}^1_{\mathscr{F}}(D,D)$. Let $M_1\in\mathrm{Ext}^1(D,D_1)$ be the preimage of $\widetilde{D}$. Consider the composition

$$\mathrm{Ext}^1(D,D_1) \lhook\joinrel\longrightarrow \mathrm{Ext}^1(D,D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}),D) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}),C_1).$$

It is straightforward to see it is equal to the composition

$$\mathrm{Ext}^1(D,D_1) \longrightarrow \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}),D_1) \xrightarrow{\iota_D} \mathrm{Ext}^1(\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_1}),C_1). \tag{2.17}$$

So $\widetilde{D}$ lies in the kernel of (2.16) if and only if M_1 is sent to zero via (2.17). However, using dévissage and [11, Lem. 5.1.1], the push-forward map ι_D in (2.17) is injective. We see (under the assumption $\psi=0$) that (1) is equivalent to that M_1 lies in the kernel of the first map of (2.17), which is equal to $\mathrm{Ext}^1(C_1,D_1)$ by dévissage. This is furthermore equivalent to that $\widetilde{D}_1$ lies in the image of the composition

$$\mathrm{Ext}^1(C_1,D_1) \lhook\joinrel\longrightarrow \mathrm{Ext}^1(D,D_1) \longrightarrow \mathrm{Ext}^1(D_1,D_1), \tag{2.18}$$

which is no other than ι_D^-. The other parts are straightforward. □

Corollary 2.33. *We have* $\dim_E(\mathrm{Ext}^1_{\mathscr{F}}(D,D)\cap\mathrm{Ext}^1_{\mathscr{G}}(D,D))=1+(n^2-2n+2)d_K$. *Consequently, the following natural map is surjective:*

$$\mathrm{Ext}^1_{\mathscr{F}}(D,D)\oplus\mathrm{Ext}^1_{\mathscr{G}}(D,D) \twoheadrightarrow \mathrm{Ext}^1(D,D). \tag{2.19}$$

Proof. The dimension part follows from Theorem 2.32, Proposition 2.29 and Proposition 2.14. The second part follows from the first part and Proposition 2.14 by comparing dimensions. □

Let

$$V(D_1,C_1) := \big(\overline{\mathrm{Ext}}^1(D_1,D_1)\times\mathrm{Hom}(K^\times,E)\big)\oplus\big(\overline{\mathrm{Ext}}^1(C_1,C_1)\times\mathrm{Hom}(K^\times,E)\big)$$
$$\Big(\xleftarrow[\sim]{(\kappa_{\mathscr{F}},\kappa_{\mathscr{G}})} \overline{\mathrm{Ext}}^1_{\mathscr{F}}(D,D)\oplus\overline{\mathrm{Ext}}^1_{\mathscr{G}}(D,D)\Big),$$

and $\mathcal{L}(D,D_1,C_1)$ be the subspace consisting of those $\big((x,\psi),(y,\psi)\big)\in V(D_1,C_1)$ such that there exist $\widetilde{D}_1\in\mathrm{Ext}^1(D_1,D_1)$ and $\widetilde{C}_1\in\mathrm{Ext}^1(C_1,C_1)$ such that $\widetilde{D}_1\equiv x$, $\widetilde{C}_1\equiv y$, and

$$\big(\widetilde{D}_1\otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}}\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1-\psi\epsilon),\widetilde{C}_1\otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}}\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1-\psi\epsilon)\big)\in\mathscr{I}_D.$$

Corollary 2.34. *(1) Let* $D,D'\in\Phi\Gamma_{\mathrm{nc}}(D_1,C_1,\phi_n)$, *then* $\mathcal{L}(D',D_1,C_1)=\mathcal{L}(D,D_1,C_1)$ *if and only if* $\iota_{D'}=a\iota_D$ *for some* $a\in E^\times$. *When* $K=\mathbb{Q}_p$, *this is equivalent to* $D\cong D'$.

(2) For $D\in\Phi\Gamma_{\mathrm{nc}}(D_1,C_1,\phi_n)$, *there is a natural exact sequence*

$$0\longrightarrow\mathcal{L}(D,D_1,C_1)\longrightarrow V(D_1,C_1)\longrightarrow\overline{\mathrm{Ext}}^1(D,D)\longrightarrow 0. \tag{2.20}$$

Proof. (1): Suppose $\mathcal{L}(D', D_1, C_1) = \mathcal{L}(D, D_1, C_1)$, we show $\mathrm{Ext}^1_{\iota_D}(D_1, D_1) \subset \mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1)$ hence $\iota_{D'} \in E^\times \iota_D$ by Proposition 2.29. Let $x \in \mathrm{Ext}^1_{\iota_D}(D_1, D_1)$, $M \in \mathrm{Ext}^1(C_1, D_1)$ be a preimage of x and $y := \iota_D^+(M) \in \mathrm{Ext}^1_{\iota_D}(C_1, C_1)$. We have $\big((\bar{x}, 0), (\bar{y}, 0)\big) \in \mathcal{L}(D, D_1, C_1) = \mathcal{L}(D', D_1, C_1)$. There exist hence $x' \in \mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1)$, $y' \in \mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1)$ such that $x' - x \in \mathrm{Ext}^1_0(D_1, D_1)$ (and $y' - y \in \mathrm{Ext}^1_0(C_1, C_1)$). As $\mathrm{Ext}^1_0(D_1, D_1) \subset \mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1)$, this implies $x \in \mathrm{Ext}^1_{\iota'_D}(D_1, D_1)$. (1) follows.

(2) follows from Theorem 2.32 and Corollary 2.33, noting $\mathcal{L}(D, D_1, C_1)$ is no other than the image of the Ker(2.19) in $\overline{\mathrm{Ext}}^1_{\mathscr{F}}(D, D) \oplus \overline{\mathrm{Ext}}^1_{\mathscr{G}}(D, D) \cong V(D_1, C_1)$. □

Now we consider $\Sigma_K \setminus \{\sigma\}$-de Rham deformations. We first consider higher intertwining for deformations of $D_\sigma = \mathfrak{T}_\sigma(D)$. We only consider $\Sigma_K \setminus \{\sigma\}$-de Rham deformations. Let $D_{1,\sigma} = \mathfrak{T}_\sigma(D_1)$ and $C_{1,\sigma} = \mathfrak{T}_\sigma(C_1)$. Let $\iota_\sigma \in \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma})$. We have similar maps as in (2.15), which induce, by restricting to $\Sigma_K \setminus \{\sigma\}$-de Rham extension groups,

$$\iota_\sigma^- : \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma}) \longrightarrow \mathrm{Ext}^1_\sigma(D_{1,\sigma}, D_{1,\sigma}),\ \ \iota_\sigma^+ : \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma}) \longrightarrow \mathrm{Ext}^1_\sigma(C_{1,\sigma}, C_{1,\sigma}).$$

Let $\mathrm{Ext}^1_{\iota_\sigma}(D_1, D_1) := \iota_\sigma^-(\mathrm{Ext}^1_\sigma(C_1, D_1))$ and $\mathrm{Ext}^1_{\iota_\sigma}(C_1, C_1) := \iota_\sigma^+(\mathrm{Ext}^1_\sigma(C_1, D_1))$. Denote by

$$\mathscr{I}_{\iota_\sigma} := \{(\widetilde{D}_{1,\sigma}, \widetilde{C}_{1,\sigma}) \in \mathrm{Ext}^1_{\iota_\sigma}(D_1, D_1) \times \mathrm{Ext}^1_{\iota_\sigma}(C_1, C_1) \mid \\ \exists M \in \mathrm{Ext}^1_\sigma(C_1, D_1) \text{ with } \iota_\sigma^-(M) = \widetilde{D}_1, \iota_\sigma^+(M) = \widetilde{C}_1\}.$$

Similarly as in Proposition 2.29, we have:

Proposition 2.35. *Let $\iota_\sigma \in \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma})$ be an injection.*

(1) $\dim_E \mathrm{Ext}^1_{\iota_\sigma}(D_{1,\sigma}, D_{1,\sigma}) = \dim_E \mathrm{Ext}^1_{\iota_\sigma}(C_{1,\sigma}, C_{1,\sigma}) = 1 + (n-1)(n-2)$.

(2) $\mathrm{Ext}^1_g(D_{1,\sigma}, D_{1,\sigma}) \subset \mathrm{Ext}^1_{\iota_\sigma}(D_{1,\sigma}, D_{1,\sigma})$ and $\mathrm{Ext}^1_g(C_{1,\sigma}, C_{1,\sigma}) \subset \mathrm{Ext}^1_{\iota_\sigma}(C_{1,\sigma}, C_{1,\sigma})$.

(3) For $\iota'_\sigma \in \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma})$, $\mathrm{Ext}^1_{\iota'_\sigma}(D_{1,\sigma}, D_{1,\sigma}) = \mathrm{Ext}^1_{\iota_\sigma}(D_{1,\sigma}, D_{1,\sigma})$ if and only if $\mathrm{Ext}^1_{\iota'_\sigma}(C_{1,\sigma}, C_{1,\sigma}) = \mathrm{Ext}^1_{\iota_\sigma}(C_{1,\sigma}, C_{1,\sigma})$ if and only if $\iota'_\sigma = a\iota_\sigma$ for some $a \in E^\times$.

Proof. We still only prove the statements for $D_{1,\sigma}$. By [23, Cor. A.4], $\dim_E \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma}) = (n-1)^2$. By similar arguments as in the proof of Proposition 2.29 (1), the kernel of $\mathrm{Ext}^1(C_{1,\sigma}, D_{1,\sigma}) \to \mathrm{Ext}^1(D_{1,\sigma}, D_{1,\sigma})$ has dimension $(n-1)-1$, which is clearly the same as $\mathrm{Ker}\, \iota_\sigma^- = \mathrm{Ker}\, \iota_\sigma^-|_{\mathrm{Ext}^1_g(C_{1,\sigma}, D_{1,\sigma})}$. (1) follows. Using [23, Cor. A.4], $\dim_E \mathrm{Ext}^1_g(C_{1,\sigma}, D_{1,\sigma}) = \frac{n(n-1)}{2}$. By comparing dimensions, we see the induced map $\mathrm{Ext}^1_g(C_{1,\sigma}, D_{1,\sigma}) \to \mathrm{Ext}^1_g(D_{1,\sigma}, D_{1,\sigma})$ is surjective. (2) follows. (3) follows from similar arguments as in the proof of Proposition 2.29 (3) using an analogue of Lemma 2.28. We leave the details to the reader. □

For $D_\sigma \in \Phi\Gamma_{\mathrm{nc}}(D_{1,\sigma}, C_{1,\sigma}, \phi_n)$ (which is the subset of $\Phi\Gamma_{\mathrm{nc}}(\phi, \mathscr{T}_\sigma(\mathbf{h}))$ of isomorphism classes of (φ, Γ)-modules D_σ such that $\mathrm{Hom}(D_{1,\sigma}, D_\sigma) = \mathrm{Hom}(D_\sigma, C_{1,\sigma}) = E$), set $\mathscr{I}_{D_\sigma}$ to be the following set

$$\{(\widetilde{D}_{1,\sigma}, \widetilde{C}_{1,\sigma}) \in \mathrm{Ext}^1_{\iota_{D_\sigma}}(D_{1,\sigma}, D_{1,\sigma}) \times \mathrm{Ext}^1_{\iota_{D_\sigma}}(C_{1,\sigma}, C_{1,\sigma}) \mid \\ \exists M \in \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma}) \text{ with } \iota_{D_\sigma}^-(M) = \widetilde{D}_{1,\sigma}, \iota_{D_\sigma}^+(M) = \widetilde{C}_{1,\sigma}\}.$$

We have by Proposition 2.35 (3) and Proposition 2.9:

Corollary 2.36. *For D_σ, $D'_\sigma \in \Phi\Gamma_{\mathrm{nc}}(D_{1,\sigma}, C_{1,\sigma}, \phi_n)$, we have $\mathscr{I}_{D_\sigma} = \mathscr{I}_{D'_\sigma}$ if and only if $D_\sigma \cong D'_\sigma$.*

The following theorem follows by the same argument as in Theorem 2.32.

Theorem 2.37. *Let $\widetilde{D}_\sigma \in \mathrm{Ext}^1_{\sigma,\mathscr{F}}(D_\sigma, D_\sigma)$ with $\kappa_{\mathscr{F}}(\widetilde{D}_\sigma) = (\widetilde{D}_{\sigma,1}, \psi)$. The followings are equivalent:*

1. $\widetilde{D}_\sigma \in \mathrm{Ext}^1_{\sigma,\mathscr{F}}(D_\sigma, D_\sigma) \cap \mathrm{Ext}^1_{\sigma,\mathscr{G}}(D_\sigma, D_\sigma)$.
2. $\widetilde{D}_{1,\sigma} \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(1-\psi\epsilon) \in \mathrm{Ext}^1_{\iota_{D_\sigma}}(D_{1,\sigma}, D_{1,\sigma})$.

Moreover, if the equivalent conditions hold, then $\kappa_{\mathscr{G},2}(\widetilde{D}_\sigma) = \psi$ and there exists $M \in \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma})$ such that $\widetilde{D}_{1,\sigma} \cong \iota^-_{D_\sigma}(M) \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1+\psi\epsilon)$ and $\kappa_{\mathscr{G},1}(\widetilde{D}_\sigma) = \iota^+_{D_\sigma}(M) \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1+\psi\epsilon)$.

Let $\mathrm{Hom}_\sigma(K^\times, E)$ be the subspace of locally σ-analytic characters of $K^\times$ (see for example [22, § 1.3.1]). Set

$$V(D_{1,\sigma}, C_{1,\sigma})_\sigma := \big(\overline{\mathrm{Ext}}^1_\sigma(D_{1,\sigma}, D_{1,\sigma}) \times \mathrm{Hom}_\sigma(K^\times, E)\big) \oplus \big(\overline{\mathrm{Ext}}^1_\sigma(C_1, C_1) \times \mathrm{Hom}_\sigma(K^\times, E)\big) \quad (2.21)$$

and $\mathcal{L}(D_\sigma, D_{1,\sigma}, D_{2,\sigma})_\sigma$ to be the the subspace consisting of those $\big((x,\psi),(y,\psi)\big) \in V(D_{1,\sigma}, C_{1,\sigma})_\sigma$ such that there exist $\widetilde{D}_{1,\sigma} \in \mathrm{Ext}^1_\sigma(D_{1,\sigma}, D_{1,\sigma})$ and $\widetilde{C}_{1,\sigma} \in \mathrm{Ext}^1_\sigma(C_{1,\sigma}, C_{1,\sigma})$ satisfying $\widetilde{D}_{1,\sigma} \equiv x$, $\widetilde{C}_{1,\sigma} \equiv y$, and

$$\big(\widetilde{D}_{1,\sigma} \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1-\psi\epsilon), \widetilde{C}_{1,\sigma} \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1-\psi\epsilon)\big) \in \mathscr{I}_{D_\sigma}.$$

By Proposition 2.9 and the same arguments as in Corollary 2.34, we have:

Corollary 2.38. *(1) Let $D_\sigma, D'_\sigma \in \Phi\Gamma_{\mathrm{nc}}(D_{1,\sigma}, C_{1,\sigma}, \phi_n)$, then $\mathcal{L}(D'_\sigma, D_{1,\sigma}, C_{1,\sigma}) = \mathcal{L}(D_\sigma, D_{1,\sigma}, C_{1,\sigma})$ if and only if $D_\sigma \cong D'_\sigma$.*

(2) There is a natural exact sequence

$$0 \longrightarrow \mathcal{L}(D_\sigma, D_{1,\sigma}, C_{1,\sigma}) \longrightarrow V(D_{1,\sigma}, C_{1,\sigma})_\sigma \longrightarrow \overline{\mathrm{Ext}}^1_\sigma(D_\sigma, D_\sigma) \longrightarrow 0. \quad (2.22)$$

Let $V(D_1, C_1)_\sigma := \big(\overline{\mathrm{Ext}}^1_\sigma(D_1, D_1) \times \mathrm{Hom}_\sigma(K^\times, E)\big) \oplus \big(\overline{\mathrm{Ext}}^1_\sigma(C_1, C_1) \times \mathrm{Hom}_\sigma(K^\times, E)\big) \subset V(D_1, C_1)$, and $\mathcal{L}(D, D_1, C_1)_\sigma := \mathcal{L}(D, D_1, C_1) \cap V(D_1, C_1)_\sigma$.

Proposition 2.39. *The functor $\mathfrak{T}_\sigma$ induces a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(D, D_1, C_1)_\sigma & \longrightarrow & V(D_1, C_1)_\sigma & \longrightarrow & \overline{\mathrm{Ext}}^1_\sigma(D, D) & \longrightarrow & 0 \\ & & \mathfrak{T}_\sigma \downarrow \sim & & \mathfrak{T}_\sigma \downarrow \sim & & \mathfrak{T}_\sigma \downarrow \sim & & \\ 0 & \longrightarrow & \mathcal{L}(D_\sigma, D_{1,\sigma}, C_{1,\sigma})_\sigma & \longrightarrow & V(D_{1,\sigma}, C_{1,\sigma})_\sigma & \longrightarrow & \overline{\mathrm{Ext}}^1_\sigma(D_\sigma, D_\sigma) & \longrightarrow & 0 \end{array}$$

where the top sequence is induced by (2.20).

Proof. All the maps are clear, and we have seen in the above corollary that the bottom sequence is exact. The left exactness of the top sequence is clear. However, by Corollary 2.27, the two right vertical maps are both isomorphisms. The proposition follows. □

Corollary 2.40. *The map (2.11) is surjective. And the same holds with D replaced by D_σ.*

Proof. By the above proposition, $\overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}}(D, D) \oplus \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{G}}(D, D) \to \overline{\mathrm{Ext}}^1_\sigma(D, D)$ is surjective. By induction on the rank n, it is not difficult to deduce $\oplus_{w \in S_n} \overline{\mathrm{Ext}}^1_{\sigma,w}(D, D) \to \overline{\mathrm{Ext}}^1_\sigma(D, D)$ is surjective. As $\mathrm{Ext}^1_0(D, D) \subset \mathrm{Ext}^1_{\sigma,w}(D, D)$ for any $w \in S_n$, we see (2.11) is also surjective. The statement for D_σ follows by the same argument or using Corollary 2.27. □

3 Locally analytic crystabelline representations of $\mathrm{GL}_n(K)$

3.1 Locally analytic representations of $\mathrm{GL}_n(K)$ and extensions

3.1.1 Notation and preliminaries

We introduce some notation on the GL_n-side. Let T be the torus subgroup of GL_n , $B \supset T$ be the Borel subgroup of upper triangular matrices. For a standard parabolic subgroup P of GL_n, denote by $L_P \supset T$ its standard Levi subgroup and P^- its opposite parabolic subgroup. Denote by $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{gl}_n$ the corresponding Lie algebras over K. For $i \in \{1, \cdots, n-1\}$, let $\lambda_i := (\underbrace{1, \cdots, 1}_{i}, 0, \cdots, 0)$ be the associated fundamental weight. Let $\theta := \sum_{i=1}^{n-1} \lambda_i = (n-1, \cdots, 0)$. For a parabolic subgroup P, let $\theta_P := \sum_{i \in S(P)} \lambda_i$ where $S(P)$ is the set of simple roots of L_P, and $\theta^P := \theta - \theta_P$, that we view as an algebraic character of L_P. Let δ_P be the modulus character of $P(K)$. For simplicity, for $i \in \{1, \cdots, n-1\}$, we denote by P_i the associated maximal parabolic subgroup, and L_i its standard Levi subgroup.

For a Lie algebra $\mathfrak{g}$ over K, denote by $\mathfrak{g}_{\Sigma_K} := \mathfrak{g} \otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma \in \Sigma_K} \mathfrak{g} \otimes_{K,\sigma} E =: \prod_{\sigma \in \Sigma_K} \mathfrak{g}_\sigma$. For a weight μ of $\mathfrak{t}_{\Sigma_K}$, denote by $M^-(\mu) := \mathrm{U}(\mathfrak{gl}_{n,\Sigma_K}) \otimes_{\mathrm{U}(\mathfrak{b}^-_{\Sigma_K})} \mu$, and let $L^-(\mu)$ be its unique simple quotient. If μ is anti-dominant, then $L^-(\mu)$ is finite dimensional and isomorphic to the dual $L(-\mu)^\vee$, where $L(-\mu)$ is the algebraic representation of $\mathrm{Res}^K_{\mathbb{Q}_p} \mathrm{GL}_n$ of highest weight $-\mu$ with respect to $\mathrm{Res}^K_{\mathbb{Q}_p} B$.

