A SIMPLE PROOF OF DIEUDONNÉ-MANIN CLASSIFICATION THEOREM

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Abstract. The Dieudonné-Manin classification theorem on φ-modules (φ-isocrystals) over a perfect field plays a very important role in p-adic Hodge theory. In this note, in a more general setting we give a new proof of this result, and in the course of the proof, we also give an explicit construction of the Harder-Narasimhan filtration of a φ-module.

Let k be a perfect field of characteristic $p > 0$. The classical Dieudonné-Manin classification theorem (cf.[2]) provides a slope decomposition of a φ-module over the field $W(k)[\frac{1}{p}]$, which is loosely analogous to the eigenspace decomposition of a vector space equipped with a linear transformation. In this note, in a more general setting we give a new proof of this result, and in the course of the proof, we also give an explicit construction of the Harder-Narasimhan filtration of a φ-module. This is without the huge machinery of commutative formal groups in the original proof.

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1. Statement of Dieudonné-Manin classification theorem

Let p be a prime number. For $q = p^f$, let $\mathbb{F}_q$ be the unique finite field of q elements and $\mathbb{Z}_q = W(\mathbb{F}_q)$. Suppose k is a perfect field of characteristic p containing $\mathbb{F}_q$. Suppose E is a complete discrete valuation field of mixed characteristic, $\mathcal{O}_E$ the ring of integers of E, $\mathfrak{m}_E$ the maximal ideal of $\mathcal{O}_E$, π a uniformizing parameter of $\mathfrak{m}_K$, and $k_E = \mathcal{O}_E/\mathfrak{m}_E = F_q$ the residue field of $\mathcal{O}_E$. Recall that

$$W_{\mathcal{O}_E}(k) = \mathcal{O}_E \otimes_{\mathbb{Z}_q} W(k)$$

is the strict $\mathcal{O}_E$-ring over k, i.e., $W_{\mathcal{O}_E}(k)$ is a commutative ring together with an injective homomorphism of rings $\iota : \mathcal{O}_E \to W_{\mathcal{O}_E}(k)$, such that it is complete and

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separated by the π-adic topology and that $W_{\mathcal{O}_E}(k)/\pi W_{\mathcal{O}_E}(k) \cong k$. Set

$$K_0 = W_{\mathcal{O}_E}(k)[\frac{1}{\pi}] \supset W_{\mathcal{O}_E} = W_{\mathcal{O}_E}(k).$$

Note that if $q = p$, $\mathcal{O}_E = \mathbb{Z}_p$ and $\pi = p$, then $W_{\mathcal{O}_E}$ is the usual ring of Witt vectors of k. The Frobenius substitution $\sigma : k \to k,\ \lambda \mapsto \lambda^q$ extends by functoriality to $W_{\mathcal{O}_E}(k)$ and K_0, which we still denote by σ.

Definition 1.1. A φ-module D over K_0 is a finite dimensional K_0-vector space equipped with a bijective σ-semi-linear map φ.

Assume D is a φ-module over K_0 of dimension r. Suppose $\{e_1, \cdots, e_r\}$ is a basis of D over K_0, then $\varphi(e_i) = \sum_{j=1}^{r} a_{ij} e_j$. The matrix of φ under this basis is $A = (a_{ij})_{1\le i,j\le r} \in \mathrm{GL}_r(K_0)$. Suppose $\{e'_1, \cdots, e'_r\}$ is another basis and A' the matrix of φ under this basis, suppose the transformation matrix of these two bases is P, then $A = \sigma(P)A'P^{-1}$. Thus

$$t_N(D) = v_\pi(\det A)$$

is a well defined integer independent of the choice of basis.

Definition 1.2. The *slope* of a φ-module $D \neq 0$ of K_0 is defined to be $\mu(D) = \frac{t_N(D)}{\dim_{K_0} D}$.

A φ-module D is called *pure of slope* μ (or *isoclinic*) if there exists a $W_{\mathcal{O}_E}$-lattice M of D such that $\pi^{-d}\varphi^h(M) = M$ where $\mu = \frac{d}{h}$, $d, h \in \mathbb{Z}$ and $h \ge 1$.

Remark. (i) A φ-module pure of slope 0 is nothing but an étale φ-module over K_0 (see [1]).

(ii) Suppose $D = K_0 e_1 \oplus \cdots \oplus K_0 e_n$, $\varphi(e_i) = e_{i+1}$ for $1 \le i \le n-1$ and $\varphi(e_n) = pe_1$, then D is pure of slope $\frac{1}{n}$.