For an admissible locally $\mathbb{Q}_p$-analytic representation V of $\mathrm{GL}_n(K)$, by [43], its dual $V^\vee$ is naturally a module over the ($\mathbb{Q}_p$-analytic) distribution algebra $\mathcal{D}(\mathrm{GL}_n(K), E)$, which, equipped with the strong topology, is a coadmissible module over $\mathcal{D}(H, E)$ for a(ny) compact open subgroup H of $\mathrm{GL}_n(K)$. For admissible locally $\mathbb{Q}_p$-analytic representations V_1, V_2 of $\mathrm{GL}_n(K)$, set $\mathrm{Ext}^i_{\mathrm{GL}_n(K)}(V_1, V_2) := \mathrm{Ext}^i_{\mathcal{D}(\mathrm{GL}_n(K),E)}(V_2^\vee, V_1^\vee)$, where the latter is defined in the abelian category of abstract $\mathcal{D}(\mathrm{GL}_n(K), E)$-modules. By [9, Lem. 2.1.1], $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(V_1, V_2)$ is equal to the extension group of admissible locally $\mathbb{Q}_p$-analytic representations of V_1 by V_2. Any representation $\widetilde{V}$ in $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(V, V)$ is equipped with a natural $E[\epsilon]/\epsilon^2$ structure where ϵ acts via $\widetilde{V} \twoheadrightarrow V \xrightarrow{\mathrm{id}} V \hookrightarrow \widetilde{V}$. If V is locally algebraic, define $\mathrm{Ext}^1_g(V, V)$ to be the subgroup of locally algebraic extensions.

Let $\phi = \phi_1 \boxtimes \cdots \boxtimes \phi_n : T(K) \to E^\times$ be a smooth character. We call ϕ is generic if $\phi_i \phi_j^{-1} \neq 1, |\cdot|_K$ for $i \neq j$. Let $\eta := |\cdot|_K^{1-n} \boxtimes |\cdot|_K^{2-n} \boxtimes \cdots \boxtimes 1$. For $w \in S_n$, let $w(\phi) := \phi_{w^{-1}(1)} \boxtimes \cdots \phi_{w^{-1}(n)}$. Let $\pi_{\mathrm{sm}}(\phi) := (\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} \phi\eta)^\infty$, which is an absolutely irreducible smooth admissible representation of $\mathrm{GL}_n(K)$ when ϕ is generic. Moreover, when ϕ is generic, $\pi_{\mathrm{sm}}(\phi) \cong \pi_{\mathrm{sm}}(w(\phi))$ for all $w \in S_n$, which is in fact the smooth representation of $\mathrm{GL}_n(K)$ corresponding to the Weil-Deligne representation $\oplus_{i=1}^n \phi_i$ in the classical local Langlands correspondence.

3.1.2 Principal series

We collect some facts on the locally $\mathbb{Q}_p$-analytic principal series of $\mathrm{GL}_n(K)$.

Let $\mathbf{h}$ be a strictly dominant weight of $\mathfrak{t}_{\Sigma_K}$, put $\lambda := \mathbf{h} - \theta^{[K:\mathbb{Q}_p]} = (\lambda_{i,\sigma} = h_{i,\sigma} - n + i)_{\substack{\sigma \in \Sigma_K \\ i=1,\cdots,n}}$, which is a dominant weight of $\mathfrak{t}$. Let ϕ be a generic smooth character of $T(K)$. Put $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) := \pi_{\mathrm{sm}}(\phi) \otimes_E L(\lambda)$, which is a locally algebraic representation of $\mathrm{GL}_n(K)$. For $w \in S_n$, we have

$$\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \cong \pi_{\mathrm{alg}}(w(\phi), \mathbf{h}).$$

For $w \in S_n$, put $\mathrm{PS}(w(\phi), \mathbf{h}) := (\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} w(\phi)\eta z^\lambda)^{\mathbb{Q}_p-\mathrm{an}} = \big(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} w(\phi) z^{\mathbf{h}}(\varepsilon^{-1} \circ \theta)\big)^{\mathbb{Q}_p-\mathrm{an}}$. We have (where $\mathcal{F}_{B^-}^{\mathrm{GL}_n}(-,-)$ denotes Orlik-Strauch functor [42]):

Proposition 3.1. *Let $w \in S_n$.*

(1) The irreducible constituents of $\mathrm{PS}(w(\phi), \boldsymbol{h})$ are given by

$$\big\{\mathscr{C}(w,u) := \mathcal{F}_{B^-}^{\mathrm{GL}_n}(L^-(-u\cdot\lambda), w(\phi)\eta)\big\}_{u=(u_\sigma)\in S_n^{|\Sigma_K|}},$$

which are pairwisely distinct. Moreover, if $\lg(u)=1$, then $\mathscr{C}(w,u)$ has multiplicity one.

(2) $\mathrm{soc}_{\mathrm{GL}_n(K)} \mathrm{PS}(w(\phi), \boldsymbol{h}) \cong \pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$.

(3) $\mathrm{soc}_{\mathrm{GL}_n(K)}(\mathrm{PS}(w(\phi), \boldsymbol{h})/\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})) \cong \oplus_{\substack{u\in S_n^{|\Sigma_K|},\\ \lg(u)=1}} \mathscr{C}(w,u)$.

(4) For $w' \in S_n$, and $u, u' \in S_n^{|\Sigma_K|}$ with $\lg(u)=\lg(u')=1$, $\mathscr{C}(w,u) \cong \mathscr{C}(w',u')$ if and only if $u=u'=s_{i,\sigma}$ for some $i \in \{1, \cdots, n-1\}$ and $\sigma \in \Sigma_K$, and $w(w')^{-1}$ lies in the Weyl group of L_{P_i}.

Proof. (1) and (4) follow from [42] (together with some standard facts on the constituents of the Verma module, see for example [33]). (2) (3) follow from [8, Cor. 2.5] or [41, Thm. 1]. □

For $i \in \{1, \cdots, n-1\}$, let $I \subset \{1, \cdots, n\}$ be a subset of cardinality i. We see that all the representations $\mathscr{C}(w, s_{i,\sigma})$ with $w(\{1,\cdots,i\}) = I$ are isomorphic, which we denote by $\mathscr{C}(I, s_{i,\sigma})$. By Proposition 3.1 (4), $\mathscr{C}(I, s_{i,\sigma})$ are pairwisely distinct for different I. For $w \in S_n$ such that $w(\{1,\cdots,i\}) = I$, we have (by [8, Cor. 2.5][41, Thm. 1])

$$\mathscr{C}(I, s_{i,\sigma}) \cong \mathrm{soc}_{\mathrm{GL}_n(K)}(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta)^{\mathbb{Q}_p-\mathrm{an}}. \tag{3.1}$$

Lemma 3.2. *Let $w \in S_n$ such that $w(\{1,\cdots,i\}) = I$.*

(1)We have $\mathrm{Hom}_{T(\mathbb{Q}_p)}(z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta\delta_B, J_B(\mathscr{C}(I, s_{i,\sigma}))) \cong E$.

(2) We have $I_{B^-(K)}^{\mathrm{GL}_n(K)}(z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta) \cong \mathscr{C}(I, s_{i,\sigma})$, where $I_{B^-(K)}^{\mathrm{GL}_n(K)}(-)$ is Emerton's induction functor [27].

Proof. By [42], it is easy to see any irreducible constituent of $\mathrm{PS}(w(\phi), \mathbf{h})$ is a subrepresentation of a certain locally $\mathbb{Q}_p$-analytic principal series, hence is very strongly admissible by [27, Prop. 2.1.2]. (1) then follows by [7, Thm. 4.3, Rem. 4.4 (i)]. By *loc. cit.* and [41, Thm. 1], we have

$$\mathrm{Hom}_{T(\mathbb{Q}_p)}(z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta\delta_B, J_B(\mathscr{C})) = 0$$

for any irreducible constituent $\mathscr{C}$ of $(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta)^{\mathbb{Q}_p-\mathrm{an}}$ with $\mathscr{C} \neq \mathscr{C}(I, s_{i,\sigma})$. Hence the natural map

$$z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta\delta_B \longrightarrow J_B\big((\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta)^{\mathbb{Q}_p-\mathrm{an}}\big)$$

has image contained in $J_B(\mathscr{C}(I, s_{i,\sigma}))$. (2) follows. □

Let $\mathrm{PS}_1(w(\phi), \mathbf{h})$ be the unique subrepresentation of $\mathrm{PS}(w(\phi), \mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ and cosocle $\oplus_{\substack{i=1,\cdots,n-1\\ \sigma\in\Sigma_K}} \mathscr{C}(w, s_{i,\sigma})$ (with the tautological injection $\mathrm{PS}_1(w(\phi), \mathbf{h}) \hookrightarrow \mathrm{PS}(w(\phi), \mathbf{h})$). Consider the amalgamated sum $\oplus_{\pi_{\mathrm{alg}}(\phi,\lambda)}^{w\in S_n} \mathrm{PS}_1(w(\phi), \mathbf{h})$. It admits a unique quotient, denoted by $\pi_1(\phi, \mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$. By Lemma 3.1 (3) (4), $\pi_1(\phi, \mathbf{h})$ is an extension of $\oplus_{\substack{i=1,\cdots,n-1,\sigma\in\Sigma_K\\ I\subset\{1,\cdots,n\},\#I=i}} \mathscr{C}(I, s_{i,\sigma})$ ($(2^n-2)d_K$ constituents in total) by $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$. We fix the tautological injection $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \hookrightarrow \pi_1(\phi, \mathbf{h})$. We study the extension group of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by $\pi_1(\phi, \mathbf{h})$. First we have:

Proposition 3.3. *(1) For $w\in S_n$, the following natural map is a bijection:*

$$\zeta_w:\mathrm{Hom}_{g'}(T(K),E)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(K)}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big),\ \psi\mapsto I_{B^-(K)}^{\mathrm{GL}_n(K)}(w(\phi)\eta z^{\lambda}(1+\psi\epsilon)),\tag{3.2}$$

and induces a bijection $\mathrm{Hom}_{\mathrm{sm}}(T(K),E)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{lalg}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big)$. In particular, we have $\dim_E\mathrm{Ext}^1_g\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big)=n$, and $\dim_E\mathrm{Ext}^1_{\mathrm{lalg}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big)=n+d_K$.

(2) For $w_1,w_2\in S_n$, the following diagram commutes:

$$\begin{array}{ccc}\mathrm{Hom}_{g'}(T(K),E) & \xrightarrow[\sim]{\zeta_{w_1}} & \mathrm{Ext}^1_{\mathrm{GL}_n(K)}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big)\\ {\scriptstyle w_2w_1^{-1}}\downarrow\wr & & \|\\ \mathrm{Hom}_{g'}(T(K),E) & \xrightarrow[\sim]{\zeta_{w_2}} & \mathrm{Ext}^1_{\mathrm{GL}_n(K)}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big).\end{array}$$

Proof. For $\psi=\psi_1+\psi_0\circ\det$ with $\psi_1\in\mathrm{Hom}_{\mathrm{sm}}(T(K),E)$, it is easy to see the natural map

$$w(\phi)\eta z^{\lambda}(1+\psi\epsilon)\delta_B\hookrightarrow J_B\big((\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)}w(\phi)\eta z^{\lambda}(1+\psi\epsilon))^{\mathbb{Q}_p-\mathrm{an}}\big)\hookrightarrow(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)}w(\phi)z^{\lambda}\eta(1+\psi\epsilon))^{\mathbb{Q}_p-\mathrm{an}}$$

factors through the subrepresentation $(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)}w(\phi)\eta(1+\psi_1\epsilon))^{\mathrm{sm}}\otimes_E L(\lambda)\otimes_{E[\epsilon]/\epsilon^2}(1+\psi_0\circ\det)$. By definition ([27]), we see

$$I_{B^-(K)}^{\mathrm{GL}_n(K)}(w(\phi)\eta z^{\lambda}(1+\psi\epsilon))\cong(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)}w(\phi)\eta(1+\psi_1\epsilon))^{\mathrm{sm}}\otimes_E L(\lambda)\otimes_{E[\epsilon]/\epsilon^2}(1+\psi_0\circ\det),$$

which clearly lies in $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$. So ζ_w is well-defined. By [44, Prop. 4.7], we have (where the subscript "Z" stands for fixing central character)

$$\mathrm{Ext}^1_{\mathrm{lalg},Z}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(K),Z}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h})).\tag{3.3}$$

By classical smooth representation theory, $\dim\mathrm{Ext}^1_{\mathrm{lalg},Z}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))=n-1$. Using similar arguments as in [10, Lem. 3.16], $\dim_E\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h})=n+d_K$. By the smooth representation theory, $\zeta_w|_{\mathrm{Hom}_{\mathrm{sm}}(T(K),E)}$ is a bijection. It is easy to deduce (3.2) is bijective. For (2), it suffices to prove the statement for g' replaced by "sm". But this is a classical fact (e.g. using Bernstein centre). □

Lemma 3.4. *For any $\mathscr{C}(I,s_{i,\sigma})$, we have:*

(1) $\dim_E\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(I,s_{i,\sigma}))=\dim_E\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\mathscr{C}(I,s_{i,\sigma}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))=1$.

(2) Let $\tilde{\pi}_{\mathrm{alg}}(\phi,\boldsymbol{h})\in\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))$ be non-split, then the pull-back map (via $\tilde{\pi}_{\mathrm{alg}}(\phi,\boldsymbol{h})\twoheadrightarrow\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})$)

$$\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(I,s_{i,\sigma}))\longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\tilde{\pi}_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(I,s_{i,\sigma}))$$

is a bijection.

Proof. By Schraen's spectral sequence [45, Cor. 4.9] (noting the separatedness assumption is satisfied for $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by the same argument below [45, Cor. 4.9]), (3.1) and [24, Lem. 2.26],

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(I,s_{i,\sigma}))=1.$$

The second equality in (1) is proved in [11, Cor. 5.2.6]. We have

$$\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\tilde{\pi}_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(I,s_{i,\sigma}))\cong \mathrm{Ext}^1_{\mathrm{GL}_n(K)}\big(\tilde{\pi}_{\mathrm{alg}}(\phi,\mathbf{h}),(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} z^{-s_{i,\sigma}\cdot\lambda}w(\phi)\eta)^{\mathbb{Q}_p-\mathrm{an}}\big)$$
$$\cong \mathrm{Hom}_{T(K)}\big(w(\phi)\eta z^{-s_{i,\sigma}\cdot\lambda}(1+\psi\epsilon),w(\phi)\eta z^{-s_{i,\sigma}\cdot\lambda}\big),$$

where $\psi=\zeta_w^{-1}\big(\tilde{\pi}_{\mathrm{alg}}(\phi,\mathbf{h})\big)\in \mathrm{Hom}_{\mathfrak{g}'}(T(K),E)$, and where the first isomorphism follows from [24, Lem. 2.26] (and an easy dévissage using (3.1)), and the second isomorphism follows from Schraen's spectral sequence [45, Cor. 4.9]. (2) follows. □

For $w\in S_n$, consider the natural map

$$\mathrm{Hom}(T(K),E)\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}\big(\mathrm{PS}(w(\phi),\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h})\big),\ \psi\mapsto \big(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} w(\phi)\eta z^{\lambda}(1+\psi\epsilon)\big)^{\mathbb{Q}_p-\mathrm{an}}.$$

Composed with the pull-back map for an injection $\jmath:\pi_{\mathrm{alg}}(\phi,\mathbf{h})\hookrightarrow \mathrm{PS}_1(w(\phi),\mathbf{h})$ and using [24, Lem. 2.26], it induces a map

$$\mathrm{Hom}(T(K),E)\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1(w(\phi),\mathbf{h})). \tag{3.4}$$

Composed furthermore with the push-forward map via the injection $\mathrm{PS}_1(w(\phi),\mathbf{h})\hookrightarrow \pi_1(\phi,\mathbf{h})$ (associated to $\jmath$), we finally obtain a map

$$\zeta_w:\mathrm{Hom}(T(K),E)\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})). \tag{3.5}$$

Note that the map ζ_w does not depend on the choice of $\jmath$.

Proposition 3.5. *(1) For $w\in S_n$, the map (3.4) is bijective. In particular,*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathrm{PS}_1(w(\phi),\boldsymbol{h}))=n+nd_K.$$

(2) For $w\in S_n$, $\zeta_w|_{\mathrm{Hom}_{\mathfrak{g}'}(T(K),E)}$ is equal to the composition of (3.2) with the injective push-forward map $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))$.

Proof. (1) follows by Schraen's spectral sequence [44, Cor. 4.9] and [23, Lem. 2.26]. (2) is clear (see Remark 3.6 below). □

Remark 3.6. *The map ζ_w can also be obtained by using Emerton's functor $I_{B^-(K)}^{\mathrm{GL}_n(K)}(-)$. In fact, by definition (and using [24, Lem. 2.26]), it is not difficult to see for $\psi\in\mathrm{Hom}(T(K),E)$, $I_{B^-(K)}^{\mathrm{GL}_n(K)}(w(\phi)\eta z^{\lambda}(1+\psi\epsilon))\subset (\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} w(\phi)\eta z^{\lambda}(1+\psi\epsilon))^{\mathbb{Q}_p-\mathrm{an}}$ is an extension of $\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})$ by a certain subrepresentation V of $\mathrm{PS}_1(w(\phi),\boldsymbol{h})$. Then $\zeta_w(\psi)$ is just the image of $I_{B^-(K)}^{\mathrm{GL}_n(K)}(w(\phi)\eta z^{\lambda}(1+\psi\epsilon))$ of the push-forward map via $V\hookrightarrow \mathrm{PS}_1(w(\phi),\boldsymbol{h})\hookrightarrow\pi_1(\phi,\boldsymbol{h})$.*

Proposition 3.7. *(1) We have an exact sequence*

$$0\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))$$
$$\longrightarrow \oplus_{\substack{i=1,\cdots,n-1,\sigma\in\Sigma_K\\ I\subset\{1,\cdots,n-1\},\#I=i}} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(I,s_{i,\sigma}))\longrightarrow 0. \tag{3.6}$$

In particular, $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))=n+(2^n-1)d_K$.

(2) The following map is surjective:

$$t_{\phi,\boldsymbol{h}}:\oplus_{w\in S_n}\mathrm{Hom}(T(K),E)\xrightarrow{(\zeta_w)} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})). \tag{3.7}$$

Proof. The sequence follows by dévissage, and it suffices to prove the second last map in (3.6) is surjective. For $w \in S_n$, using dévissage, we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{PS}_1(w(\phi), \mathbf{h})) \longrightarrow \oplus_{\substack{i=1,\cdots,n-1 \\ \sigma\in\Sigma_K}} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathscr{C}(w, s_{i,\sigma})). \quad (3.8)$$

By comparing dimensions (using Proposition 3.3 (1), Proposition 3.5 (1) and Lemma 3.4 (1)), the last map in (3.8) is surjective. The following diagram clearly commutes

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{PS}_1(w(\phi), \mathbf{h})) & \longrightarrow & \oplus_{\substack{i=1,\cdots,n-1 \\ \sigma\in\Sigma_K}} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathscr{C}(w, s_{i,\sigma})) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) & \longrightarrow & \oplus_{\substack{i=1,\cdots,n-1,\sigma\in\Sigma_K \\ I\subset\{1,\cdots,n-1\},\#I=i}} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathscr{C}(I, s_{i,\sigma})) \end{array} \quad (3.9)$$

where all the vertical maps are injective. With w varying, it is easy to deduce the bottom map is surjective. The dimension part follows then from Lemma 3.4 (1) and Proposition 3.5 (1). Finally, varying w, the image of the right vertical map in (3.9) can "cover" the target. (2) follows. □

Remark 3.8. *By Proposition 3.5 (1) and [24, Lem. 2.26], for $w \in S_n$, we have*

$$\zeta_w : \mathrm{Hom}(T(K), E) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \mathrm{PS}(w(\phi), \boldsymbol{h})).$$

Denote by $\pi(\phi, \boldsymbol{h})$ the unique quotient of $\oplus^{w\in S_n}_{\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})} \mathrm{PS}(w(\phi), \boldsymbol{h})$ of socle $\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$ (cf. [15], which is the representation $\pi(\rho)^{\mathrm{fs}}$ of loc. cit.). The representation $\pi_1(\phi, \boldsymbol{h})$ is in fact the first two layers in the socle filtration of $\pi(\phi, \boldsymbol{h})$. Moreover, using again [24, Lem. 2.26], we have

$$\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi(\phi, \boldsymbol{h})). \quad (3.10)$$

Proposition 3.7 (2) hence holds with $\pi_1(\phi, \boldsymbol{h})$ replaced by $\pi(\phi, \boldsymbol{h})$.

Denote by

$$\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \subset \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \subset \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$$

the respective image of $\mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h}))$, $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h}))$, and $\mathrm{Im}(\zeta_w)$ for $w \in S_n$. We also use the notation $\mathrm{Ext}^1_{\mathscr{T}_w}$ for Ext^1_w whenever it is convenient for the context where $\mathscr{T}_w$ is the B-filtration of $\oplus_{i=1}^n \phi_i$ associated to w. We have hence an isomorphism

$$\zeta_w : \mathrm{Hom}(T(K), E) \xrightarrow{\sim} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})).$$

By Proposition 3.3 (2), for $w_1, w_2 \in S_n$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{g'}(T(K), E) & \xrightarrow[\sim]{\zeta_{w_1}} & \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \\ {\scriptstyle w_2 w_1^{-1}} \downarrow \wr & & \| \\ \mathrm{Hom}_{g'}(T(K), E) & \xrightarrow[\sim]{\zeta_{w_2}} & \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})). \end{array} \quad (3.11)$$

3.1.3 Parabolic inductions

Let $P \supset B$ be a standard parabolic subgroup of GL_n of $L_P = \mathrm{diag}(\mathrm{GL}_{n_1}, \cdots, \mathrm{GL}_{n_r})$. Let $\mathscr{W}_P$ be the Weyl group of L_P. Let $\mathscr{F}_P$ be a P-filtration of $\oplus_{i=1}^n \phi_i$ and $\phi_{\mathscr{F}_P,i} := \otimes \phi_j$ for $\phi_j \in \mathrm{gr}_i \mathscr{F}_P$ (where the order of these ϕ_j does not matter here). For $i = 1, \cdots, r$, let $\mathbf{h}^i := (\mathbf{h}_{n_1+\cdots n_{i-1}+1}, \cdots, \mathbf{h}_{n_1+\cdots+n_i})$. Applying the constructions in § 3.1.2 to $(\phi_{\mathscr{F}_{P,i}}, \mathbf{h}^i)$, we obtain $\mathrm{GL}_{n_i}(K)$-representations $\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i)$, $\pi_1(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i)$ etc. Consider the parabolic induction

$$\big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)} (\widehat{\boxtimes}_{i=1}^r \pi_1(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i)) \varepsilon^{-1} \circ \theta^P\big)^{\mathbb{Q}_p-\mathrm{an}} \supset \pi_{\mathrm{alg}}(\phi, \mathbf{h}). \tag{3.12}$$

Lemma 3.9. *For $i = 1, \cdots, n-1$, $\sigma \in \Sigma_K$ and $I \subset \{1, \cdots, n\}$, $\#I = i$, $\mathscr{C}(I, s_{i,\sigma})$ appears as an irreducible constituent of $\big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)} (\widehat{\boxtimes}_{i=1}^r \pi_1(\phi_{\mathscr{F}_P,i}, \boldsymbol{h}^i)) \varepsilon^{-1} \circ \theta^P\big)^{\mathbb{Q}_p-\mathrm{an}}$ if and only if one of the following conditions holds:*

(1) there exists $k \in \{1, \cdots, r\}$ such that $(n_1 + \cdots + n_{k-1}) + 1 \leq i \leq (n_1 + \cdots + n_k) - 1$ and $\{j \mid \phi_j \in \mathrm{Fil}_{\mathscr{F}_P,k-1}\} \subset I \subset \{j \mid \phi_j \in \mathrm{Fil}_{\mathscr{F}_P,k}\}$,

(2) $i = n_1 + \cdots + n_k$ for some $k = 1, \cdots, r-1$, and $I = \{j \mid \phi_j \in \mathrm{Fil}_k \mathscr{F}_P\}$.

Moreover, each of such constituents has multiplicity one, and lies in the socle of

$$\big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)} (\widehat{\boxtimes}_{i=1}^r \pi_1(\phi_{\mathscr{F}_P,i}, \boldsymbol{h}^i)) \varepsilon^{-1} \circ \theta^P\big)^{\mathbb{Q}_p-\mathrm{an}} / \pi_{\mathrm{alg}}(\phi, \boldsymbol{h}). \tag{3.13}$$

Proof. Using [42], we see the constituents for i in (2) appear with multiplicity one in

$$\big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)} (\boxtimes_{i=1}^r \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i)) \varepsilon^{-1} \circ \theta^P\big)^{\mathbb{Q}_p-\mathrm{an}} \Big(\longhookrightarrow \big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)} (\widehat{\boxtimes}_{i=1}^r \pi_1(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i)) \varepsilon^{-1} \circ \theta^P\big)^{\mathbb{Q}_p-\mathrm{an}}\Big).$$

For i in (1), consider the following subquotient of (3.12):

$$\big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)} \big((\boxtimes_{\substack{i=1,\cdots,r \\ i \neq k}} \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i) \widehat{\boxtimes} \mathscr{C}(I, s_{i,\sigma})_k\big) \varepsilon^{-1} \circ \theta^P\big)^{\mathbb{Q}_p-\mathrm{an}} \tag{3.14}$$

where $\mathscr{C}(I, s_{i,\sigma})_k$ denotes the corresponding representation in the cosocle of $\pi_1(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i)$. Using (3.1) for $\mathscr{C}(I, s_{i,\sigma})_k$, the transitivity of parabolic inductions, [42, Thm.] and [41, Thm. 1], we see that the socle of (3.14) is just $\mathscr{C}(I, s_{i,\sigma})$ with multiplicity one. It is not difficult to see these give all the $\mathscr{C}(I, s_{i,\sigma})$ appearing in (1), and they all have multiplicity one. It rests to show all these constituents lie in the socle of (3.13). By the definition of $\pi_1(\phi_{\mathscr{F},i}, \mathbf{h}^i)$ (and the transitivity of parabolic induction), it is easy to see all such constituents come from certain principal series. We can then use Proposition 3.1 to conclude. $\square$

Denote by $S_{\mathscr{F}_P}$ the subset of the constituents $\mathscr{C}(I, s_{i,\sigma})$, those that satisfy one of the conditions in Lemma 3.9. Then $\big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)} (\widehat{\boxtimes}_{i=1}^r \pi_1(\phi_{\mathscr{F}_P,i}, \mathbf{h}^i)) \varepsilon^{-1} \circ \theta^P\big)^{\mathbb{Q}_p-\mathrm{an}}$ contains a unique subrepresentation $\pi_{\mathscr{F}_P}(\phi, \mathbf{h})$ such that $\mathrm{soc}_{\mathrm{GL}_n(K)} \pi_{\mathscr{F}_P}(\phi, \mathbf{h}) \cong \pi_{\mathrm{alg}}(\phi, \mathbf{h})$ and $\pi_{\mathscr{F}_P}(\phi, \mathbf{h}) / \pi_{\mathrm{alg}}(\phi, \mathbf{h}) \cong \oplus_{\mathscr{C} \in S_{\mathscr{F}_P}} \mathscr{C}$. It is easy to see the (tautological injection) $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \hookrightarrow \pi_{\mathscr{F}_P}(\phi, \mathbf{h})$ uniquely extends to $\pi_{\mathscr{F}_P}(\phi, \mathbf{h}) \hookrightarrow \pi_1(\phi, \mathbf{h})$.

Proposition 3.10. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathscr{F}_P}(\phi, \boldsymbol{h})) = n + d_K r + d_K \sum_{i=1}^r (2^{n_i} - 2).$$

And the following push-forward map is injective

$$\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathscr{F}_P}(\phi, \boldsymbol{h})) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})). \tag{3.15}$$

Proof. We have an exact sequence by dévissage

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_{\mathscr{F}_P}(\phi,\mathbf{h})) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \oplus_{\mathscr{C}\notin S_{\mathscr{F}_P}}\mathscr{C}). \quad (3.16)$$

(3.15) follows. By Proposition 3.7 (1), the last map in (3.16) is surjective. The first part follows then by a direct calculation using Proposition 3.7 (1), Lemma 3.4 (1) and Lemma 3.9. □

Set $\mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h}))$ to be the image of (3.15). Note by [24, Lem. 2.26], the following natural map is bijective:

$$\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_{\mathscr{F}_P}(\phi,\mathbf{h})) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}\Big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)}(\widehat{\boxtimes}_{i=1}^r\pi_1(\phi_{\mathscr{F}_P,i},\mathbf{h}^i))\varepsilon^{-1}\circ\theta^P\big)^{\mathbb{Q}_p-\mathrm{an}}\Big). \quad (3.17)$$

By Schraen's spectral sequence [45, Cor. 4.9], there is a bijection

$$\mathrm{Ext}^1_{L_P(K)}\big(\boxtimes_{i=1}^r \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\mathbf{h}^i), \widehat{\boxtimes}_{i=1}^r\pi_1(\phi_{\mathscr{F}_P,i},\mathbf{h}^i)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}\Big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \big(\mathrm{Ind}_{P^-(K)}^{\mathrm{GL}_n(K)}(\widehat{\boxtimes}_{i=1}^r\pi_1(\phi_{\mathscr{F}_P,i},\mathbf{h}^i))\varepsilon^{-1}\circ\theta^P\big)^{\mathbb{Q}_p-\mathrm{an}}\Big).$$

Using (3.17) and (3.15) and the natural map

$$\prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(K)}\big(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\mathbf{h}^i), \pi_1(\phi_{\mathscr{F}_P,i},\mathbf{h}^i)\big) \to \mathrm{Ext}^1_{L_P(K)}\big(\boxtimes_{i=1}^r \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\mathbf{h}^i), \widehat{\boxtimes}_{i=1}^r\pi_1(\phi_{\mathscr{F}_P,i},\mathbf{h}^i)\big) \quad (3.18)$$

sending $(\widetilde{\pi}_i)$ to the completed tensor product $\widehat{\boxtimes}_{i=1}^r\widetilde{\pi}_i$ over $E[\epsilon]/\epsilon^2$, we finally obtain a map

$$\zeta_{\mathscr{F}_P}: \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(K)}\big(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\mathbf{h}^i), \pi_1(\phi_{\mathscr{F}_P,i},\mathbf{h}^i)\big) \longrightarrow \mathrm{Ext}^1_{\mathscr{F}_P}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})\big). \quad (3.19)$$

For $w\in S_n$, let $\mathscr{T}_w$ be the B-filtration of $\oplus_{i=1}^n\phi_i$ associated to w. Suppose $\mathscr{T}_w$ is compatible with $\mathscr{F}_P$. It is clear that $\mathrm{PS}_1(w(\phi),\mathbf{h})$ is a subrepresentation of $\pi_{\mathscr{F}_P}(\phi,\mathbf{h})$ (e.g. by comparing constituents and using Lemma 3.4 (1)), hence (by dévissage) $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})) \hookrightarrow \mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h}))$.

Proposition 3.11. *The map $\zeta_{\mathscr{F}_P}$ is bijective. Moreover, for any w such that the associated B-filtration $\mathscr{T}_w$ is compatible with $\mathscr{F}_P$, the following diagram commutes*

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Hom}(T(K)\cap L_{P,i}(K), E) & \xrightarrow{\sim} & \mathrm{Hom}(T(K), E) \\ \sim\downarrow & & \sim\downarrow \\ \prod_{i=1}^r \mathrm{Ext}^1_{\mathscr{T}_{w,i}}\big(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i), \pi_1(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i)\big) & \longrightarrow & \mathrm{Ext}^1_w\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}), \pi_1(\phi,\boldsymbol{h})\big) \end{array} \quad (3.20)$$

where $\mathscr{T}_{w,i}$ is the induced $B\cap L_P$-filtration on $\mathrm{gr}_i\,\mathscr{F}_P$.

Proof. The commutativity of the diagram follows by definition and the transitivity of the parabolic induction. By similar arguments as in the proof of Proposition 3.7, one sees

$$\bigoplus_{\mathscr{T}_w \text{ compatible with } \mathscr{F}_P} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})) \longrightarrow \mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})) \quad (3.21)$$

is surjective, hence so is $\zeta_{\mathscr{F}_P}$. By comparing dimensions, $\zeta_{\mathscr{F}_P}$ is bijective. □

Remark 3.12. *We see the map (3.18) is actually bijective.*

Denote by $\mathrm{Ext}^1_{\mathscr{F}_P,g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ the image of $\prod_{i=1}^r \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\mathbf{h}^i),\pi_1(\phi_{\mathscr{F}_p,i},\mathbf{h}^i))$ under $\zeta_{\mathscr{F}_P}$. By Proposition 3.11, and (3.11), the following diagram commutes (assuming $\mathscr{T}_{w_i}$ compatible with $\mathscr{F}_P$):

$$\begin{array}{ccc} \mathrm{Hom}_{P,g'}(T(K),E) & \xrightarrow[\sim]{\zeta_{w_1}} & \mathrm{Ext}^1_{\mathscr{F}_P,g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \\ {\scriptstyle w_2w_1^{-1}}\big\downarrow\wr & & \big\| \\ \mathrm{Hom}_{P,g'}(T(K),E) & \xrightarrow[\sim]{\zeta_{w_2}} & \mathrm{Ext}^1_{\mathscr{F}_P,g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \end{array} \tag{3.22}$$

We finally discuss some intertwining properties related to § 2.4. Let $\phi^1 := \phi_1 \boxtimes \cdots \boxtimes \phi_{n-1} : T_{n-1}(K) \to E^\times$, $\mathbf{h}^1 := (\mathbf{h}_1,\cdots,\mathbf{h}_{n-1})$ and $\mathbf{h}^2 := (\mathbf{h}_2,\cdots,\mathbf{h}_n)$ which are dominant weights of $\mathfrak{t}_{n-1,\Sigma_K}$. We have locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_{n-1}(K)$:

$$\pi_{\mathrm{alg}}(\phi^1,\mathbf{h}^1)\subset\pi_1(\phi^1,\mathbf{h}^1),\ \pi_{\mathrm{alg}}(\phi^2,\mathbf{h}^2)\subset\pi_1(\phi^1,\mathbf{h}^2),$$

and parabolic inductions

$$\big(\mathrm{Ind}_{P_1^-(K)}^{\mathrm{GL}_n(K)}(\pi_1(\phi^1,\mathbf{h}^1)\otimes\varepsilon)\boxtimes\phi_n z^{\mathbf{h}_n}\big)^{\mathbb{Q}_p-\mathrm{an}} \text{ and } \big(\mathrm{Ind}_{P_2^-(K)}^{\mathrm{GL}_n(K)}\phi_n z^{\mathbf{h}_1}\varepsilon^{n-1}\boxtimes\pi_1(\phi^1,\mathbf{h}^2)\big)^{\mathbb{Q}_p-\mathrm{an}}.$$

Let $\mathscr{F}$ be the filtration $\oplus_{i=1}^{n-1}\phi_i\subset\oplus_{i=1}^n\phi_i$ and $\mathscr{G}$ be the filtration $\phi_n\subset\oplus_{i=1}^n\phi_i$. By Lemma 3.9, $C(I,s_{i,\sigma})$ appears in $\pi_{\mathscr{F}}(\phi,\mathbf{h})$ (resp. in $\pi_{\mathscr{G}}(\phi,\mathbf{h})$) if and only if $i=1,\cdots,n-1$, $\sigma\in\Sigma_K$ and $I\subset\{1,\cdots,n-1\}$, $\#I=i$ (resp. $I=I_1\cup\{n\}$ with $I_1\subset\{1,\cdots,n-1\}$ and $\#I_1=i-1$). In particular,

$$\big(\pi_{\mathscr{F}}(\phi,\mathbf{h})/\pi_{\mathrm{alg}}(\phi,\mathbf{h})\big)\cap\big(\pi_{\mathscr{G}}(\phi,\mathbf{h})/\pi_{\mathrm{alg}}(\phi,\mathbf{h})\big)=0.$$

The following proposition is straightforward (where the exactness of the last sequence follows by comparing dimensions)

Proposition 3.13. *There is a natural exact sequence*

$$0\longrightarrow\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\longrightarrow\pi_{\mathscr{F}}(\phi,\boldsymbol{h})\oplus\pi_{\mathscr{G}}(\phi,\boldsymbol{h})\longrightarrow\pi_1(\phi,\boldsymbol{h})\longrightarrow 0.$$

Consequently, we have a natural exact sequence

$$\begin{aligned}0\longrightarrow \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))\longrightarrow \\ \mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))\oplus\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) \\ \longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))\longrightarrow 0.\end{aligned}$$

Remark 3.14. *Applying Proposition 3.11 and Proposition 3.10 to $\mathscr{F}$ and $\mathscr{G}$, we have*

$$\zeta_{\mathscr{F}}:\mathrm{Ext}^1_{\mathrm{GL}_{n-1}(K)}(\pi_{\mathrm{alg}}(\phi^1,\boldsymbol{h}^1),\pi_1(\phi^1,\boldsymbol{h}^1))\times\mathrm{Hom}(K^\times,E)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})),$$

$$\zeta_{\mathscr{G}}:\mathrm{Ext}^1_{\mathrm{GL}_{n-1}(K)}(\pi_{\mathrm{alg}}(\phi^1,\boldsymbol{h}^2),\pi_1(\phi^1,\boldsymbol{h}^2))\times\mathrm{Hom}(K^\times,E)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})).$$

and $\dim_E\mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))=\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))=n+2^{n-1}d_K.$

3.1.4 Locally σ-analytic parabolic inductions

Let $\sigma\in\Sigma_K$. Let λ_σ be the σ-component of λ, and $\lambda^\sigma:=(\lambda_\tau)_{\tau\neq\sigma}$. We also view them as weights of $\mathfrak{t}_{\Sigma_K}$ in the obvious way. For $i=1,\cdots,n-1$, $I\subset\{1,\cdots,n\}$, $\#I=i$, let $w\in S_n$ such that $w(\{1,\cdots,i\})=I$. We have

$$\mathscr{C}(I,s_{i,\sigma})\cong\mathcal{F}_{B^-}^{\mathrm{GL}_n}(L^-(-s_{i,\sigma}\cdot\lambda),w(\phi)\eta))\cong\mathcal{F}_{B^-}^{\mathrm{GL}_n}(L^-(-s_{i,\sigma}\cdot\lambda_\sigma),w(\phi)\eta))\otimes_E L(\lambda^\sigma).$$

Note we have $\mathcal{F}_{B^-}^{\mathrm{GL}_n}(L_\sigma^-(-s_{i,\sigma}\cdot\lambda_\sigma),w(\phi)\eta))\hookrightarrow(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)}w(\phi)\eta z^{s_{i,\sigma}\cdot\lambda_\sigma})^{\sigma-\mathrm{an}}$, which is hence locally σ-analytic, i.e. is locally $\mathbb{Q}_p$-analytic and the induced $\mathfrak{gl}_{n,\Sigma_K}$-action factors through $\mathfrak{gl}_{n,\sigma}$. Let $\pi_{1,\sigma}(\phi,\mathbf{h})$ be the subrepresentation of $\pi_1(\phi,\mathbf{h})$ given by the extension of $\oplus_{\substack{i=1,\cdots,n-1\\ I\subset\{1,\cdots,n\},\#I=i}}\mathscr{C}(I,s_{i,\sigma})$ by $\pi_{\mathrm{alg}}(\phi,\lambda)$. Similarly, for $w\in S_n$, we let $\mathrm{PS}_{1,\sigma}(w(\phi),\mathbf{h})\subset\mathrm{PS}_1(w(\phi),\mathbf{h})$ be the subrepresentation consisting of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ and $\mathscr{C}(w,s_{i,\sigma})$ for $i=1,\cdots,n-1$. It is easy to see

$$\mathrm{PS}_{1,\sigma}(w(\phi),\mathbf{h})=\mathrm{PS}_1(w(\phi),\mathbf{h})\cap(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)}w(\phi)\eta z^{\lambda_\sigma})^{\sigma-\mathrm{an}}\otimes_E L(\lambda^\sigma)\longrightarrow\mathrm{PS}(w(\phi),\mathbf{h}).$$

Moreover, $\pi_{1,\sigma}(\phi,\mathbf{h})$ is the unique quotient of $\oplus^{w\in S_n}_{\pi_{\mathrm{alg}}(\phi,\lambda)}\mathrm{PS}_{1,\sigma}(w(\phi),\mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. In particular, $\pi_{1,\sigma}(\phi,\mathbf{h})$ is $\mathrm{U}(\mathfrak{g}_{\Sigma_K\setminus\{\sigma\}})$-finite, i.e. $\mathrm{U}(\mathfrak{g}_{\Sigma_K\setminus\{\sigma\}})v$ is finite dimensional for any $v\in\pi_{1,\sigma}(\phi,\mathbf{h})$. In fact, $\pi_{1,\sigma}(\phi,\mathbf{h})$ is no other than the maximal $\mathrm{U}(\mathfrak{g}_{\Sigma_K\setminus\{\sigma\}})$-finite subrepresentation of $\pi_1(\phi,\mathbf{h})$.