The aim of this note is to give a simple proof of the following theorem of Dieudonné-Manin which classifies all φ-modules.

Theorem 1.3 (Dieudonné-Manin)**.** *For a φ-module D over K_0, then*

$$D = \bigoplus_{\mu\in\mathbb{Q}} D_\mu,$$

where D_μ is the part of D pure of slope μ and $D_\mu = 0$ for all but finitely many μ. Hence $\mu \dim_{K_0} D_\mu \in \mathbb{Z}$ and

$$t_N(D) = \sum_{\mu\in\mathbb{Q}} \mu \dim_{K_0} D_\mu.$$

Remark. By Fontaine's theory of mod-p representations (cf. [1], Chapter 2), if k is algebraically closed and if D is pure of slope $\mu = \frac{d}{h}$ with $d, h \in \mathbb{Z}, h \ge 1$, then $D \cong K_0 \otimes_{\mathbb{Q}_{p^h}} D_{\varphi^h = p^d}$.

2. Proof of the classification Theorem

Suppose D is a φ-module. For $h, d \in \mathbb{Z}$ and $h \ge 1$, we write $\varphi_{h,d} = \pi^{-d}\varphi^h$. Then $\varphi_{h,d}$ is bijective in D. Let M be a $W_{\mathcal{O}_E}$-lattice of D, we set $M_{h,d} = \cap_{n\ge 0}\varphi_{h,d}^{-n}(M)$ and $D^\mu = M_{h,d}[\frac{1}{\pi}]$ where $\mu = d/h \in \mathbb{Q}$. Clearly by definition $M_{h,d}$ is a sub-$W_{\mathcal{O}_E}$-module of M stable under $\varphi_{h,d}$.

Proposition 2.1. *Suppose D is a φ-module over K_0, $\mu = \frac{d}{h} \in \mathbb{Q}$. Then*

(1) D^μ is independent of the choices of the lattice M and the pair (h, d).

(2) $x \in D^\mu$ if and only if the $W_{\mathcal{O}_E}$-module $W_{\mathcal{O}_E}[x, \varphi_{h,d}(x) \cdots, \varphi^n_{h,d}(x), \cdots]$ is a finite $W_{\mathcal{O}_E}$-module, in particular D^μ is a φ-submodule of D.

(3) $\{D^\mu\}_{\mu\in\mathbb{Q}}$ forms a decreasing filtration of D which is separate and exhaustive, in other words,

- (i) *If $\mu \leq \mu'$, then $D^\mu \supset D^{\mu'}$;*
- (ii) *$D^\mu = D$ for $\mu \ll 0$ and $D^\mu = 0$ for $\mu \gg 0$.*

Proof. (1) Suppose $M' = TM$ is another lattice of D where $T \in \mathrm{GL}(D)$. We choose $k \in \mathbb{N}$ such that $TM \supset \pi^k M$. For $x \in M_{h,d}[\frac{1}{\pi}]$, suppose $\pi^a x \in M_{h,d}$, then $\varphi^n_{h,d}(\pi^a x) \in M$ for all $n \in \mathbb{N}$ and $\varphi^n_{h,d}(\pi^{a+k}x) \in \pi^k M \subset M'$ for all $n \in \mathbb{N}$, thus $\pi^{a+k}x \in M'_{h,d}$ and $x \in M'_{h,d}[\frac{1}{\pi}]$. This proves the independence of M.

Now for $(h', d') = (kh, kd)$, we let $M' = \cap_{0\leq j\leq k-1}\varphi^j_{h,d}(M)$. Then M' is a lattice in D and $M'_{kh,kd} = M_{h,d}$. Thus $M_{kh,kd}[\frac{1}{\pi}] = M'_{kh,kd}[\frac{1}{\pi}] = M_{h,d}[\frac{1}{\pi}]$. This proves the independence of the pair (h, d).