For $\mathrm{U}(\mathfrak{g}_{\Sigma_K\setminus\{\sigma\}})$-finite representations V, W, we denote by $\mathrm{Ext}^1_\sigma(V,W)\subset\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(V,W)$ the subspace of extensions, those that are $\mathrm{U}(\mathfrak{g}_{\Sigma_K\setminus\{\sigma\}})$-finite. Let $\mathrm{Hom}_{\sigma,g'}(T(K),E):=\mathrm{Hom}_{g'}(T(K),E)\cap\mathrm{Hom}_\sigma(T(K),E)$ (where $\mathrm{Hom}_\sigma(T(K),E)$ is the subspace of locally σ-analytic characters).

Lemma 3.15. *We have* $\dim_E\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))=n+1$, *and (3.2) induces an isomorphism*

$$\mathrm{Hom}_{\sigma,g'}(T(K),E)\xrightarrow{\sim}\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})).$$

Proof. As $\mathrm{Ext}^1_{\mathrm{GL}_n(K),Z}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))\subset\mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$ (by (3.3)), we have an exact sequence (similarly as in [10, Lem. 3.16])

$$0\longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_n(K),Z}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))\longrightarrow\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))\longrightarrow\mathrm{Hom}_\sigma(Z(K),E)\longrightarrow 0.$$

The first part follows. The second isomorphism follows by comparing dimensions. □

Proposition 3.16. *Let $w\in S_n$, the map (3.4) induces an isomorphism*

$$\mathrm{Hom}_\sigma(T(K),E)\xrightarrow{\sim}\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathrm{PS}_{1,\sigma}(w(\phi),\boldsymbol{h})).\tag{3.23}$$

Proof. For $\psi\in\mathrm{Hom}_\sigma(T(K),E)$, by similar arguments as in the proof of Proposition 3.3, we see $I_{B^-(K)}^{\mathrm{GL}_n(K)}\big(w(\phi)\eta z^\lambda(1+\psi\epsilon)\big)$ is a subrepresentation of

$$(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)}w(\phi)\eta z^{\lambda_\sigma}(1+\psi\epsilon))^{\sigma-\mathrm{an}}\otimes_E L(\lambda^\sigma),$$

hence is $\mathrm{U}(\mathfrak{g}_{\Sigma_K\setminus\{\sigma\}})$-finite. Together with Remark 3.6, we obtain the map (3.23). By the above lemma, Lemma 3.4 (1), and an easy dévissage

$$0\longrightarrow\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))\longrightarrow\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_{1,\sigma}(w(\phi),\mathbf{h}))\\ \longrightarrow\oplus_{i=1}^{n-1}\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(w(\phi),s_{i,\sigma}))$$

we see $\dim_E\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_{1,\sigma}(w(\phi),\mathbf{h}))\leq 2n$. By comparing dimensions (noting (3.23) is clearly injective), (3.23) is bijective. □

Remark 3.17. *By the proof and comparing dimensions (using Lemma 3.4 (1)), we see*

$$\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(w(\phi),s_{i,\sigma})) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(w(\phi),s_{i,\sigma})).$$

Denote by $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ (resp. $\mathrm{Ext}^1_{\sigma,g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$) the image of $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{1,\sigma}(\phi,\mathbf{h}))$ (resp. $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$) via the (injective) push-forward map. It is easy to see

$$\mathrm{Ext}^1_{\sigma,g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))=\mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\cap \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})).$$

Proposition 3.18. *(1) We have an exact sequence*

$$0\longrightarrow \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))\longrightarrow \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))$$
$$\longrightarrow \oplus_{\substack{i=1,\cdots,n-1,\sigma\in\Sigma_K\\ I\subset\{1,\cdots,n-1\},\#I=i}} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(I,s_{i,\sigma}))\longrightarrow 0. \quad (3.24)$$

And $\dim_E \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))=n+2^n-1$.

(2) The map (3.7) induces a surjection

$$t_{\phi,\boldsymbol{h}}:\oplus_{w\in S_n}\mathrm{Hom}_\sigma(T(K),E)\twoheadrightarrow \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})).$$

(3) The following map is surjective

$$\oplus_{\tau\in\Sigma_K}\mathrm{Ext}^1_\tau(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})),$$

and induces an isomorphism

$$\oplus_{\tau\in\Sigma_K}\big(\mathrm{Ext}^1_\tau(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))/\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))\big)$$
$$\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))/\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})).$$

Proof. (1) The left exactness is clear. For $w\in S_n$, we have a commutative diagram

$$\begin{array}{ccc}\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_{1,\sigma}(\phi,\mathbf{h})) & \longrightarrow & \oplus_{i=1,\cdots,n-1}\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(w(\phi),s_{i,\sigma}))\\ \downarrow & & \|\\ \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{1,\sigma}(\phi,\mathbf{h})) & \longrightarrow & \oplus_{i=1,\cdots,n-1}\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(w(\phi),s_{i,\sigma})).\end{array}$$

The top map is surjective by the proof of Proposition 3.16 (and Remark 3.17), so is the bottom map. The first part follows. The second part is clear.

(2) follows by the same argument as in the proof of Proposition 3.7 (2).

(3) The first part follows easily by comparing the exact sequences (3.6) and (3.24). The second part follows by comparing dimensions (the both sides having dimension $(2^n-1)d_K$). ◻

Now let P be a standard parabolic subgroup of GL_n, and $\mathscr{F}_P$ be a P-filtration on ϕ. We use the notation in § 3.1.3. Let $\pi_{\mathscr{F}_P,\sigma}(\phi,\mathbf{h}):=\pi_{\mathscr{F}_P}(\phi,\mathbf{h})\cap\pi_{1,\sigma}(\phi,\mathbf{h})$, which is the maximal $\mathrm{U}(\mathfrak{g}_{\Sigma_K\setminus\{\sigma\}})$-finite subrepresentation of $\pi_{\mathscr{F}_P}(\phi,\mathbf{h})$. Then $\pi_{\mathscr{F}_P,\sigma}(\phi,\mathbf{h})$ is an extension of the direct sum of $\mathscr{C}(I,s_{i,\sigma})\in S_{\mathscr{F}_P}$ (for the fixed σ) by $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. We denote by $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ the image of $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathscr{F}_P,\sigma}(\phi,\mathbf{h}))$ via the (injective) push-forward map. As previously, we also write $\mathrm{Ext}^1_{\sigma,w}$ for $\mathrm{Ext}^1_{\sigma,\mathscr{T}_w}$. One easily sees

$$\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))=\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\cap\mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})).$$

Proposition 3.19. *(1) We have* $\dim_E \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) = n + r + \sum_{i=1}^r (2^{n_i} - 2)$.

(2) (3.19) induces an isomorphism

$$\prod_{i=1}^r \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i),\pi_1(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i)) \xrightarrow{\sim} \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\lambda)). \tag{3.25}$$

Moreover, for any w such that the associated B-filtration $\mathscr{T}_w$ is compatible with $\mathscr{F}_P$, the following diagram commutes

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Hom}_\sigma(T(K)\cap L_{P,i}(K),E) & \xrightarrow{\sim} & \mathrm{Hom}_\sigma(T(K),E) \\ \sim\downarrow & & \sim\downarrow \\ \prod_{i=1}^r \mathrm{Ext}^1_{\sigma,\mathscr{T}_{w,i}}\left(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i),\pi_{1,\sigma}(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i)\right) & \longrightarrow & \mathrm{Ext}^1_{\sigma,w}\left(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{1,\sigma}(\phi,\boldsymbol{h})\right) \end{array}$$

where $\mathscr{T}_{w,i}$ is the induced $B\cap L_P$-filtration on $\mathrm{gr}_i\,\mathscr{F}_P$.

Proof. By dévissage and Lemma 3.9, we have $\dim_E \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \leq n+r+\sum_{i=1}^r(2^{n_i}-2)$. By similar arguments as in the proof of Proposition 3.16, we see (3.19) restricts to an injective map as in (3.25). By comparing dimensions, (3.25) is bijective and (1) follows. The second part of (2) follows easily from (3.20). □

Finally, let $\mathscr{F}$ and $\mathscr{G}$ be as in Proposition 3.13. We have

Proposition 3.20. *There is a natural exact sequence*

$$0 \longrightarrow \pi_{\mathrm{alg}}(\phi,\boldsymbol{h}) \longrightarrow \pi_{\mathscr{F},\sigma}(\phi,\boldsymbol{h}) \oplus \pi_{\mathscr{G},\sigma}(\phi,\boldsymbol{h}) \longrightarrow \pi_{1,\sigma}(\phi,\boldsymbol{h}) \longrightarrow 0.$$

Consequently, we have a natural exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_{\sigma,g'}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) \longrightarrow & \\ \mathrm{Ext}^1_{\sigma,\mathscr{F}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) \oplus \mathrm{Ext}^1_{\sigma,\mathscr{G}}&(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) \\ &\longrightarrow \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) \longrightarrow 0. \end{aligned} \tag{3.26}$$

Proof. We only need to show the surjectivity of the second last map in (3.26). But it follows by comparing dimensions. □

3.2 A p-adic Langlands correspondence in the crystabelline case

3.2.1 Construction and properties

In this section, we associate to $D \in \Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$ a locally $\mathbb{Q}_p$-analytic representation $\pi_{\min}(D)$ of $\mathrm{GL}_n(K)$ over E, which determines the Hodge parameters of D, discussed in § 2.2 (hence determines D when $K=\mathbb{Q}_p$).

Consider the following composition

$$\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow[\sim]{(\kappa_w)} \oplus_{w\in S_n}\mathrm{Hom}(T(K),E) \xrightarrow{t_{\phi,\mathbf{h}}} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})). \tag{3.27}$$

Theorem 3.21. *The natural map* $\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(D,D)\twoheadrightarrow \overline{\mathrm{Ext}}^1(D,D)$ *factors through (3.27), i.e. there exists a unique map*

$$t_D:\mathrm{Ext}^1_{\mathrm{GL}_n(K)}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\twoheadrightarrow \overline{\mathrm{Ext}}^1(D,D)$$

such that $\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(D,D)\twoheadrightarrow \overline{\mathrm{Ext}}^1(D,D)$ *is equal to* t_D *composed with (3.27).*

Proof. We prove the theorem by induction on n. It is trivial for $n=1$. Suppose it holds for $n-1$. As in § 2.4, let $D_1\in\Phi\Gamma_{\mathrm{nc}}(\phi^1,\mathbf{h}^1)$ (resp. $C_1\in\Phi\Gamma_{\mathrm{nc}}(\phi^1,\mathbf{h}^2)$) be the corresponding saturate (φ,Γ)-submodule (resp. quotient) of D, and $\mathscr{F}$, $\mathscr{G}$ be the associated filtrations on D. For $w\in S_{n-1}$, the following diagram commutes (cf. (2.9)):

$$\begin{array}{ccc}
\mathrm{Hom}(T(K),E) & \xrightarrow{\sim} & \mathrm{Hom}(T_1(K),E)\times\mathrm{Hom}(K^\times,E)\\
{\scriptstyle\kappa_w}\uparrow\wr & & {\scriptstyle(\kappa_w,\mathrm{id})}\uparrow\wr\\
\overline{\mathrm{Ext}}^1_w(D,D) & \xrightarrow{\sim} & \overline{\mathrm{Ext}}^1_w(D_1,D_1)\times\mathrm{Hom}(K^\times,E)\\
\downarrow & & \downarrow\\
\overline{\mathrm{Ext}}^1_{\mathscr{F}}(D,D) & \xrightarrow{\sim} & \overline{\mathrm{Ext}}^1(D_1,D_1)\times\mathrm{Hom}(K^\times,E).
\end{array}$$

By induction hypothesis, the map $\oplus_{w\in S_{n-1}}\overline{\mathrm{Ext}}^1_w(D_1,D_1)\twoheadrightarrow\overline{\mathrm{Ext}}^1(D_1,D_1)$ factors through the following map (defined similarly as in (3.27))

$$t_{\phi^1,\mathbf{h}^1}:\oplus_{w\in S_{n-1}}\overline{\mathrm{Ext}}^1_w(D_1,D_1)\twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_{n-1}(K)}(\pi_{\mathrm{alg}}(\phi^1,\mathbf{h}^1),\pi_1(\phi^1,\mathbf{h}^1)).$$

Using Proposition 3.11, we deduce that $\oplus_{w\in S_{n-1}}\overline{\mathrm{Ext}}^1_w(D,D)\twoheadrightarrow\overline{\mathrm{Ext}}^1_{\mathscr{F}}(D,D)$ $(\hookrightarrow\overline{\mathrm{Ext}}^1(D,D))$ factors through $\mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$. Let $S'_{n-1}:=\{w\in S_n\mid w(n)=1\}$, that is a subset of S_n of cardinality $(n-1)!$. By a similar discussion with D_1 replaced by C_1, the map $\oplus_{w\in S'_{n-1}}\overline{\mathrm{Ext}}^1_w(D,D)\twoheadrightarrow\overline{\mathrm{Ext}}^1_{\mathscr{G}}(D,D)\hookrightarrow\overline{\mathrm{Ext}}^1(D,D)$ factors through $\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$. By (3.11) and (2.6) (with g replaced by g'), it is not difficult to see the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) & \longrightarrow & \mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\\
\downarrow & & \downarrow\\
\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) & \longrightarrow & \overline{\mathrm{Ext}}^1(D,D).
\end{array}$$

Hence by Proposition 3.13, the composition

$$\mathrm{Ext}^1_{\mathscr{F}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\oplus\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\to\overline{\mathrm{Ext}}^1_{\mathscr{F}}(D,D)\oplus\overline{\mathrm{Ext}}^1_{\mathscr{G}}(D,D)\to\overline{\mathrm{Ext}}^1(D,D).$$

factors though a map

$$t_D:\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\twoheadrightarrow\overline{\mathrm{Ext}}^1(D,D). \tag{3.28}$$

Next, we show t_D satisfies the property in the theorem. By construction, the map

$$\oplus_{w\in S_{n-1}\cup S'_{n-1}}\overline{\mathrm{Ext}}^1_w(D,D)\longrightarrow\overline{\mathrm{Ext}}^1(D,D)$$

factors through t_D. It suffices to show for the other $w \in S_n$, $\overline{\mathrm{Ext}}^1_w(D,D) \to \overline{\mathrm{Ext}}^1(D,D)$ also factors as

$$\overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow[\sim]{\kappa_w} \mathrm{Hom}(T(K),E) \xrightarrow{\zeta_w} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \xrightarrow{(3.28)} \overline{\mathrm{Ext}}^1(D,D). \quad (3.29)$$

Suppose hence $w(n)=i$ with $1<i<n$. We have

$$\mathrm{Hom}(T(K),E) \cong \oplus_{j=1}^{n-1} \mathrm{Hom}(Z_{s_j}(K),E) \oplus \mathrm{Hom}(Z(K),E), \quad (3.30)$$

where $Z_{s_j} \subset T$ is the centre of the Levi subgroup (containing T) of the standard maximal parabolic subgroup P_{s_j} such that $s_j \notin \mathscr{W}_{P_{s_j}}$. For any $j=1,\cdots,n-1$, $\kappa_w^{-1}(\mathrm{Hom}(L_{s_j}(K),E)) \subset \overline{\mathrm{Ext}}^1_{\mathscr{F}_{P_{s_j}},g'}(D,D)$ (cf. Corollary 2.16), where $\mathscr{F}_{P_{s_j}}$ is the P_{s_j}-filtration associated to the B-filtration $\mathscr{T}_w$ (such that $\mathscr{T}_w$ is compatible with $\mathscr{F}_{P_{s_j}}$). Let w_j be an element in the Weyl group of $L_{P_{s_j}}$ such that $w_j(i)=1$ or $w_j(i)=n$ (whose existence is clear). By Corollary 2.16 (2) and (3.22), we have a commutative diagram

$$\begin{array}{ccccc}
\overline{\mathrm{Ext}}^1_{\mathscr{F}_{P_{s_j}},g'}(D,D) & \xrightarrow{\kappa_w} & \mathrm{Hom}_{P_{s_j},g'}(T(K),E) & \xrightarrow{\zeta_w} & \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \\
\| & & w_j \downarrow \sim & & \| \\
\overline{\mathrm{Ext}}^1_{\mathscr{F}_{P_{s_j}},g'}(D,D) & \xrightarrow{\kappa_{w_j w}} & \mathrm{Hom}_{P_{s_j},g'}(T(K),E) & \xrightarrow{\zeta_{w_j w}} & \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})).
\end{array}$$

It is clear that $w_j w \in S_{n-1} \cup S'_{n-1}$, hence the map $\overline{\mathrm{Ext}}^1_{w_j w}(D,D) \to \overline{\mathrm{Ext}}^1(D,D)$ is equal to $t_D \circ (\zeta_{w_j w} \circ \kappa_{w_j w})$. In particular, its restriction on $\overline{\mathrm{Ext}}^1_{\mathscr{F}_{P_{s_j}},g'}(D,D)$ is equal to $t_D \circ (\zeta_{w_j w} \circ \kappa_{w_j w}) = t_D \circ (\zeta_w \circ \kappa_w)$ by the above commutative diagram. As $\overline{\mathrm{Ext}}^1_w(D,D)$ is spanned by $\overline{\mathrm{Ext}}^1_{\mathscr{F}_{P_{s_j}},g'}(D,D)$ and $\mathrm{Hom}(Z(K),E)$ (e.g. using (3.30)). We obtain the factorisation as in (3.29). This concludes the proof. ∎

Remark 3.22. *(1) When $n \geq 2$, $\mathrm{Ext}^1(D,D) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1(D,D)$ and t_D is bijective.*

(2) The same argument holds with $\pi_1(\phi,\boldsymbol{h})$ replaced by $\pi(\phi,\boldsymbol{h})$ (with the same t_D under the isomorphism (3.10)).

Let $\mathcal{L}(D) := \mathrm{Ker}(t_D)$. By comparing dimensions (Proposition 2.11, Proposition 3.7 (1)), we have

$$\dim_E \mathcal{L}(D) = (2^n - \frac{n(n+1)}{2} - 1)d_K.$$

The following lemma is clear.

Lemma 3.23. *For any $w \in S_n$, $\mathcal{L}(D) \cap \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))=0$.*

Let $\pi_{\min}(D)$ (resp. $\pi_{\mathrm{fs}}(D)$) be the extension of $\mathcal{L}(D)\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h})$ ($\cong \pi_{\mathrm{alg}}(\phi,\mathbf{h})^{\oplus(2^n-\frac{n(n+1)}{2}-1)d_K}$) by $\pi_1(\phi,\mathbf{h})$ (resp. $\pi(\phi,\mathbf{h})$) associated to $\mathcal{L}(D)$ (see also Remark 3.22 (2)). We have

$$\pi_{\mathrm{fs}}(D) \cong \pi_{\min}(D) \oplus_{\pi_1(\phi,\mathbf{h})} \pi(\phi,\mathbf{h}). \quad (3.31)$$

In the sequel, we will mainly work with $\pi_{\min}(D)$, noting that most of the statements generalize to $\pi_{\mathrm{fs}}(D)$ without effort. We have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\lambda),\mathcal{L}(D)\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h})) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\lambda),\pi_1(\phi,\mathbf{h})) \\ \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\lambda),\pi_{\min}(D)).$$

By Lemma 3.4 (2), one sees the last map is surjective. For a P-filtration $\mathscr{F}_P$ on D, we denote by $\mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\lambda),\pi_{\min}(D))$ the image of $\mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\lambda),\pi_1(\phi,\mathbf{h}))$. The following proposition is then a direct consequence of Theorem 3.21.

Corollary 3.24. *Let $D\in\Phi\Gamma_{\mathrm{nc}}(\phi,\boldsymbol{h})$. The representation $\pi_{\min}(D)$ is the unique extension of $\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})^{\oplus(2^n-\frac{n(n+1)}{2}-1)d_K}$ by $\pi_1(\phi,\boldsymbol{h})$ satisfying the following properties:*

- *We have $\mathrm{soc}_{\mathrm{GL}_n(K)}\pi_{\min}(D)\cong\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})$, and*

$$\mathrm{soc}_{\mathrm{GL}_n(K)}\big(\pi_{\min}(D)/\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big)\cong\mathrm{soc}_{\mathrm{GL}_n(K)}\big(\pi_1(\phi,\boldsymbol{h})/\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big).$$

- *There is a bijection*

$$t_D:\overline{\mathrm{Ext}}^1(D,D)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\min}(D))$$

which is compatible with trianguline deformations: for any $w\in S_n$, we have

$$\begin{array}{ccc}\overline{\mathrm{Ext}}^1_w(D,D) & \xrightarrow{t_D} & \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\min}(D))\\ {\scriptstyle\kappa_w}\downarrow\wr & & \uparrow{\scriptstyle\zeta_w}\\ \mathrm{Hom}(T(K),E) & \xrightarrow{\mathrm{id}} & \mathrm{Hom}(T(K),E).\end{array}$$

Denote by $\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\min}(D))$ the image of $\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$. Using Lemma 3.23, we have:

Corollary 3.25. *The map t_D induces isomorphisms*

$$\overline{\mathrm{Ext}}^1_g(D,D)\xrightarrow{\sim}\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\min}(D))$$

and $\overline{\mathrm{Ext}}^1_w(D,D)\xrightarrow{\sim}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\min}(D))$ for all $w\in S_n$.

Let χ_D be the character $z^{\sum_{i=1}^n\lambda_n}|\cdot|_K^{-\frac{n(n-1)}{2}}\prod_{i=1}^n\phi_i$ of $K^\times$. We have $\wedge^n D\cong\mathcal{R}_{K,E}(\chi_D\varepsilon^{\frac{n(n-1)}{2}})$. Let ξ_λ be the central character of $\mathrm{U}(\mathfrak{gl}_{n,\Sigma_K})$ associated to the weight λ.

Proposition 3.26. *The representation $\pi_{\min}(D)$ has central character χ_D and infinitesimal character ξ_λ.*

Proof. We only prove the statement for the infinitesimal character, the central character being similar. Let $\mathcal{Z}_K$ be the centre of $\mathrm{U}(\mathfrak{gl}_{n,\Sigma_K})$. Recall we have the Harish-Chandra isomorphism $\mathrm{HC}:\mathcal{Z}_K\xrightarrow{\sim}\mathrm{U}(\mathfrak{t}_{\Sigma_K})^{\mathscr{W}_{n,K}}$, where $\mathscr{W}_{n,K}$ is the Weyl group of $\mathrm{Res}^K_{\mathbb{Q}_p}\mathrm{GL}_n$, isomorphic to $S_n^{d_K}$, and where we normalize the map such that a weight μ of $\mathfrak{t}_{\Sigma_K}$ corresponds to $\xi_{\mu+\theta_K}$. In particular, the weight $\mathbf{h}$ corresponds to χ_λ. Let X_{ξ_λ} (resp. $X_{\mathbf{h}}$) be the tangent space of $\mathcal{Z}_K$ (resp. $\mathrm{U}(\mathfrak{t}_{\Sigma_K})$) at ξ_λ (resp. at $\mathbf{h}$). The map HC induces a bijection $\mathrm{HC}:X_{\mathbf{h}}\xrightarrow{\sim}X_{\xi_\lambda}$.

For $\widetilde{D}\in\mathrm{Ext}^1(D,D)$, the Sen weights of $\widetilde{D}$ (over $E[\epsilon]/\epsilon^2$) have the form $(h_{i,\sigma}+a_{i,\sigma}\epsilon)_{\substack{\sigma\in\Sigma_K\\ i=1,\cdots,n}}$. We obtain hence an E-linear map $\mathrm{Ext}^1(D,D)\to X_{\mathbf{h}}$, $\widetilde{D}\mapsto(a_{i,\sigma})$. The map is trivial on $\mathrm{Ext}^1_g(D,D)$, thus induces an E-linear map

$$d_{\mathrm{Sen}}:\overline{\mathrm{Ext}}^1(D,D)\longrightarrow X_{\mathbf{h}}.$$

By considering the $\mathcal{Z}_K$-action, we have a natural map (for example using similar arguments as in [10, Lem. 3.16])

$$d_{\inf}: \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \longrightarrow X_{\xi_\lambda}.$$

The proposition (for the infinitesimal character) will be a direct consequence of the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) & \xrightarrow{d_{\inf}} & X_{\xi_\lambda} \\ {\scriptstyle t_D}\downarrow & & \downarrow{\scriptstyle \mathrm{HC}^{-1}} \\ \overline{\mathrm{Ext}}^1(D,D) & \xrightarrow{d_{\mathrm{Sen}}} & X_{\mathbf{h}}. \end{array} \tag{3.32}$$

However, for each $w\in S_n$, it is easy to see from the explicit construction that the following diagram commutes

$$\begin{array}{ccc} \overline{\mathrm{Ext}}^1_w(D,D) & \xrightarrow{d_{\mathrm{Sen}}} & X_{\mathbf{h}} \\ \sim\downarrow & & \downarrow{\scriptstyle \mathrm{HC}} \\ \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) & \xrightarrow{d_{\inf}} & X_{\xi_\lambda}. \end{array}$$

From which and Theorem 3.21, the commutativity of (3.32) follows. □

We next discuss the compatibility of t_D (and $\pi_{\min}(D)$) with parabolic inductions. Let $P\supset B$ be a standard parabolic subgroup of GL_n with L_P equal to $\mathrm{diag}(\mathrm{GL}_{n_1},\cdots,\mathrm{GL}_{n_r})$. Let $\mathscr{F}_P$ be a P-filtration of D, $M_i:=\mathrm{gr}_i\,\mathscr{F}_P$, which is a (φ,Γ)-module of rank n_i, for $i=1,\cdots,r$.

Proposition 3.27. *Keep the above notation, t_D restricts to a surjection*

$$t_{D,\mathscr{F}_P}: \mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) \twoheadrightarrow \overline{\mathrm{Ext}}^1_{\mathscr{F}_P}(D,D).$$

Moreover, the following diagram commutes

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(K)}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i),\pi_1(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i)) & \xrightarrow{(t_{M_i})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(M_i,M_i) \\ \sim\downarrow{\scriptstyle (3.19)} & & \sim\downarrow \\ \mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) & \xrightarrow{t_{D,\mathscr{F}_P}} & \overline{\mathrm{Ext}}^1_{\mathscr{F}_P}(D,D). \end{array} \tag{3.33}$$

In particular, the parabolic induction (3.19) induces a natural isomorphism

$$\oplus_{i=1}^r \mathcal{L}(M_i) \xrightarrow{\sim} \mathcal{L}(D)_{\mathscr{F}_P} := \mathcal{L}(D)\cap \mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})). \tag{3.34}$$

Proof. The first part follows from (3.21). The commutativity of the diagram follows from (2.9) and (3.20). □

Remark 3.28. *Let $\pi_{\min,\mathscr{F}_P}(D)\subset \pi_{\min}(D)$ be the extension of $\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})^{\oplus \sum_{i=1}^r (d_K(2^{n_i}-\frac{n_i(n_i+1)}{2}-1))}$ by $\pi_1(\phi,\boldsymbol{h})$ associated to $\mathcal{L}(D)_{\mathscr{F}_P}$. By Proposition 3.27, $\pi_{\min,\mathscr{F}_P}(D)$ is the maximal subrepresentation of $\pi_{\min}(D)$ which comes from the push-forward of extensions of $\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})$ by $\pi_{\mathscr{F}_P}(\phi,\boldsymbol{h})$ via $\pi_{\mathscr{F}_P}(\phi,\boldsymbol{h})\hookrightarrow \pi_1(\phi,\boldsymbol{h})$. We have $I^{\mathrm{GL}_n(K)}_{P^-(K)}(\widehat{\boxtimes}_{i=1}^r \pi(M_i)\otimes_E \varepsilon^{-1}\circ\theta_P)\hookrightarrow \pi_{\min,\mathscr{F}_P}(D)$. Moreover, (3.33) induces a commutative diagram*

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(K)}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i),\pi_1(M_i)) & \xrightarrow[\sim]{(t_{M_i})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(M_i,M_i) \\ \sim\downarrow & & \sim\downarrow \\ \mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\min,\mathscr{F}_P}(D)) & \xrightarrow{t_{D,\mathscr{F}_P}} & \overline{\mathrm{Ext}}^1_{\mathscr{F}_P}(D,D), \end{array}$$

where $\mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\min,\mathscr{F}_P}(D))$ *is the image of* $\mathrm{Ext}^1_{\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))$ *via the push-forward map, and where the left vertical map is obtained in a similar way as (3.19).*

Let $\sigma\in\Sigma_K$. By Proposition 3.18 and Corollary 2.19, (3.27) restricts to a surjection

$$\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_{\sigma,w}(D,D)\twoheadrightarrow \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})). \tag{3.35}$$

We have the following corollary:

Corollary 3.29. *Let* $D\in\Phi\Gamma_{\mathrm{nc}}(\phi,\boldsymbol{h})$, *and* $\sigma\in\Sigma_K$.

(1) The map (2.11) factors through the restriction of t_D*:*

$$t_{D,\sigma}:\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))\twoheadrightarrow\overline{\mathrm{Ext}}^1_\sigma(D,D).$$

(2) Let P *be a standard parabolic subgroup and* $\mathscr{F}_P$ *be a* P*-filtration on* D*. Let* $t_{D,\mathscr{F}_P,\sigma}$ *be the restriction of* $t_{D,\sigma}$ *to* $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))$*, we have a commutative diagram*

$$\begin{array}{ccc}\prod_{i=1}^r\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i),\pi_1(\phi_{\mathscr{F}_P,i},\boldsymbol{h}^i)) & \xrightarrow{(t_{M_i,\sigma})} & \prod_{i=1}^r\overline{\mathrm{Ext}}^1_\sigma(M_i,M_i)\\ \sim\downarrow(3.25) & & \sim\downarrow\\ \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) & \xrightarrow{t_{D,\mathscr{F}_P,\sigma}} & \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_P}(D,D).\end{array}$$

In particular, $\mathrm{Ker}\,t_{D,\mathscr{F}_P,\sigma}\cong\oplus_{i=1}^r\mathrm{Ker}\,t_{M_i,\sigma}$*,*

Proof. (1) follows by Theorem 3.21 (and Corollary 2.40). (2) follows from (3.33). □

Let $\mathcal{L}(D)_\sigma:=\ker t_{D,\sigma}$. It is clear that $\mathcal{L}(D)_\sigma=\mathcal{L}(D)\cap\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$.

Corollary 3.30. *We have* $\oplus_{\sigma\in\Sigma_K}\mathcal{L}(D)_\sigma\xrightarrow{\sim}\mathcal{L}(D)$.

Proof. Consider the induced map

$$\bar{t}_D:\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))/\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\twoheadrightarrow\overline{\mathrm{Ext}}^1(D,D)/\overline{\mathrm{Ext}}^1_g(D,D),$$

$$\bar{t}_{D,\sigma}:\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))/\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\twoheadrightarrow\overline{\mathrm{Ext}}^1_\sigma(D,D)/\overline{\mathrm{Ext}}^1_g(D,D).$$

As $\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\cap\mathcal{L}(D)=0$, we have isomorphisms $\mathcal{L}(D)\xrightarrow{\sim}\mathrm{Ker}\,\bar{t}_D$ and $\mathcal{L}(D)_\sigma\xrightarrow{\sim}\mathrm{Ker}\,\bar{t}_{D,\sigma}$. Using Proposition 3.18 (3), the map $\oplus_{\sigma\in\Sigma_K}\mathrm{Ker}\,\bar{t}_{D,\sigma}\to\mathrm{Ker}\,\bar{t}_D$ is injective. By comparing dimensions, it is actually bijective. We deduce the natural map $\oplus_{\sigma\in\Sigma_K}\mathcal{L}(D)_\sigma\to\mathcal{L}(D)$ is injective. Again by comparing dimensions, it is bijective. □

Using the isomorphism $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{1,\sigma}(\phi,\mathbf{h}))\xrightarrow{\sim}\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$, we view $\mathcal{L}(D)_\sigma$ as subspace of $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{1,\sigma}(\phi,\mathbf{h}))$. Set $\pi_{\min}(D)_\sigma$ to be the extension of $\mathcal{L}(D)_\sigma\otimes_E\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by $\pi_{1,\sigma}(\phi,\mathbf{h})$. We have hence

$$\pi_{\min}(D)\cong\bigoplus^{\sigma\in\Sigma_K}_{\pi_{\mathrm{alg}}(\phi,\mathbf{h})}\pi_{\min}(D)_\sigma.$$

And $\pi_{\min}(D)_\sigma$ is exactly the maximal $\mathrm{U}(\mathfrak{gl}_{n,\Sigma_K\setminus\{\sigma\}})$-finite subrepresentation of $\pi_{\min}(D)$. Let P be a standard parabolic subgroup and $\mathscr{F}_P$ be a P-filtration, then $\pi_{\min,\mathscr{F}_P}(D)_\sigma:=\pi_{\min,\mathscr{F}_P}(D)\cap\pi_{\min}(D)_\sigma$ is just the extension of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})\otimes_E\mathrm{Ker}\,t_{D,\mathscr{F}_P,\sigma}$ by $\pi_{1,\sigma}(\phi,\mathbf{h})$.

Let $D_\sigma = \mathfrak{T}_\sigma(D)$ (2.3). Consider the composition

$$t_{\phi,\mathbf{h}_\sigma} : \oplus_{w\in S_n} \overline{\mathrm{Ext}}^1_{w,\sigma}(D_\sigma, D_\sigma) \xrightarrow[\sim]{(\kappa_w)} \oplus_{w\in S_n} \mathrm{Hom}_\sigma(T(K), E) \xrightarrow{(\zeta_w)} \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_{1,\sigma}(\phi,\mathbf{h})).$$

Let $t_{D_\sigma} = \mathfrak{T}_\sigma \circ t_{D,\sigma}$, hence $\mathrm{Ker}\, t_{D_\sigma} = \mathrm{Ker}\, t_{D,\sigma}$ by Corollary 2.27 (1). The following corollary is an easy consequence of Corollary 3.29 and Corollary 2.27:

Corollary 3.31. *(1) The natural surjection* $\oplus_{w\in S_n} \overline{\mathrm{Ext}}^1_{w,\sigma}(D_\sigma, D_\sigma) \twoheadrightarrow \overline{\mathrm{Ext}}^1_\sigma(D_\sigma, D_\sigma)$ *(cf. Corollary 2.40) factors through*

$$\oplus_{w\in S_n} \overline{\mathrm{Ext}}^1_{w,\sigma}(D_\sigma, D_\sigma) \xrightarrow{t_{\phi,\boldsymbol{h}_\sigma}} \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}), \pi_{1,\sigma}(\phi,\boldsymbol{h})) \xrightarrow{t_{D_\sigma}} \overline{\mathrm{Ext}}^1_\sigma(D_\sigma, D_\sigma)$$

and t_{D_σ} *induces an isomorphism*

$$\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}), \pi_{\min}(D)_\sigma) \xrightarrow[\sim]{t_{D_\sigma}} \overline{\mathrm{Ext}}^1_\sigma(D_\sigma, D_\sigma). \tag{3.36}$$

(2) Let P be a standard parabolic subgroup, $\mathscr{F}_P$ be a P-filtration on D_σ (which corresponds to a P-filtration on D, still denoted by $\mathscr{F}_P$). We have a commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i}, \boldsymbol{h}^i), \pi_{1,\sigma}(\phi_{\mathscr{F}_P,i}, \boldsymbol{h}^i)) & \xrightarrow{(t_{M_{i,\sigma}})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(M_{i,\sigma}, M_{i,\sigma}) \\ \sim\downarrow (3.25) & & \sim\downarrow \\ \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}), \pi_{1,\sigma}(\phi,\boldsymbol{h})) & \xrightarrow{t_{D_\sigma,\mathscr{F}_P}} & \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_P}(D_\sigma, D_\sigma). \end{array} \tag{3.37}$$

where $t_{D_\sigma,\mathscr{F}_P}$ is the restriction t_{D_σ}. In particular, (3.25) induces an isomorphism

$$\oplus_{i=1}^r \mathrm{Ker}\, t_{M_{i,\sigma}} \xrightarrow{\sim} \mathrm{Ker}\, t_{D_\sigma,\mathscr{F}_P}. \tag{3.38}$$

Remark 3.32. *Similarly as in Remark 3.28, (3.37) induces a commutative diagram*

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_P,i}, \boldsymbol{h}^i), \pi_{\min}(M_i)_\sigma) & \xrightarrow[\sim]{(t_{M_{i,\sigma}})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(M_{i,\sigma}, M_{i,\sigma}) \\ \sim\downarrow & & \sim\downarrow \\ \mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}), \pi_{\min,\mathscr{F}_P}(D)_\sigma) & \xrightarrow[\sim]{t_{D_\sigma,\mathscr{F}_P}} & \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_P}(D_\sigma, D_\sigma) \end{array} \tag{3.39}$$

where $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}), \pi_{\min,\mathscr{F}_P}(D)_\sigma)$ *is the image of* $\mathrm{Ext}^1_{\sigma,\mathscr{F}_P}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}), \pi_{1,\sigma}(\phi,\boldsymbol{h}))$ *via the push-forward map, and where the left vertical map is obtained in a similar way as (3.19).*

Theorem 3.33. *Let D, $D' \in \Phi\Gamma_{\mathrm{nc}}(\phi,\boldsymbol{h})$, and $\sigma \in \Sigma_K$. Then $\pi_{\min}(D)_\sigma \cong \pi_{\min}(D')_\sigma$ if and only if $D_\sigma \cong D'_\sigma$. Consequently, $\pi_{\min}(D) \cong \pi_{\min}(D')$ if and only if $D_\sigma \cong D'_\sigma$ for all $\sigma \in \Sigma_K$.*

Proof. We prove the theorem by induction on n. The case where $n \leq 2$ is trivial. Indeed, in this case, $\pi_{\min}(D)_\sigma$ are all isomorphic, and D_σ are all isomorphic as well, for $D \in \Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$. Suppose it holds for $n-1$. Let D_1 (resp. D'_1) be the saturated (φ,Γ)-submodule of D (resp. of D') of rank $n-1$, and C_1 (resp. C'_1) be the quotient of D (resp. of D'), both with the associated Weil-Deligne representation $\oplus_{i=1}^{n-1}\phi_i$. Let $\mathscr{F}$ (resp. $\mathscr{F}'$, resp. $\mathscr{G}$, resp. $\mathscr{G}'$) be the filtration $D_1 \subset D$ (resp. $D'_1 \subset D'$, resp. $\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}) \subset C_1$, resp. $\mathcal{R}_{K,E}(\phi_n z^{\mathbf{h}_n}) \subset D'$). As $\pi_{\min}(D)_\sigma \cong \pi_{\min}(D')_\sigma =: \pi$, we have $\pi_{\min,\mathscr{F}}(D)_\sigma \cong \pi_{\min,\mathscr{F}'}(D)_\sigma =: \pi^-$ and $\pi_{\min,\mathscr{G}}(D)_\sigma \cong \pi_{\min,\mathscr{G}'}(D)_\sigma =: \pi^+$ (see Remark 3.28 and the discussion below Corollary 3.30). Hence by Corollary 3.31, in particular the isomorphism (3.38), we have $\pi_{\min}(D_1)_\sigma \cong \pi_{\min}(D_1')_\sigma$ and $\pi_{\min}(C_1)_\sigma \cong \pi_{\min}(C_1')_\sigma$ (as $\mathrm{GL}_{n-1}(K)$-representations). By induction hypothesis, $D_{1,\sigma} \cong D'_{1,\sigma}$ and $C_{1,\sigma} \cong C'_{1,\sigma}$.