(2) Let $\mu = \frac{d}{h}$. Suppose M is a lattice in D. Then $x \in D^\mu$ means that there exists $k \in \mathbb{N}$, $\pi^k x \in M_{h,d}$, or equivalently $\varphi^n_{h,d}(\pi^k x) \in M$ for $n \in \mathbb{N}$, so $W_{\mathcal{O}_E}[x, \varphi_{h,d}(x) \cdots, \varphi^n_{h,d}(x), \cdots] \supset \pi^{-k}M$ is a finite $W_{\mathcal{O}_E}$-module. Conversely, if the $W_{\mathcal{O}_E}$-module $W_{\mathcal{O}_E}[x, \varphi_{h,d}(x) \cdots, \varphi^n_{h,d}(x), \cdots]$ is a finite $W_{\mathcal{O}_E}$-module, we extend it to a $W_{\mathcal{O}_E}$-lattice M of D, then $x \in M_{h,d} \subset D^\mu$.

(3) If $d < d'$, then by definition $M_{h,d} \supset M_{h,d'}$, this proves (i). Suppose $\pi^{d_2}M \subset \varphi(M) \subset \pi^{d_1}M$, then for $d > d_2$, $M_{1,d} = 0$ and for $d < d_1$, $M_{1,d} = M$, this proves (ii). □

Remark. Suppose $D = D_{a,b} = K_0e_1 \oplus K_0e_2$, $\varphi(e_1) = e_2$ and $\varphi(e_2) = ae_1 + be_2$, a natural question is to compute $D^\mu_{a,b}$ for $\mu \in \mathbb{Q}$. At present we don't know the answer.

Lemma 2.2. *Suppose $0 \to D_1 \to D \to D_2 \to 0$ is a short exact sequence of φ-modules, then*

(1) the sequence $0 \to D^\mu_1 \to D^\mu \to D^\mu_2$ is exact;

(2) if moreover $D_1 = D^{\mu_0}$ for some μ_0, then $0 \to D^\mu_1 \to D^\mu \to D^\mu_2 \to 0$ is exact.

Proof. (1) follows easily from Proposition 2.1(2).

(2) The case $\mu > \mu_0$ follows from the case $\mu = \mu_0$. So we need only to prove the exactness in the case $\mu \leq \mu_0$. We first show the case $\mu = \mu_0$, which is equivalent to the claim $(D/D^{\mu_0})^{\mu_0} = 0$. We assume $D = D^\lambda$, $\mu_0 = \frac{d_0}{h}$ and $\lambda = \frac{d}{h}$.

We claim there exists a $W_{\mathcal{O}_E}$-lattice M in D such that M is stable under $\varphi_{h,d}$ and $M \cap D^{\mu_0}$ is stable under φ_{h,d_0}. To see this, we first find a $W_{\mathcal{O}_E}$-lattice L in D which is stable under $\varphi_{h,d}$, then the image of L in D/D^{μ_0} is a $W_{\mathcal{O}_E}$-lattice. Suppose it is generated by $\bar{e}_1, \bar{e}_2, \cdots \bar{e}_r$. For each i, take a preimages of $\bar{e}_i$ in L, denoted by e_i. Choose a $W_{\mathcal{O}_E}$-lattice L_0 in D^{μ_0} which is stable under φ_{h,d_0}. Then there exists $N \in \mathbb{N}$, such that $L \cap D^{\mu_0} \subseteq \pi^{-N}L_0$. Take $e_{r+1}, e_{r+2}, \cdots e_n$ as a basis of $\pi^{-N}L_0$. (Note that $\pi^{-N}L_0$ is still stable under φ_{h,d_0}). Then the lattice M generated by $e_1, e_2, \cdots e_n$ is what we need. That's because $\varphi_{h,d}(e_i) \in L \subseteq M$ when $i \leq r$, and $\varphi_{h,d}(e_i) = \pi^{d_0-d}\varphi_{h,d_0}(e_i) \in \pi^{-N}L_0 \subseteq M$ when $i \geq r + 1$.

If $(D/D^{\mu_0})^{\mu_0} \neq 0$, then there exists $x \in D, x \notin D^{\mu_0}$, $\varphi^n_{h,d_0}(x) \in M + D^{\mu_0}$ for any n. For $n \geq 1$, let k_n be the smallest integer such that $\varphi^n_{h,d_0}(x) = x_n + \pi^{-k_n}y_n$

where $x_n \in M$, $y_n \in M \cap D^{\mu_0}$ (if $\varphi^n_{h,d_0}(x) \in M$, let $k_n = 0$). In fact, k_n is also the smallest integer such that $\varphi^n_{h,d_0}(x) \in \pi^{-k_n} M$.