Let $\mathcal{L}$ be the kernel of the following natural map (induced by $\pi^{\pm} \hookrightarrow \pi$)

$$\overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi^-) \oplus \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi^+) \twoheadrightarrow \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi). \tag{3.40}$$

We have a commutative diagram of exact sequences (see (2.22) for the bottom one)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi^-) \oplus \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi^+) & \longrightarrow & \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi) & \longrightarrow & 0 \\ & & \downarrow & & \sim\downarrow & & t_{D_\sigma}\downarrow\sim & & \\ 0 & \longrightarrow & \mathcal{L}(D_\sigma, D_{1,\sigma}, C_{1,\sigma}) & \longrightarrow & V(D_{1,\sigma},C_{1,\sigma})_\sigma & \longrightarrow & \overline{\mathrm{Ext}}^1_\sigma(D_\sigma,D_\sigma) & \longrightarrow & 0 \end{array} \tag{3.41}$$

where the middle (bijective) map is induced by (3.39). We deduce $\mathcal{L} \xrightarrow{\sim} \mathcal{L}(D_\sigma, D_{1,\sigma}, C_{1,\sigma})$. Similarly, replacing D_σ by D'_σ, we obtain $\mathcal{L} \xrightarrow{\sim} \mathcal{L}(D_\sigma, D_{1,\sigma}, C_{1,\sigma})$. Note the middle map in (3.41) does not change when D_σ is replaced by D'_σ. Hence $\mathcal{L}(D_\sigma, D_{1,\sigma}, C_{1,\sigma}) \cong \mathcal{L}(D'_\sigma, D_{1,\sigma}, C_{1,\sigma})$ as subspace of $V(D_{1,\sigma},C_{1,\sigma})_\sigma$. But this implies $D_\sigma \cong D'_\sigma$ by Corollary 2.38 and Proposition 2.39. □

3.2.2 Universal extensions

We give a reformulation of Theorem 3.21 using deformation rings of (φ,Γ)-modules, which will be useful in our proof of the local-global compatibility.

Let $D \in \Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$. Let R_D be the universal deformation ring of deformations of D over local artinian E-algebras. Let $R_{D,w}$ be the universal deformation ring of $\mathscr{T}_w$-deformations of D (i.e. the trianguline deformations of D with respect to the refinement $w(\phi)$), and $R_{D,g}$ be the universal deformation ring of de Rham deformations. For a continuous character δ of $T(K)$, denote by R_δ the universal deformation ring of deformations of δ over local artinian E-algebras. If δ is locally algebraic, denote by $R_{\delta,g}$ the universal deformation ring of locally algebraic deformations of δ. All of these rings are formally smooth completed local Noetherian E-algebras. For a completed local Noetherian E-algebra R, we use $\mathfrak{m}_R$ to denote its maximal ideal and we will use $\mathfrak{m}$ for simplicity when it does not cause confusion. We have natural surjections

$$R_D \twoheadrightarrow R_{D,w} \twoheadrightarrow R_{D,g}, \; R_\delta \twoheadrightarrow R_{\delta,g}.$$

For $w \in S_n$, we have a commutative Cartesian diagram (of local Artinian E-algebras) induced by κ_w (2.5)

$$\begin{array}{ccc} R_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2 & \longrightarrow & R_{w(\phi)z^{\mathbf{h}},g}/\mathfrak{m}^2 \\ \downarrow & & \downarrow \\ R_{D,w}/\mathfrak{m}^2 & \longrightarrow & R_{D,g}/\mathfrak{m}^2. \end{array}$$

Let $\mathcal{H}$ be the Bernstein centre over E associated to $\pi_{\mathrm{sm}}(\phi)$ (see for example [18, § 3.12]), and $\widehat{\mathcal{H}}_\phi$ be the completion of $\mathcal{H}$ at $\pi(\phi)$. We have isomorphisms

$$\widehat{\mathcal{H}}_\phi \xrightarrow{\sim} R_{w(\phi)\eta,g} \xrightarrow{\sim} R_{w(\phi)z^{\mathbf{h}},g}$$

where the first sends a smooth deformation χ of $w(\phi)\eta$ to $(\mathrm{Ind}_{B^-(K)}^{\mathrm{GL}_n(K)} \chi)^\infty$, and the second is given by twisting $\eta z^{-\mathbf{h}}$. By Lemma 2.12 and Proposition 3.3 (2), the composition

$$A_0 := \widehat{\mathcal{H}}_\phi/\mathfrak{m}^2 \xrightarrow{\sim} R_{w(\phi)z^{\mathbf{h}},g}/\mathfrak{m}^2 \hookrightarrow R_{D,g}/\mathfrak{m}^2$$

is independent of the choice of w. We let $A_D := R_D/\mathfrak{m}^2 \times_{R_{D,g}} A_0$ and $A_w := R_{D,w}/\mathfrak{m}^2 \times_{R_{D,g}} A_0$. The tangent space of A_D (resp. A_w) is hence naturally isomorphic to $\overline{\mathrm{Ext}}^1(D,D)$ (resp. $\overline{\mathrm{Ext}}^1_w(D,D)$). By Proposition 2.13, the natural morphism $A_D \to \prod_w A_w$ is injective. We always identify $\mathrm{Ext}^1_{T(K)}(\delta,\delta)$ with $\mathrm{Hom}(T(K),E)$. Hence the tangent space of A_w is naturally isomorphic to $\mathrm{Hom}(T(K),E)$. We let $\mathcal{I}_w$ be the kernel of $A_D \twoheadrightarrow A_w$.

Let $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}$ (resp. $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}_w$) be the (universal) extension of $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h})$ $\big($resp. of $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h})\big)$ by $\pi_1(\phi,\mathbf{h})$. For $w\in S_n$, denote by $\delta_w := w(\phi)z^{\mathbf{h}}\varepsilon^{-1}\circ\theta$, and $\widetilde{\delta}_w^{\mathrm{univ}}$ the universal extension of $\mathrm{Ext}^1_{T(K)}(\delta_w,\delta_w)\otimes_E\delta_w$ ($\cong \mathrm{Hom}(T(K),E)\otimes_E \delta_w$) by δ_w.

Lemma 3.34. *The induced representation $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}}$ is the universal extension of $\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\otimes_E \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathrm{PS}_1(w(\phi),\boldsymbol{h}))$ by $\mathrm{PS}_1(w(\phi),\boldsymbol{h})$.*

Proof. By Remark 3.6, $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}}$ is an extension of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})^{\oplus(n(d_K+1))}$ by a certain subrepresentation V of $\mathrm{PS}_1(w(\phi),\mathbf{h})$. However, using Proposition 3.5 (1) and the surjectivity of the last map in (3.8), we see V has to be the entire $\mathrm{PS}_1(w(\phi),\mathbf{h})$. Using again Proposition 3.5 (1), we see $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}}$ is in fact the universal extension. □

We have hence an isomorphism of $\mathrm{GL}_n(K)$-representations

$$I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}} \oplus_{\mathrm{PS}_1(w(\phi),\mathbf{h})} \pi_1(\phi,\mathbf{h}) \xrightarrow{\sim} \pi_1(\phi,\mathbf{h})_w^{\mathrm{univ}}. \tag{3.42}$$

There is a natural action of $A_w\cong R_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2$ on $\widetilde{\delta}_w^{\mathrm{univ}}$ where $x\in \mathfrak{m}_{R_{w(\phi)z^{\mathbf{h}}}}/\mathfrak{m}^2_{R_{w(\phi)z^{\mathbf{h}}}}\cong \mathrm{Hom}(T(K),E)^\vee$ acts via

$$x:\widetilde{\delta}_w^{\mathrm{univ}} \twoheadrightarrow \mathrm{Hom}(T(K),E)\otimes_E\delta_w \xrightarrow{x} \delta_w \hookrightarrow \widetilde{\delta}_w^{\mathrm{univ}}.$$

Hence $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}}$ is equipped with an induced $R_{\delta_w}/\mathfrak{m}^2$-action. Similarly $\pi(\phi,\mathbf{h})_w^{\mathrm{univ}}$ is equipped with an action of A_w given by

$$x:\pi_1(\phi,\mathbf{h})_w^{\mathrm{univ}} \twoheadrightarrow \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h}) \xrightarrow{x} \pi_{\mathrm{alg}}(\phi,\mathbf{h}) \hookrightarrow \pi_1(\phi,\mathbf{h})_w^{\mathrm{univ}}. \tag{3.43}$$

for $x\in\mathfrak{m}_{A_w}/\mathfrak{m}^2_{A_w}\cong \mathrm{Hom}(T(K),E)^\vee \xrightarrow[\sim]{\zeta_w} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))^\vee$. The injection $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}}\hookrightarrow \pi(\phi,\mathbf{h})_w^{\mathrm{univ}}$ (induced by (3.42)) is A_w-equivariant. The following theorem is a reformulation of Theorem 3.21.

Theorem 3.35. *There is a unique A_D-action on $\pi_1(\phi,\boldsymbol{h})^{\mathrm{univ}}$ such that for all $w\in S_n$, we have an $A_w\times\mathrm{GL}_n(K)$-equivariant injection*

$$\pi_1(\phi,\boldsymbol{h})_w^{\mathrm{univ}} \hookrightarrow \pi_1(\phi,\boldsymbol{h})^{\mathrm{univ}}[\mathcal{I}_w].$$

Proof. By Theorem 3.21, we define an A_D-action by letting $x\in\mathfrak{m}_{A_D}/\mathfrak{m}^2_{A_D}\cong \overline{\mathrm{Ext}}^1(D,D)^\vee \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))^\vee$ act via

$$x:\pi_1(\phi,\mathbf{h})^{\mathrm{univ}} \twoheadrightarrow \mathrm{Ext}^1(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h}) \xrightarrow{x} \pi_{\mathrm{alg}}(\phi,\mathbf{h}) \hookrightarrow \pi_1(\phi,\mathbf{h})_w^{\mathrm{univ}}. \tag{3.44}$$

It is clear that the action satisfies the property in the theorem. The uniqueness follows from the fact $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}$ is generated by $\pi_1(\phi,\mathbf{h})_w^{\mathrm{univ}}$. □

Corollary 3.36. *We have $\pi_{\min}(D)\cong\pi_1(\phi,\boldsymbol{h})^{\mathrm{univ}}[\mathfrak{m}_{A_D}]$.*

4 Local-global compatibility

4.1 The patched setting

Let Π_∞ be the patched Banach representation in [18] (for $\mathrm{GL}_n(F) = \mathrm{GL}_n(K)$), which is equipped with an action of the patched Galois deformation ring $R_\infty \cong R_{\overline{\rho}}^{\square} \widehat{\otimes}_{\mathcal{O}_E} R_\infty^{\wp}$ (where $\wp$ is "$\widetilde{\mathfrak{p}}$" and $\overline{\rho}$ is the local Galois representation $\overline{r}$ of *loc. cit.*). We refer to *loc. cit.* for details. Let

$$\mathcal{E} \longrightarrow (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T} \times (\mathrm{Spf}\, R_\infty^{\wp})^{\mathrm{rig}} \tag{4.1}$$

be the associated patched eigenvariety (see [24, § 4.1.2], that is an easy variation of the patched eigenvariety introduced in [13]), $\mathcal{M}$ be the natural coherent sheaf on $\mathcal{E}$ such that there is an $T(K) \times R_\infty$-equivariant isomorphism (see [13, § 3.1] for "$R_\infty - \mathrm{an}$")

$$\Gamma(\mathcal{E}, \mathcal{M})^\vee \cong J_B(\Pi_\infty^{R_\infty-\mathrm{an}}).$$

Recall a point $x = (\rho_{x,\wp}, \delta_x, \mathfrak{m}_x^{\mathfrak{p}}) \in (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T} \times (\mathrm{Spf}\, R_\infty^{\wp})^{\mathrm{rig}}$ lies in $\mathcal{E}$ if and only if

$$\mathrm{Hom}_{T(K)}\left(\delta_x, J_B(\Pi_\infty^{R_\infty-\mathrm{an}})[\mathfrak{m}_x]\right) \neq 0$$

where $\mathfrak{m}_x = (\rho_{x,\wp}, \mathfrak{m}_x^{\mathfrak{p}})$ is the associated maximal ideal of $R_\infty[1/p]$.

Let $X_{\mathrm{tri}}^{\square}(\overline{\rho}) \hookrightarrow (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T}$ be the trianguline variety [13, § 2.2], and ι_p be the twisting map

$$\iota_p : (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T} \xrightarrow{\sim} (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}} \times \widehat{T}, (\rho_p, \chi) \mapsto (\rho_p, \chi\delta_B(\varepsilon \circ \theta)).$$

Recall (4.1) factors through an embedding (cf. [13, Thm. 1.1], see also [34][38]).

$$\mathcal{E} \longrightarrow \iota_p^{-1}(X_{\mathrm{tri}}^{\square}(\overline{\rho})) \times (\mathrm{Spf}\, R_\infty^{\wp})^{\mathrm{rig}}, \tag{4.2}$$

which identifies $\mathcal{E}$ with a union of irreducible components of the latter. Recall $\dim X_{\mathrm{tri}}^{\square}(\overline{\rho}) = n^2 + d_K \frac{n(n+1)}{2}$ (cf. [13, Thm. 2.6]).

Let $\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(E)$ be a continuous representation such that ρ has a modulo p reduction equal to $\overline{\rho}$ and that $D := D_{\mathrm{rig}}(\rho) \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ (ϕ, $\mathbf{h}$ given as in § 2.2). Let $\mathfrak{m}_\rho \subset R_{\overline{\rho}}^{\square}[1/p]$ be the maximal ideal associated to ρ, and suppose there exists a maximal ideal $\mathfrak{m}^{\wp}$ of $R_\infty^{\wp}[1/p]$ such that $\Pi_\infty[\mathfrak{m}]^{\mathrm{lalg}} \neq 0$ for $\mathfrak{m} = (\mathfrak{m}_\rho, \mathfrak{m}^{\wp})$, the corresponding maximal ideal of $R_\infty[1/p]$. By [18, § 4], we have $\Pi_\infty[\mathfrak{m}]^{\mathrm{lalg}} \cong \pi_{\mathrm{alg}}(\phi, \mathbf{h})$. This implies that for any refinement $w(\phi)$,

$$x_w := (x_{w,\wp}, \mathfrak{m}^{\wp}) = (\rho, \delta_w \delta_B = w(\phi) z^{\mathbf{h}} \delta_B(\varepsilon^{-1} \circ \theta), \mathfrak{m}^{\wp}) \in \mathcal{E}.$$

As D is non-critical, all these (classical) points x_w are non-critical. In particular, $X_{\mathrm{tri}}^{\square}(\overline{\rho})$ is smooth at the points $\iota_p(x_{w,\wp})$ (cf. [13, Thm. 2.6 (iii)]) and (4.2) is a local isomorphism (noting it actually holds for all points of regular integer weights by [14]). As $(\mathrm{Spf}\, R_\infty^{\wp})^{\mathrm{rig}}$ is also smooth at $\mathfrak{m}^{\wp}$ (e.g. see the proof of [23, Cor. 4.4]), $\mathcal{E}$ is smooth at all x_w. By [12, Lem. 3.8] and the multiplicity one property in the construction in [18], we see $\mathcal{M}$ is locally free of rank one at all x_w.

Let $\mathcal{I} := (\mathfrak{m}_\rho^2, \mathfrak{m}^{\wp}) \subset R_\infty[1/p]$, and we study $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$. For $w \in S_n$, let $U = U_\wp \times U^{\wp} \subset \iota_p^{-1}(X_{\mathrm{tri}}^{\square}(\overline{\rho})) \times (\mathrm{Spf}\, R_\infty^{\wp})^{\mathrm{rig}}$ be a smooth affinoid neighbourhood of x_w such that $x_{w'} \notin U$ for $w' \neq w$. Let $\mathfrak{m}_{x_{w,\wp}}$ be the maximal ideal of $\mathcal{O}(U_\wp)$ at $x_{w,\wp}$. Consider $\mathcal{M}_{\tilde{x}_w} := \mathcal{M}/(\mathfrak{m}_{x_{w,\wp}}^2 + \mathfrak{m}^{\wp})$, which is a finite dimensional E-vector space and equipped with a natural $T(K)$-action. In fact, $\mathfrak{m}_{x_{w,\wp}} \subset \mathcal{O}(U_\wp)$

is the ideal generated by $\mathfrak{m}_\rho \subset R_{\overline{\rho}}^\square[1/p]$ hence $\mathfrak{m}_{x_{w,\wp}}^2 + \mathfrak{m}^\wp \subset \mathcal{O}(U)$ is the ideal generated by $\mathcal{I}$. By definition, there is a natural $T(K) \times R_\infty$-equivariant map

$$\mathcal{M}_{\tilde{x}_w}^\vee \hookrightarrow J_B(\Pi_\infty^{R_\infty-\mathrm{an}})[\mathcal{I}]. \tag{4.3}$$

We also have a $T(K) \times R_\infty$-equivariant exact sequence

$$0 \longrightarrow \mathcal{M}_{x_w}^\vee \longrightarrow \mathcal{M}_{\tilde{x}_w}^\vee \longrightarrow (\mathcal{M}_{x_w}^\vee)^{\oplus n^2+d_K\frac{n(n+1)}{2}} \longrightarrow 0.$$

In particular, as $T(K)$-representation, $\mathcal{M}_{\tilde{x}_w}^\vee$ is isomorphic to an extension of $(\delta_w\delta_B)^{n^2+d_K\frac{n(n+1)}{2}}$ by $\delta_w\delta_B$.

Lemma 4.1. *The map (4.3) is balanced, hence by [27, Thm. 0.13] induces a* $\mathrm{GL}_n(K) \times R_\infty$-*equivariant injection*

$$\iota_w : I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w}^\vee \otimes_E \delta_B^{-1}) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}].$$

Proof. The lemma follows by the same argument as in [24, Lem. 4.11], based on the fact x_w is non-critical (hence does not have companion points of non-dominant weight). □

Recall the completion of $R_{\overline{\rho}}^\square$ at ρ is isomorphic to $R_\rho^\square$ (cf. [35]). For $w \in S_n$, denote by $R_{D,w}^\square := R_\rho^\square \widehat{\otimes}_{R_\rho} R_{D,w}$ (recalling $R_D \cong R_\rho$). As x_w is non-critical, the completion of $X_{\mathrm{tri}}^\square(\overline{\rho})$ at x_w is naturally isomorphic to $R_{D,w}^\square$. Then $\mathcal{M}_{\tilde{x}_w}$ is a free $R_{D,w}^\square/\mathfrak{m}^2$-module of rank 1, hence is isomorphic to $\mathcal{N}_w \otimes_{A_w} R_{D,w}^\square/\mathfrak{m}_{R_{D,w}^\square}^2$ for a rank one free A_w-module $\mathcal{N}_w$. Equipping $\mathcal{N}_w$ with the natural $T(K)$-action (using $T(K) \to A_w$) twisted by δ_B, then $\mathcal{N}_w^\vee \cong \widetilde{\delta}_w^{\mathrm{univ}}\delta_B$. The induced surjection $\mathcal{M}_{\tilde{x}_w}^\vee \twoheadrightarrow \mathcal{N}_w^\vee$ is $A_w \times T(K)$-equivariant, and induces an $A_w \times \mathrm{GL}_n(K)$-equivariant surjection

$$f_w : I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w}^\vee \otimes_E \delta_B^{-1}) \twoheadrightarrow I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}}.$$

Proposition 4.2. *(1)* $\mathrm{Ker}\, f_w \cong \pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus(n^2-n+\frac{n(n-1)}{2}d_K)}$.

(2) $\iota_w(\mathrm{Ker}\, f_w) \subset \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$ *is independent of the choice of* w.

Proof. Similarly as in Remark 3.6, we see $I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w}^\vee \otimes_E \delta_B^{-1})$ is isomorphic to an extension of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus(n^2+\frac{n(n+1)}{2}d_K)}$ by a certain subrepresentation V of $\mathrm{PS}_1(w(\phi), \mathbf{h})$. As f_w is surjective, we see by Lemma 3.34 that $V \cong \mathrm{PS}_1(w(\phi), \mathbf{h})$. By comparing the multiplicities of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$, (1) follows.

Consider $\mathcal{M}_{\tilde{x}_w,g} := \mathcal{M}_{\tilde{x}_w} \otimes_{R_{D,w}/\mathfrak{m}^2} R_{D,g}/\mathfrak{m}^2$ (equipped with the natural $R_\infty \times T(K)$-action), and $\mathcal{N}_{w,g} := \mathcal{N}_w \otimes_{A_w} A_0$ (equipped with the natural $A_0 \times T(K)$-action, where the $T(K)$-action is given by $T(K) \to A_w \to A_0$). Set $\widetilde{\delta}_{w,g}^{\mathrm{univ}}$ to be the universal extension of $\mathrm{Ext}^1_{\mathrm{lalg}}(\delta_w, \delta_w) \otimes_E \delta_w$ by δ_w, then we have $\mathcal{N}_{w,g}^\vee \cong \widetilde{\delta}_{w,g}^{\mathrm{univ}}\delta_B$. We have a $T(K) \times R_\infty$-equivariant isomorphism $\mathcal{M}_{\tilde{x}_w,g} \cong \mathcal{N}_{w,g} \otimes_{A_0} R_{D,g}/\mathfrak{m}^2$. By Lemma 4.1, the $T(K) \times R_\infty$-equivariant map $\mathcal{M}_{\tilde{x}_w,g}^\vee \hookrightarrow J_B(\Pi_\infty^{R_\infty-\mathrm{an}})[\mathcal{I}]$ is also balanced. We have hence a commutative diagram

$$\begin{array}{ccccc} I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w,g}^\vee \otimes_E \delta_B^{-1}) & \longrightarrow & I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w}^\vee \otimes_E \delta_B^{-1}) & \xrightarrow{\iota_w} & \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}] \\ \downarrow{\scriptstyle f_{w,g}} & & \downarrow{\scriptstyle f_w} & & \\ I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_{w,g}^{\mathrm{univ}} & \longrightarrow & I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}} & & \end{array}$$

Note that $I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w,g}^\vee)$ is an extension of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})^{\oplus(n^2+\frac{n(n-1)}{2}d_K)}$ by $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. It is easy to see $\mathrm{Ker}\, f_{w,g} \xrightarrow{\sim} \mathrm{Ker}\, f_w$ (e.g. by comparing the multiplicity of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$). On the other hand, using [18, Lem. 4.18, Thm. 4.19], we see the image of $I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w,g}^\vee \otimes_E \delta_B^{-1})$ under ι_w is exactly $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]^{\mathrm{lalg}}$, the locally algebraic subrepresentation of $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$.

Recall we have a commutative diagram of rings

$$\begin{array}{ccc} \widehat{\mathcal{H}}_\phi/\mathfrak{m}^2 & \longrightarrow & R_{D,g}/\mathfrak{m}^2 \\ \sim\downarrow & & \Vert \\ A_{w,g} & \longrightarrow & R_{D,g}/\mathfrak{m}^2. \end{array}$$

By [18, § 4] and the theory of Bernstein centre, the top map induces a surjection

$$f_0 : \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]^{\mathrm{lalg}} \twoheadrightarrow \big((\text{c-ind}_{\mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n(K)}\Theta)^\infty \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m}_\phi^2\big) \otimes_E L(\lambda) \cong \pi_{\mathrm{alg}}(\phi,\mathbf{h})^{\mathrm{univ}}$$

where Θ is the (1-dimensional) smooth irreducible representation of $\mathrm{GL}_n(\mathcal{O}_K)$ associated to the Bernstein component of $\pi_{\mathrm{sm}}(\phi)$ and recall $\mathcal{H} \cong \mathrm{End}_{\mathrm{GL}_n(K)}\big((\text{c-ind}_{\mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n(K)}\Theta)^\infty\big)$. We see the following diagram commutes

$$\begin{array}{ccc} I_{B^-(K)}^{\mathrm{GL}_n(K)}(\mathcal{M}_{\tilde{x}_w,g}^\vee \otimes_E \delta_B^{-1}) & \xrightarrow[\sim]{\iota_w} & \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]^{\mathrm{lalg}} \\ f_{w,g}\downarrow & & f_0\downarrow \\ I_{B^-(K)}^{\mathrm{GL}_n(K)}\delta_{w,g}^{\mathrm{univ}} & \xrightarrow{\sim} & \pi_{\mathrm{alg}}(\phi,\mathbf{h})^{\mathrm{univ}}. \end{array}$$

Thus $\iota_w(\mathrm{Ker}\, f_w)$ is equal to $\mathrm{Ker}\, f_0$, and (2) follows. □

Corollary 4.3. *We have* $(\mathrm{Im}\,\iota_w)[\mathfrak{m}] \cong \mathrm{PS}_1(w(\phi),\boldsymbol{h})$.

Proof. By the proof of Proposition 4.2, the injection $\pi_{\mathrm{alg}}(\phi,\mathbf{h}) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}] \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$ extends to

$$\mathrm{PS}_1(w(\phi),\mathbf{h}) \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]. \tag{4.4}$$

As x_w is non-critical, all the irreducible constituents $\mathrm{PS}_1(w(\phi),\mathbf{h})$ except $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ can not be a subrepresentation of $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$. We deduce the image of (4.2) lies in $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$ (noting this also follows from results of [15]). By [12, Lem. 4.16], we have (where $\{-\}$ denotes the generalized eigenspace):

$$J_B(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}])[T(K)=\delta_w\delta_B] \xrightarrow{\sim} J_B(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}])\{T(K)=\delta_w\delta_B\}$$

which is hence one dimensional. We then deduce the injection $\mathrm{PS}_1(w(\phi),\mathbf{h}) \hookrightarrow (\mathrm{Im}\,\iota_w)[\mathfrak{m}]$ is bijective. □

Let $\widetilde{\pi}$ be the subrepresentation of $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$ generated by $\mathrm{Im}\,\iota_w$ for all $w \in S_n$, and $V_0 := \mathrm{Ker}\, f_w$ (for any w). It is clear that $\widetilde{\pi}$ is stabilized by R_∞.

Theorem 4.4. *We have an isomorphism equivariant under the action of* $\mathrm{GL}_n(K) \times A_D$*:* $\tilde{\pi}/V_0 \cong \pi_1(\phi,\boldsymbol{h})^{\mathrm{univ}}$*. Moreover, the isomorphism induces* $\tilde{\pi}[\mathfrak{m}] \xrightarrow{\sim} \pi(\phi,\boldsymbol{h})^{\mathrm{univ}}[\mathfrak{m}_{A_D}] = \pi_{\min}(D)$*.*

Proof. We first show $\widetilde{\pi}/V_0 \cong \pi_1(\phi, \mathbf{h})^{\mathrm{univ}}$ as $\mathrm{GL}_n(K)$-representation. The injection $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$ uniquely extends to an injection $\pi_1(\phi, \mathbf{h}) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$ (using Corollary 4.3, or [15]). By Corollary 4.3, its image is contained in $\widetilde{\pi}$. The composition

$$\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \longrightarrow \widetilde{\pi}[\mathfrak{m}] \longrightarrow \widetilde{\pi} \longrightarrow\!\!\!\!\!\rightarrow \widetilde{\pi}/V_0$$

is injective. Hence we obtain

$$V_1 := V_0 \oplus \pi_1(\phi, \mathbf{h}) \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}].$$

For $w \in S_n$, it is clear $V_1 \cap \mathrm{Im}\,\iota_w \cong V_0 \oplus \mathrm{PS}_1(w(\phi), \mathbf{h})$ (using Corollary 4.3). We deduce that $\widetilde{\pi}$ is isomorphic to an extension of a certain copy of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by V_1. We claim

$$\mathrm{soc}_{\mathrm{GL}_n(K)}\left(\widetilde{\pi}/V_0\right) \cong \pi_{\mathrm{alg}}(\phi, \mathbf{h}).$$

Indeed, if not, there will be an extension of the form V_0—$\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ lies in $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]$. Applying the Jacquet-Emerton functor, the extension will produce an extension of $\delta_w\delta_B$ by $(\delta_w\delta_B)^{\oplus(n^2-n+\frac{n(n-1)}{2}d_K)} \subset J_B(V_0)$ (for any w), which has to lie in $M_{\tilde{x}_w}^\vee$. We deduce the extension V_0—$\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ lies in $\mathrm{Im}(\iota_w)$. However, we have

$$\mathrm{soc}_{\mathrm{GL}_n(K)}(\mathrm{Im}\,\iota_w/V_0) \cong \mathrm{soc}_{\mathrm{GL}_n(K)}\,\pi_1(\phi, \mathbf{h})_w^{\mathrm{univ}} \cong \pi_{\mathrm{alg}}(\phi, \mathbf{h}).$$

Consequently, $\widetilde{\pi}/V_0$ has to be a subrepresentation of $\pi_1(\phi, \mathbf{h})^{\mathrm{univ}}$. As $\widetilde{\pi}/V_0$ contains all $I_{B^-(K)}^{\mathrm{GL}_n(K)}\delta_w^{\mathrm{univ}}$, it is not difficult to see $\widetilde{\pi}/V_0 \xrightarrow{\sim} \pi_1(\phi, \mathbf{h})^{\mathrm{univ}}$.

Now consider the A_D-action on $\widetilde{\pi}$ induced from the natural $R_D^\square/\mathfrak{m}_D^2$-action (inherited from the R_∞-action). By [18, § 4] (and the proof of Proposition 4.2), V_0 is annihilated by $\mathfrak{m}_{A_D}$ in particular A_D-stable. The quotient $\widetilde{\pi}/V_0$ hence also has an A_D-action. Remark this action comes from the R_∞-action hence has a global nature. We need to show this action coincides with the local one given in Theorem 3.35. However, for each $w \in S_n$, we have $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}_w^{\mathrm{univ}} \hookrightarrow (\widetilde{\pi}/V_0)[\mathcal{I}_w]$, (where $\mathcal{I}_w$ is the kernel of $A_D \twoheadrightarrow A_{D,w}$), and the injection is $A_{D,w}$-equivariant (by Lemma 4.1). By the uniqueness part in Theorem 3.35, we deduce $\pi_1(\phi, \mathbf{h})^{\mathrm{univ}} \xrightarrow{\sim} \widetilde{\pi}/V_0$ is indeed A_D-equivariant.

We prove the second part. Using $\widetilde{\pi}[\mathfrak{m}] \cap V_0 = 0$, we see the surjection $\widetilde{\pi} \twoheadrightarrow \pi_1(\phi, \mathbf{h})^{\mathrm{univ}}$ induces an injection $\widetilde{\pi}[\mathfrak{m}] \hookrightarrow \pi_1(\phi, \mathbf{h})^{\mathrm{univ}}[\mathfrak{m}_{A_D}] \cong \pi_{\min}(D)$. We show it is a bijection by comparing the multiplicities of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ of both sides. We have a natural exact sequence

$$0 \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}] \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}] \xrightarrow{f} \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]^{\oplus(n^2+n^2d_K)}.$$

It is easy to see $\widetilde{\pi} \subset f^{-1}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus(n^2+n^2d_K)}\big)$. We have hence an exact sequence

$$0 \longrightarrow \widetilde{\pi}[\mathfrak{m}] \longrightarrow \widetilde{\pi} \longrightarrow \pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus(n^2+n^2d_K)} \tag{4.5}$$

Hence the multiplicity of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ in $\widetilde{\pi}[\mathfrak{m}]$ is at least

$$1 + (n^2 - n + \frac{n(n-1)}{2}d_K) + (n + (2^n - 1)d_K) - (n^2 + n^2 d_K) = 1 + (2^n - \frac{n(n+1)}{2} - 1)d_K.$$

Hence $\widetilde{\pi}[\mathfrak{m}] \hookrightarrow \pi_{\min}(D)$ has to be an isomorphism. $\square$

Remark 4.5. *(1) By the proof, the last map in (4.5) is surjective.*

(2) By [15], the map $\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$ *uniquely extends to* $\pi(\phi, \boldsymbol{h}) \hookrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$*. Using (3.31), we deduce an injection*

$$\pi_{\mathrm{fs}}(D) \longrightarrow \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}].$$

Remark that $\pi_{\mathrm{fs}}(D)$ *should still be far from the entire* $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$*.*

Corollary 4.6. *Keep the situation as in Theorem 4.4. The representation* $\pi_{\min}(D)$ *is the maximal subrepresentation of* $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$ *given by extensions of* $\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$ *by* $\pi_1(\phi, \boldsymbol{h})$*.*

Proof. It suffices to show any such extension (with $\pi_1(\phi, \mathbf{h}) \subset \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$) is contained in $\widetilde{\pi}$. But it is clear as $\widetilde{\pi}$ contains $\pi_1(\phi, \mathbf{h})^{\mathrm{univ}}$ and $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathcal{I}]^{\mathrm{lalg}}$. □

By the corollary and Theorem 3.33, we have

Corollary 4.7. *Keep the situation, then* $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]$ *determines* $\{D_\sigma\}_{\sigma\in\Sigma_K}$ *for* $D = D_{\mathrm{rig}}(\rho)$*. In particular, when* $K = \mathbb{Q}_p$*, it determines* ρ*.*

4.2 Some other cases

We discuss the local-global compatibility in the space of p-adic automorphic representations for certain definite unitary groups (with fewer global hypotheses than §4.1).

4.2.1 Some formal results

We first discuss some corollaries of the results in 3.2. Let $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ and $\mathrm{Ext}^1_U(D, D)$ be a certain subspace of $\mathrm{Ext}^1(D, D)$. For $w \in S_n$, set $\mathrm{Ext}^1_{U,w}(D, D) := \mathrm{Ext}^1_U(D, D) \cap \mathrm{Ext}^1_w(D, D)$. We assume the following hypotheses.

Hypothesis 4.8. *(1)* $\dim_E \mathrm{Ext}^1_U(D, D) = \frac{n(n+1)}{2} d_K$*, and* $\mathrm{Ext}^1_U(D, D) \cap \mathrm{Ext}^1_g(D, D) = 0$*.*

(2) For $w \in S_n$*,* $\dim_E \mathrm{Ext}^1_{U,w}(D, D) = n d_K$ *and the following induced map is surjective*

$$\oplus_{w\in S_n} \mathrm{Ext}^1_{w,U}(D, D) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Ext}^1_U(D, D).$$

Denote by $\mathrm{Ext}^1_{U,w}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ the image of the composition

$$\mathrm{Ext}^1_{U,w}(D, D) \longrightarrow \overline{\mathrm{Ext}}^1_w(D, D) \xrightarrow{\sim} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$$

which is injective by Hypothesis 4.8 (1). Denote by $\mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ the subspace generated by $\mathrm{Ext}^1_{U,w}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ for all $w \in S_n$.

Corollary 4.9. *(1) The map*

$$t_{\phi,\boldsymbol{h}} : \oplus_{w\in S_n} \mathrm{Ext}^1_{U,w}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})) \cong \oplus_{w\in S_n} \mathrm{Ext}^1_{U,w}(D, D) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Ext}^1_U(D, D)$$

(uniquely) factors through a surjection

$$t_{D,U} : \mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Ext}^1_U(D, D).$$

(2) We have $\mathrm{Ker}\, t_{D,U} \xrightarrow{\sim} \mathrm{Ker}\, t_D$*.*

Proof. By Hypothesis 4.8 (1), the following composition is injective:

$$\mathrm{Ext}^1_U(D,D) \hookrightarrow \mathrm{Ext}^1(D,D) \twoheadrightarrow \overline{\mathrm{Ext}}^1(D,D).$$

As the following diagram obviously commutes

$$\begin{array}{ccccc} \oplus_w \mathrm{Ext}^1_{U,w}(D,D) & \longrightarrow & \oplus_w \overline{\mathrm{Ext}}^1_w(D,D) & \xrightarrow{\sim} & \oplus_w \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}^1_U(D,D) & \longrightarrow & \overline{\mathrm{Ext}}^1(D,D) & \xleftarrow{t_D} & \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \end{array}$$

(1) follows. It is clear that $\mathrm{Ker}\, t_{D,U} = \mathrm{Ker}\, t_D \cap \mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \subset \mathrm{Ker}\, t_D$. By Hypothesis 4.8 (1) (2), the composition

$$\mathrm{Ext}^1_{U,w}(D,D) \xrightarrow{\kappa_w} \mathrm{Hom}(T(K),E) \twoheadrightarrow \mathrm{Hom}(T(\mathcal{O}_K),E) \tag{4.6}$$

is an isomorphism. Let $\mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1(w(\phi),\mathbf{h}))$ be the image of $\mathrm{Ext}^1_{U,w}(D,D)$ under $\zeta_w\circ\kappa_w$, which has dimension nd_K. We have by dévissage an exact sequence

$$0 \longrightarrow W \longrightarrow \mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1(w(\phi),\mathbf{h})) \longrightarrow \oplus_{\substack{i=1,\cdots,n-1\\ \sigma\in\Sigma_K}} \mathrm{Ext}^1(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(w(\phi),s_{i,\sigma})), \tag{4.7}$$

where W is a subspace of $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$. As (4.6) is bijective, it is not difficult to see the last map in (4.7) is surjective hence $\dim_E W = d_K$. Similarly, $\mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ lies in an exact sequence

$$0 \longrightarrow W' \longrightarrow \mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \longrightarrow \oplus_{\substack{i=1,\cdots,n-1,\sigma\in\Sigma_K\\ I\subset\{1,\cdots,n-1\},\#I=i}} \mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(I,s_{i,\sigma})), \tag{4.8}$$

where $W'\supset W$ is a subspace of $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$. By the surjectivity of the last map in (4.7), we deduce the last map of (4.8) is surjective as well. Hence $\dim_E \mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \geq nd_K + (2^n-2)d_K$. Consequently $\dim_E \mathrm{Ker}\, t_{D,U} \geq (2^n - \frac{n(n+1)}{2} - 1)d_K = \dim_E \mathrm{Ker}\, t_D$. (2) follows. □

Set $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}_U$ (resp. $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}_{U,w}$) to be the tautological extension of

$$\mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h}) \ \big(\text{resp. } \mathrm{Ext}^1_{U,w}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_E \pi_{\mathrm{alg}}(\phi,\mathbf{h})\big)$$

by $\pi_1(\phi,\mathbf{h})$. Let $R_{D,U}/\mathfrak{m}^2$ (resp. $R_{D,U,w}/\mathfrak{m}^2$) be the quotient of $R_D/\mathfrak{m}^2$ (resp. $R_{D,w}/\mathfrak{m}^2$) associated to $\mathrm{Ext}^1_U(D,D)$ (resp. $\mathrm{Ext}^1_{U,w}(D,D)$). Let $R_{w(\phi)z^{\mathbf{h}},U}/\mathfrak{m}^2$ be the quotient of $R_{w(\phi)z^{\mathbf{h}}}$ associated to the image of $\mathrm{Ext}^1_{U,w}(D,D)\hookrightarrow \mathrm{Ext}^1_{T(K)}(w(\phi)z^{\mathbf{h}},w(\phi)z^{\mathbf{h}})$. We have a natural isomorphism

$$R_{w(\phi)z^{\mathbf{h}},U}/\mathfrak{m}^2 \xrightarrow{\sim} R_{D,U,w}/\mathfrak{m}^2. \tag{4.9}$$

Consider $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}^{\mathrm{univ}}_{w,U}$. By similar argument as in the proof of Lemma 3.34 and (the surjectivity of the last map in) (4.7), $I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}^{\mathrm{univ}}_{w,U}$ is isomorphic to the universal extension of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})\otimes_E \mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1(w(\phi),\mathbf{h}))$ by $\mathrm{PS}_1(w(\phi),\mathbf{h})$. Moreover, similarly as in § 3.2.2, we have a $\mathrm{GL}_n(K)\times R_{D,U,w}/\mathfrak{m}^2$-equivariant injection

$$I_{B^-(K)}^{\mathrm{GL}_n(K)}\widetilde{\delta}^{\mathrm{univ}}_{w,U} \longrightarrow \pi_1(\phi,\mathbf{h})^{\mathrm{univ}}_{U,w},$$

where the $R_{D,U,w}$-action on the left hand side is induced by its action on $\widetilde{\delta}^{\mathrm{univ}}_{w,U}$ using (4.9) and the $R_{D,U,w}$-action on the right hand side is given in a similar way as in (3.43). The following corollary follows by similar arguments as in Theorem 3.35 and Corollary 3.36.