We have $\varphi_{h,d_0}(x_n + \pi^{-k_n} y_n) = x_{n+1} + \pi^{-k_{n+1}} y_{n+1} = \varphi_{h,d_0}(x_n) + \pi^{-k_n} z_n$, where $z_n \in M \cap D^{\mu_0}$. Since $\varphi_{h,d_0}(M) \subseteq \pi^{-(d_0-d)} M$, it's easy to see $k_{n+1} \le \max(k_n, d_0 - d)$. Take $N = \max(k_1, d_0 - d)$, then $k_n \le N$ is bounded. This implies that $\pi^N x \in \cap_{n \ge 0} \varphi^{-n}_{h,d_0}(M)$. Hence $\pi^N x$ and $x \in D^{\mu_0}$, a contradiction. Thus we have shown $(D/D^{\mu_0})^{\mu_0}$=0.

Now for the case $\mu < \mu_0$, if $D^\mu = D$, then by (1), $D/D^{\mu_0} \supseteq (D/D^{\mu_0})^\mu \supseteq D^\mu/(D^{\mu_0})^\mu = D/D^{\mu_0}$, so all must be equal. In the general case, the exact sequence

$$0 \to D^\mu/D^{\mu_0} \to D/D^{\mu_0} \to D/D^\mu \to 0.$$

and the fact $(D/D^\mu)^\mu = 0$ implies that $(D^\mu/D^{\mu_0})^\mu = (D/D^{\mu_0})^\mu$. Together with $(D^\mu/D^{\mu_0})^\mu = D^\mu/D^{\mu_0}$, we get $(D/D^{\mu_0})^\mu = D^\mu/D^{\mu_0}$. □

For any $\mu \in \mathbb{Q}$, we let $D^{>\mu}$ be the union of all $D^{\mu'}$ for $\mu' > \mu$ and $D^{<\mu}$ be the intersection of all $D^{\mu'}$ for $\mu' < \mu$.

Lemma 2.3. *(1) For any μ, there exists $\mu' < \mu$, $D^{\mu'} = D^\mu$. In particular, the filtration $\{D^\mu\}$ is left continuous, i.e., $D^{<\mu} = D^\mu$.*

(2) For $\mu = \frac{d}{h}$ and $\dim_{K_0} D^\mu$=l, if $D^\mu = D^{>\mu}$, then $D^{\mu'} = D^\mu$ where $\mu' = \frac{ld+1}{lh}$.

Proof. (1) By Lemma 2.2(2), we can replace D by D/D^μ and assume $D^\mu = 0$. Let $\mu = \frac{d}{h}$. Take a lattice M in D, then $\bigcap_{i=0}^\infty \varphi^{-i}_{h,d}(M) = 0$, and there exists k such that $\bigcap_{i=0}^k \varphi^{-i}_{h,d}(M) \subseteq \pi^2 M$. One can show easily that $\bigcap_{i=0}^{Nk} \varphi^{-i}_{h,d}(M) \subseteq \pi^{2N} M$ for $N \ge 1$ by induction.

Let L be the lattice $\bigcap_{i=0}^k \varphi^{-i}_{h,d}(M)$, Then $\varphi^{-j}_{kh,kd}(L) = \bigcap_{i=kj}^{k(j+1)} \varphi^{-i}_{h,d}(M)$ and

$$\bigcap_{i=0}^{j} \varphi^{-i}_{kh,kd}(L) = \bigcap_{i=0}^{k(j+1)} \varphi^{-i}_{h,d}(M) \subseteq \pi^{2(j+1)} M.$$

So we have

$$\bigcap_{i=0}^{j} \varphi^{-i}_{kh,kd-1}(L) = \bigcap_{i=0}^{j} \pi^{-i} \varphi^{-i}_{kh,kd}(L) \subseteq \bigcap_{i=0}^{j} \pi^{-j} \varphi^{-i}_{kh,kd}(L) \subseteq \pi^j M.$$

As a consequence $\bigcap_{i=0}^\infty \varphi^{-i}_{kh,kd-1}(L) = 0$, which implies that $D^{\mu'} = 0$ for $\mu' = \frac{kd-1}{kh}$.

(2) By Lemma 2.2(1), we can replace D by $D \cap D^\mu$ and assume $D = D^\mu$. The fact $D^{>\mu} = D$ implies that there exists $\alpha \in \mathbb{N}$, $D^{\frac{\alpha d+1}{\alpha h}} = D$. Therefore we have a lattice M which is stable under $\varphi_{\alpha h, \alpha d+1}$, and consequently stable under $\varphi_{\alpha h, \alpha d}$. It's easy to see $\varphi^n_{\alpha h, \alpha d}(M) = \varphi^n_{\alpha h, \alpha d+1}(\pi^n M) \to 0$ as $n \to \infty$. Therefore for any lattice L stable under $\varphi_{h,d}$, $\varphi^n_{h,d}(L) \to 0$ as $n \to \infty$; in particular, $\varphi^n_{h,d}(L) \subset \pi L$ when n is sufficiently large.

If L is stable under $\varphi_{h,d}$, then $\varphi^i_{h,d}(L) \supset \varphi^{i+1}_{h,d}(L)$, and there exists a chain of sub-k-vector spaces of $L/\pi L$

$$\frac{L}{\pi L} \supset \dots \frac{\varphi^{i-1}_{h,d}(L)}{\varphi^i_{h,d}(L) \cap \pi L} \supset \frac{\varphi^i_{h,d}(L)}{\varphi^i_{h,d}(L) \cap \pi L} \supset \frac{\varphi^{i+1}_{h,d}(L)}{\varphi^{i+1}_{h,d}(L) \cap \pi L} \supset \cdots.$$

It's easy to check that if $\dim_k \frac{\varphi^i_{h,d}(L)}{\varphi^i_{h,d}(L)\cap \pi L} = \dim_k \frac{\varphi^{i+1}_{h,d}(L)}{\varphi^{i+1}_{h,d}(L)\cap \pi L}$, then $\dim_k \frac{\varphi^j_{h,d}(L)}{\varphi^j_{h,d}(L)\cap \pi L} = \dim_k \frac{\varphi^i_{h,d}(L)}{\varphi^i_{h,d}(L)\cap \pi L}$ for any $j > i$. Since $\dim_k \frac{\varphi^j_{h,d}(L)}{\varphi^j_{h,d}(L)\cap \pi L} = 0$ when j is sufficiently large, the fact $\dim_k \frac{L}{\pi L} = l$ implies that $\varphi^l_{h,d}(L) \subseteq \pi L$. This means that L is stable under $\varphi_{lh,ld+1}$ and hence $D^{\frac{ld+1}{lh}} = D$. □

Corollary 2.4. *Let $a = \sup\{\lambda \in \mathbb{Q} : D^\lambda = D\}$, then a is a rational number and $D^a = D$.*

Proof. Suppose $\dim_{K_0} D = l$. If a is not rational, by Dirichlet's Approximation Theorem, there exist infinitely many pairs of integers (p, q) such that $\frac{p}{q} < a < \frac{p}{q} + \frac{1}{q^2}$. Choose $q > l$ and let $(p, q) = (d, h)$. By the above Lemma, $D^{\frac{d}{h}+\frac{1}{lh}} = D^{\frac{d}{h}} = D$ and hence $\frac{d}{h} + \frac{1}{lh} < a$, a contradiction.

The second part of the corollary follows from Lemma 2.3(1). □

Proposition 2.5. *Set $\mathrm{gr}_\mu D = D^\mu/D^{>\mu}$, then $\mathrm{gr}_\mu D$ is pure of slope μ.*

Proof. By Lemma 2.2, we can replace D by $D^\mu/D^{>\mu}$, and assume $D^\mu = D$ and $D^{>\mu} = 0$.

Let $\mu = \frac{d}{h}$, then there exists a $W_{\mathcal{O}_E}$-lattice M of $D^\mu = D$ which is stable under $\varphi_{h,d}$. The filtration of sub-k-vector spaces

$$\cdots \subseteq \frac{\varphi^n_{h,d}(M)}{\varphi^n_{h,d}(M)\cap \pi M} \subseteq \frac{\varphi^n_{h,d}(M)}{\varphi^n_{h,d}(M)\cap \pi M} \subseteq \cdots \subseteq \frac{M}{\pi M}$$

of $M/\pi M$ is stable since $\dim_k M/\pi M = \dim_{K_0} D$ is finite.