Corollary 4.10. *(1) There is a unique $R_{D,U}$-action on $\pi_1(\phi, \boldsymbol{h})_U^{\mathrm{univ}}$ such that for all $w \in S_n$, there is a $\mathrm{GL}_n(K) \times R_{D,U,w}$-equivariant injection $\pi_1(\phi, \boldsymbol{h})_{U,w}^{\mathrm{univ}} \hookrightarrow \pi_1(\phi, \boldsymbol{h})_U^{\mathrm{univ}}[I_w]$.*

(2) We have $\pi_1(\phi, \boldsymbol{h})_U^{\mathrm{univ}}[\mathfrak{m}_{R_{D,U}}] \cong \pi_{\min}(D)$.

4.2.2 Local-global compatibility

We prove a local-global compatibility result in a non-patched setting. We briefly introduce the setup and some notation.

Let F/F^+ be a CM extension and G/F^+ be a unitary group attached to the quadratic extension F/F^+ as in [2, § 6.2.2] such that $G \times_{F^+} F \cong \mathrm{GL}_n$ ($n \geq 2$) and $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact. For a finite place v of F^+ which is totally split in F and $\tilde{v}$ a place of F dividing v, we have isomorphisms $G(F_v^+) \xrightarrow{\sim} G(F_{\tilde{v}}) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}})$. We let S_p denote the set of places of F^+ dividing p and we assume that each place in S_p is split in F.

We fix a place $\wp$ of F^+ above p, set $K := F_\wp^+$. We have thus an isomorphism $G(F_\wp^+) \xrightarrow{\sim} \mathrm{GL}_n(K)$. For each $v \in S_p$, $v \neq \wp$, let ξ_v be a dominant weight of $\mathrm{Res}_{\mathbb{Q}_p}^{F_v^+} \mathrm{GL}_n$, and $\tau_v : I_{F_v^+} \to \mathrm{GL}_n(E)$ be an inertial type. Let $W_{\xi,\tau}$ be the representation of $\prod_{v \in S_p, v \neq \wp} G(F_v^+)$ over $\mathcal{O}_E$ associated to $\xi = (\xi_v)$ and $\tau = (\tau_v)$.

Let $U^\wp = U^p U_p^\wp = \prod_{v \nmid p} U_v \times \prod_{v \in S_p \setminus \{\wp\}} U_v$ be a sufficiently small (cf. [20]) compact open subgroup of $G(\mathbb{A}_{F^+}^{\infty,\wp})$ with $U_v \cong \mathrm{GL}_n(\mathcal{O}_{F_v^+})$ for $v \in S_p \setminus \{\wp\}$. We also assume that U_v is hyperspecial if v is inert in F. Let $S(U^\wp)$ be the union of S_p and of the places $v \notin S_p$ such that U_v is not hyperspecial. For each $v \in S(U^\wp)$ (which splits in F), fix a place $\widetilde{v}$ of F dividing v, and let $\widetilde{S}(U^\wp)$ be the set of such $\widetilde{v}$.

For $k \in \mathbb{Z}_{\geq 1}$ and a compact open subgroup $U_\wp$ of $G(\mathcal{O}_{F_\wp^+})$, consider the $\mathcal{O}_E/\varpi_E^k$-module

$$S_{\xi,\tau}(U^\wp U_\wp, \mathcal{O}_E/\varpi_E^k) = \{f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \to W_{\xi,\tau}/\varpi_E^k \mid f(gu) = u^{-1} f(g),\ \forall g \in G(\mathbb{A}_{F^+}^\infty), u \in U^\wp U_\wp\}$$

where $U^\wp U_\wp$ acts on $W_{\xi,\tau}/\varpi_E^k$ via the projection $U^\wp U_\wp \to \prod_{v \in S_p \setminus \{\wp\}} U_v$. Put

$$\widehat{S}_{\xi,\tau}(U^\wp, \mathcal{O}_E) := \varprojlim_k S_{\xi,\tau}(U^\wp, \mathcal{O}_E/\varpi_E^k) := \varprojlim_k \varinjlim_{U_\wp} S_{\xi,\tau}(U^\wp U_\wp, \mathcal{O}_E/\varpi_E^k),$$

and $\widehat{S}_{\xi,\tau}(U^\wp, E) := \widehat{S}_{\xi,\tau}(U^\wp, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E$. Then $\widehat{S}_{\xi,\tau}(U^\wp, E)$ is an admissible unitary Banach representation of $\mathrm{GL}_n(K)$. Recall that $\widehat{S}_{\xi,\tau}(U^\wp, E)$ is equipped with a natural action of $\mathbb{T}(U^\wp)$ commuting with $\mathrm{GL}_n(K)$, where $\mathbb{T}(U^\wp)$ is the polynomial $\mathcal{O}_E$-algebra generated by the spherical Hecke operators at places v such that $U_v \cong \mathrm{GL}_n(\mathcal{O}_{F_v^+})$. We make the following hypothesis

Hypothesis 4.11. *We have either ($p > 2$, $n \leq 3$) or ($p > 2$, F/F^+ is unramified and G is quasi-split at all finite places of F^+).*

Let $\overline{\rho} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(k_E)$ be a continuous *automorphic* representation with respect to $W_{\xi,\tau}$ (see for example [28, Def. 5.3.1]). To $\overline{\rho}$, one can associate a maximal ideal $\mathfrak{m}_{\overline{\rho}}$ of $\mathbb{T}(U^\wp)$. Let

$$\widehat{S}_{\xi,\tau}(U^\wp, E)_{\overline{\rho}} := \big(= \varprojlim_k \varinjlim_{U_\wp} S_{\xi,\tau}(U^\wp U_\wp, \mathcal{O}_E/\varpi_E^k)_{\mathfrak{m}_{\overline{\rho}}}\big)[1/p],$$

which is in fact a direct summand of $\widehat{S}_{\xi,\tau}(U^\wp, E)$. Recall the action of $\mathbb{T}(U^\wp)$ on $\widehat{S}_{\xi,\tau}(U^\wp, E)_{\overline{\rho}}$ factors through a faithful action of a certain completed Noetherian local $\mathcal{O}_E$-algebra $\widehat{\mathbb{T}}_{\xi,\tau}(U^\wp)_{\overline{\rho}}$.

Let $R_{\overline{\rho},S(U^\wp)}$ be the Galois deformation ring associated to the deformation problem (cf. [20, § 2])

$$(F/F^+, S(U^\wp), \widetilde{S}(U^\wp), \mathcal{O}_E, \overline{\rho}, \varepsilon^{1-n}\delta^n_{F/F^+}, \{R^{\square}_{\overline{\rho}_{\widetilde{v}}}\}_{v\in S(U^\wp)\setminus S_p^\wp} \cup \{R^{\xi_v,\tau_v}_{\overline{\rho}_{\widetilde{v}}}\}_{v\in S_p^\wp})$$

where $R^{\square}_{\overline{\rho}_{\widetilde{v}}}$ denotes the reduced and p-torsion free quotient of the universal framed deformation ring, and $R^{\xi_v,\tau_v}_{\overline{\rho}_{\widetilde{v}}}$ denotes the framed potentially semi-stable deformation ring of type (ξ_v, τ_v). Recall there is a natural surjection $R_{\overline{\rho},S(U^\wp)} \twoheadrightarrow \widehat{\mathbb{T}}_{\xi,\tau}(U^\wp)_{\overline{\rho}}$ (e.g. see [28, § 5.4]).

Let $\rho \in (\mathrm{Spf}\, R_{\overline{\rho},S(U^\wp)})^{\mathrm{rig}}$ such that $D := D_{\mathrm{rig}}(\rho_{\widetilde{\wp}}) \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ with $\rho_{\widetilde{\wp}} := \rho|_{F_{\widetilde{\wp}}} : \mathrm{Gal}_K \to \mathrm{GL}_n(E)$. Let $\mathfrak{m}_\rho$ be the maximal ideal of $R_{\overline{\rho},S(U^\wp)}[1/p]$ associated to ρ, and assume $\widehat{S}_{\xi,\tau}(U^\wp, E)[\mathfrak{m}_\rho]^{\mathrm{lalg}} \neq 0$. There exists hence $r \in \mathbb{Z}_{\geq 1}$ such that (see for example [17])

$$\pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus r} \xrightarrow{\sim} \widehat{S}_{\xi,\tau}(U^\wp, E)[\mathfrak{m}_\rho]^{\mathrm{lalg}}. \tag{4.10}$$

There is a natural map from R_D to the completion $\widehat{\mathcal{O}}_{(\mathrm{Spf}\, R_{\overline{\rho},S(U^\wp)})^{\mathrm{rig}},\rho}$. We denote by $\mathrm{Ext}^1_U(D, D)$ the image of the induced tangent map. We make the following assumption.

Hypothesis 4.12. *Suppose* $\mathrm{Ext}^1_U(D, D) \cap \mathrm{Ext}^1_g(D, D) = 0$.

By [40] (see also [1]), the hypothesis holds when $\overline{\rho}$ has enormous image. By similar arguments as in [32, Cor. 8.5], we have:

Proposition 4.13. *Assume Hypothesis 4.11 and 4.12. Then* $(\mathrm{Spf}\, R_{\overline{\rho},S(U^\wp)})^{\mathrm{rig}}$ *is smooth of dimension* $d_K \frac{n(n-1)}{2}$ *at* ρ *and* $\mathrm{Ext}^1_U(D, D)$ *satisfies the assumption Hypothesis 4.8.*

Under the assumption, the map $R_D \to \widehat{\mathcal{O}}_{(\mathrm{Spf}\, R_{\overline{\rho},S(U^\wp)})^{\mathrm{rig}},\rho}$ induces an isomorphism (see for example the proof of [32, Thm. 8.8])

$$R_{D,U} \xrightarrow{\sim} \widehat{\mathcal{O}}_{(\mathrm{Spf}\, R_{\overline{\rho},S(U^\wp)})^{\mathrm{rig}},\rho}/\mathfrak{m}^2_\rho.$$

We study $\widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathrm{an}}[\mathfrak{m}^2_\rho]$, which is equipped with a natural action of $\mathrm{GL}_n(K) \times R_{D,U}$.

Let $\mathcal{E} \hookrightarrow (\mathrm{Spf}\, R_{\overline{\rho},S(U^\wp)})^{\mathrm{rig}} \times \widehat{T}$ be the eigenvariety to $J_B(\widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathrm{an}}_{\overline{\rho}})$. In particular, a point $(\rho', \delta) \in \mathcal{E}$ if and only if $\mathrm{Hom}_{T(K)}(\delta, J_B(\widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathrm{an}}_{\overline{\rho}})[\mathfrak{m}_{\rho'}]) \neq 0$. There is coherent sheaf $\mathcal{M}$ over $\mathcal{E}$ such that $\Gamma(\mathcal{E}, \mathcal{M}) \cong J_B(\widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathrm{an}}_{\overline{\rho}})^\vee$. By (4.10), for $w \in S_n$, we have $x_w = (\mathfrak{m}_\rho, \delta_w\delta_B) \in \mathcal{E}$ (see § 3.2.2 for δ_w). By assumption and [19], $\mathcal{E}$ is étale over the weight space $\mathcal{W}$ (which is the rigid analytic space parametrizing continuous characters of $T(\mathcal{O}_K)$) at the points x_w, and $\mathcal{M}$ is locally free of rank r at each x_w. Let $\mathcal{U}$ be an affinoid smooth neighbourhood of x_w, and $\mathfrak{m}_{\rho,w} \subset \mathcal{O}(\mathcal{U})$ be the maximal ideal associated to x_w. Using global triangulation theory ([34][38]), the étaleness of $\mathcal{E}$ at x_w and Hypothesis 4.12, we have $\mathcal{O}(\mathcal{U})/\mathfrak{m}^2_{\rho,w} \cong R_{D,w,U}$. Moreover, it is not difficult to see there is a $T(K) \times R_{D,w,U}$-equivariant isomorphism $(\mathcal{M}/\mathfrak{m}^2_{\rho,w})^\vee \cong \widetilde{\delta}^{\mathrm{univ},\oplus r}_{w,U}$. We obtain hence a $T(K) \times R_{D,U}$-equivariant injection

$$(\widetilde{\delta}^{\mathrm{univ}}_{w,U}\delta_B)^{\oplus r} \lhook\joinrel\longrightarrow J_B(\widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathbb{Q}_p-\mathrm{an}}_{\overline{\rho}})[\mathfrak{m}^2_\rho].$$

Note that the map induces an isomorphism

$$(\widetilde{\delta}^{\mathrm{univ}}_{w,U}\delta_B)^{\oplus r} \xrightarrow{\sim} J_B(\widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathbb{Q}_p-\mathrm{an}}_{\overline{\rho}})[\mathfrak{m}^2_\rho]\{T(K) = \delta_w\delta_B\}. \tag{4.11}$$

Similarly in Lemma 4.1, the map is balanced and induces using [27, Thm. 0.13] a $\mathrm{GL}_n(K) \times R_{D,w,U}$-equivariant injection

$$\iota_w : (I^{\mathrm{GL}_n(K)}_{B^-(K)}\widetilde{\delta}^{\mathrm{univ}}_{w,U})^{\oplus r} \lhook\joinrel\longrightarrow \widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathbb{Q}_p-\mathrm{an}}_{\overline{\rho}}[\mathfrak{m}^2_\rho]. \tag{4.12}$$

Let $\widetilde{\pi}$ be the subrepresentation of $\widehat{S}_{\xi,\tau}(U^\wp, E)^{\mathbb{Q}_p-\mathrm{an}}_{\overline{\rho}}[\mathfrak{m}^2_\rho]$ generated by $\mathrm{Im}\,\iota_w$ for all w.

Theorem 4.14. *We have a* $\mathrm{GL}_n(K) \times R_{D,U}$*-equivariant isomorphism* $\widetilde{\pi} \cong \pi_1(\phi, \boldsymbol{h})_U^{\mathrm{univ}, \oplus r}$*. Consequently,* $\pi_{\min}(D)^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$*.*

Proof. We first show $\widetilde{\pi} \cong \pi_1(\phi, \mathbf{h})_U^{\mathrm{univ}, \oplus r}$ as $\mathrm{GL}_n(K)$-representation. The injection $\pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ extends uniquely to an injection $\pi_1(\phi, \mathbf{h})^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ (using [15] or (4.12) similarly as in the proof of Corollary 4.3). Note that $\mathrm{Im}\, \iota_w \cap \pi_1(\phi, \mathbf{h})^{\oplus r} \cong \mathrm{PS}_1(w(\phi), \mathbf{h})$ (by the same argument as in the proof of Corollary 4.3). As in the proof of Theorem 4.4, $\widetilde{\pi}$ is isomorphic to an extension of certain copies of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by $\pi_1(\phi, \mathbf{h})^{\oplus r}$. Using (4.12) (and the structure of $\pi_1(\phi, \mathbf{h})_U^{\mathrm{univ}}$), it is not difficult to see $\widetilde{\pi}$ has to be isomorphic to $\pi_1(\phi, \mathbf{h})_U^{\mathrm{univ}, \oplus r}$.

For the part on the $R_{D,U}$-action, it suffices to show any injection

$$\iota : \pi_1(\phi, \mathbf{h})_U^{\mathrm{univ}} \longrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho^2]$$

(extending $\pi_1(\phi, \mathbf{h}) \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$) is $R_{D,U}$-equivariant. As $\pi_1(\phi, \mathbf{h})_U^{\mathrm{univ}}$ is generated by $I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{w,U}^{\mathrm{univ}}$, it suffices to show the restriction of ι at $I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{w,U}^{\mathrm{univ}}$ is $R_{D,U}$-equivariant. It is clear that the $R_{D,U}$ action on the both sides of (4.11) factors through $R_{D,U,w}$. On the other hand, the restriction of ι on $I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{w,U}^{\mathrm{univ}}$ corresponds to a unique $T(K)$-equivariant injection

$$\widetilde{\delta}_{w,U}^{\mathrm{univ}} \delta_B \longrightarrow J_B(\widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho^2]), \tag{4.13}$$

whose image is contained in the right hand side of (4.11). However, any $T(K)$-equivariant injection $\widetilde{\delta}_{w,U}^{\mathrm{univ}} \delta_B \hookrightarrow (\widetilde{\delta}_{w,U}^{\mathrm{univ}} \delta_B)^{\oplus r}$ has to be $R_{D,U,w}$-equivariant, so is (4.13). Thus $\iota|_{I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{w,U}^{\mathrm{univ}}}$ is $R_{D,U,w}$-equivariant for all w, so ι is $R_{D,U}$-equivariant. This finishes the proof. $\square$

Remark 4.15. *As in Remark 4.5 (2) (using results in [15]),* $\pi_{\min}(D)^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ *extends uniquely to* $\pi_{\mathrm{fs}}(D)^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$*.*

Corollary 4.16. $\pi_{\min}(D)^{\oplus r}$ *is the maximal subrepresentation of* $\widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$*, which is generated by extensions of* $\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$ *by* $\pi_1(\phi, \boldsymbol{h})$*.*

Proof. It suffices to prove any extension $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by $\pi_1(\phi, \mathbf{h})$, that is contained in $\widehat{S}_{\xi,\tau}(U^{\wp}, E)[\mathfrak{m}_\rho^2]$, is contained in $\widetilde{\pi}$. Let V be such an extension. Note the composition

$$\pi_1(\phi, \mathbf{h}) \longrightarrow V \longrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)[\mathfrak{m}_\rho^2]$$

factors through $\widetilde{\pi}$, and we let ι be the induced injection. Suppose V is not contained in $\widetilde{\pi}$, we have $V \oplus_{\pi_1(\phi,\mathbf{h}),\iota} \widetilde{\pi} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho^2]$. Using the surjectivity of the last map in (4.8), there is an extension V' of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ such that $V \oplus_{\pi_1(\phi,\mathbf{h}),\iota} \widetilde{\pi} \cong V' \oplus_{\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\iota} \widetilde{\pi}$. However, using (4.11) (and $\dim_E J_B(V')\{T(K) = \delta_w\} = 2$), it is easy to see any such extension has to be contained in the image of (4.12) hence in $\widetilde{\pi}$, a contradiction. $\square$

Corollary 4.17. *For* $D' \in \Phi\Gamma_{\mathrm{nc}}(\phi, \boldsymbol{h})$*,* $\pi_{\min}(D') \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)[\mathfrak{m}_\rho]$ *if and only if* $D'_\sigma \cong D_\sigma$ *for all* $\sigma \in \Sigma_K$*. In particular, when* $K = \mathbb{Q}_p$*, the* $\mathrm{GL}_n(\mathbb{Q}_p)$*-representation* $\widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbb{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ *determines* $\rho_{\widetilde{\wp}}$*.*

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