If $\frac{\varphi^N_{h,d}(M)}{\varphi^N_{h,d}(M)\cap \pi M} = 0$ when N is sufficiently large, then $\varphi^n_{Nh,Nd}(M) \subseteq \pi^n M$ for all $n \in \mathbb{N}$, which implies that $M \subseteq \cap_{n\geq 0}\varphi^{-n}_{Nh,Nd+1}(M)$. This is not possible since $D^{>\mu} = 0$. As a consequence, when N is sufficiently large, we have a bijection of the nonzero k-vector space $\frac{\varphi^N_{h,d}(M)}{\varphi^N_{h,d}(M)\cap \pi M}$ to itself

$$\varphi^n_{h,d} : \frac{\varphi^N_{h,d}(M)}{\varphi^N_{h,d}(M)\cap \pi M} \to \frac{\varphi^N_{h,d}(M)}{\varphi^N_{h,d}(M)\cap \pi M}$$

for $n \in \mathbb{N}$. Replace (h, d) by (Nh, Nd) and still denote it by (h, d), then we get a bijection

$$\varphi^n_{h,d} : \frac{\varphi_{h,d}(M)}{\varphi_{h,d}(M)\cap \pi M} \to \frac{\varphi_{h,d}(M)}{\varphi_{h,d}(M)\cap \pi M}$$

for any $n \in \mathbb{N}$.

If $\varphi_{h,d} : M \to M$ is not bijective, then there exists x_1 satisfying $\varphi_{h,d}(x_1) \in \pi M$ and $x_1 \notin \pi M$. Indeed, if $\varphi_{h,d} : M \to M$ is not surjective, we can find an element $x \in M$ and $x \notin \varphi_{h,d}(M)$. Since $\varphi_{h,d}(M)$ is still a $W_{\mathcal{O}_E}$-lattice in D, we can find $k \in \mathbb{N}$ such that $\pi^k x \in \varphi_{h,d}(M)$, and $\pi^{k-1}x \notin \varphi_{h,d}(M)$. Then take $x_1 \in M$ to be the preimage of $\pi^k x$.

We now construct by induction a sequence (x_n) such that $x_n - x_{n-1} \in \pi^{n-1}M$ and $\varphi^i_{h,d}(x_n) \in \pi^i M$ for any $1 \leq i \leq n$. Suppose $x_1, x_2 \cdots x_n$ have been constructed and $\varphi^n_{h,d}(x_n) = \pi^n z_n$. Let $x_{n+1} = x_n + \pi^n y$. It's easy to see $\varphi^i_{h,d}(x_{n+1}) \in \pi^i M$ for $1 \leq i \leq n$ if $y \in M$. Since $\varphi^{n+1}_{h,d}(x_{n+1}) = \pi^n(\varphi_{h,d}(z_n) + \varphi^{n+1}_{h,d}(y))$, to have

$\varphi_{h,d}^{n+1}(x_{n+1}) \in \pi^{n+1}M$, it's sufficient to find $y \in M$ such that $\varphi_{h,d}(z_n) + \varphi_{h,d}^{n+1}(y) \in \pi M$, but this is guaranteed by the bijection

$$\varphi_{h,d}^n : \frac{\varphi_{h,d}(M)}{\varphi_{h,d}(M) \cap \pi M} \to \frac{\varphi_{h,d}(M)}{\varphi_{h,d}(M) \cap \pi M}.$$

Take $x = \lim_{n\to\infty} x_n$, then $x \in M$, $x \neq 0$. It's easy to see $\varphi_{h,d}^n(x) \in \pi^n M$ for any $n \geq 0$, so $x \in \cap_{n\geq 0}\varphi_{h,d+1}^{-n}(M)$ which contradicts to $D^{>\mu} = 0$. □

Since D is of finite dimension, $\mathrm{gr}_\mu D = 0$ for all but finitely many μ. Suppose $\mu_1 > \mu_2 > \cdots > \mu_r$ are all the μ's such that $\mathrm{gr}_\mu D \neq 0$. In fact we can take $\mu_1 = \sup\{\lambda \in \mathbb{Q} : D^\lambda \neq 0\}$ and $\mu_i = \sup\{\lambda \in \mathbb{Q} : D^\lambda \supsetneq D^{\mu_{i-1}}\}$ when $i > 1$. By Lemma 2.3 (1), $D^{\mu_i} \supsetneq D^{\mu_{i-1}}$, and if $\mu_i > \mu > \mu_{i+1}$, then $D^\mu = D^{\mu_i}$. We have

Proposition 2.6. *Suppose D is a φ-module. Then the filtration*

$$0 \subsetneq D^{\mu_1} = \mathrm{gr}_{\mu_1} D \subsetneq D^{\mu_2} \subsetneq \cdots \subsetneq D^{\mu_r} = D$$

is the Harder-Narasimhan filtration *of D, i.e., the unique filtration $\cdots \subsetneq D_i \subsetneq D_{i+1} \subsetneq \cdots$ of φ-modules such that the D_i/D_{i-1}'s are pure of strictly decreasing slopes.*

Proof. The existence follows from Proposition 2.5. For the uniqueness, by Lemma 2.2, for a Harder-Narasimhan filtration $0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_s = D$ of D, then $D^\mu = 0$ for $\mu > \mu(D_1)$ and $D^{\mu(D_1)} = D_1 \neq 0$. We also have $D^\mu = 0$ for $\mu > \mu_1$ and $D^{\mu_1} \neq 0$. Thus $\mu(D_1) = \mu_1$ and $D_1 = D^{\mu_1}$. Now the rest follows from induction on the length of the filtration. □

Proposition 2.7. *Suppose $0 \to D_1 \to D \to D_2 \to 0$ is a short exact sequence of φ-modules, then for every $\mu \in \mathbb{Q}$, $0 \to D_1^\mu \to D^\mu \to D_2^\mu \to 0$ is also exact.*

Proof. We prove by induction on the dimension of D. The case $\dim D = 1$ is trivial. In general, suppose $\dim D \geq 2$ and D_1 is a non-zero proper sub-object of D. We assume D' is the second to last term of the Harder-Narasimhan filtration of D, and $D'' = D/D'$, then for the exact sequence $0 \to D' \to D \to D'' \to 0$ and $\mu \in \mathbb{Q}$, the complex $0 \to D'^\mu \to D^\mu \to D''^\mu \to 0$ is always exact. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1' & \longrightarrow & D' & \longrightarrow & D_2' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i_1} & & \\
0 & \longrightarrow & D_1 & \longrightarrow & D & \longrightarrow & D_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1'' & \xrightarrow{i_2} & D'' & \longrightarrow & D_2'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $D_1' = D_1 \cap D'$ and $D_2' = D'/D_1'$ and $D_1'' = D_1/D_1'$, the injections i_1 and i_2 are defined by diagram chasing, and $D_2'' = D''/D_1'' \cong D_2/D_2'$ is obtained by snake lemma. Now take the μ-invariant of the above diagram, by induction, we have exact sequences in all rows and columns except the middle row, then the middle row must also be exact by diagram chasing. □

Proof of Theorem 1.3. We are now ready to prove the theorem of Dieudonné-Manin. Suppose D is a φ-module over k, such that

$$0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_{r-1} \subsetneq D_r = D$$

is the Harder-Narasimhan filtration of D, suppose $\mu_i = \mu(D_i/D_{i-1})$. Since φ is bijective on D, replace φ and σ by φ^{-1} and σ^{-1}, then D can be regarded as a φ^{-1}-module and we can develop the Harder-Narasimhan filtration for D as a φ^{-1}-modules, i.e., D possesses a unique filtration

$$0 = D'_0 \subsetneq D'_1 \subsetneq \cdots \subsetneq D'_{s-1} \subsetneq D'_s = D$$

such that D'_i/D'_{i-1} are pure of slope $\mu'_i = \mu'(\varphi^{-1}, D'_i/D'_{i-1})$ as φ^{-1}-modules and μ'_i's are strictly decreasing. By definition we see that a φ^{-1}-module pure of slope μ is nothing but a φ-module pure of slope $-\mu$, thus $0 = D'_0 \subsetneq D'_1 \subsetneq \cdots \subsetneq D'_{s-1} \subsetneq D'_s = D$ is the unique filtration of D such that the sequences $\mu(D'_i/D'_{i-1}) = -\mu'_i$ are strictly increasing.

It suffices to show that $D = \oplus(D_i/D_{i-1})$. We show it by induction on the length s of the (φ^{-1})- Harder-Narasimhan filtration of D. The case $s = 1$ is trivial. In general, we have $D^{\mu} = 0$ for $\mu > \mu_1$ and $D^{\mu_1} = D_1 \neq 0$. By Proposition 2.7 and induction hypothesis, we also have $D^{\mu} = 0$ for $\mu > -\mu'_s$ and $D^{-\mu'_s} \cong D/D'_{s-1} \neq 0$, thus $\mu_1 = -\mu'_s$ and $D_1 \cong D/D'_{s-1}$ is a direct summand of D. By induction, this finishes the proof of the theorem. □

References

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