# Change of weights for locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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**Abstract**

Let $D_1 \subset D_2$ be $(\varphi, \Gamma)$-modules of rank 2 over the Robba ring, and $\pi(D_1)$, $\pi(D_2)$ be the associated locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ via the $p$-adic local Langlands correspondence. We describe the relation between $\pi(D_1)$ and $\pi(D_2)$.

## Contents



## 1 Introduction and notation

Let $E$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{R}_E$ be the Robba ring of $\mathbb{Q}_p$ with $E$-coefficients. Let $D$ be an indecomposable $(\varphi, \Gamma)$-module of rank 2 over $\mathcal{R}_E$. By [5, Thm. 0.1], the (locally analytic) $p$-adic Langlands correspondence associates to $D$ a locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $E$. One phenomenon on the Galois side is that the $(\varphi, \Gamma)$-module $D$ has (infinitely) many $(\varphi, \Gamma)$-submodules $D'$, including trivial ones $\{t^i D\}_{i \in \mathbb{Z}_{\geq 0}}$ and some non-trivial ones discussed below. In this note, we describe the relation between $\pi(D)$ and $\pi(D')$. Note that the correspondence $D \mapsto \pi(D)$ is compatible with twisiting by characters. In particular, if $D' = t^i D$, then $\pi(D') \cong \pi(D) \otimes_E z^i \circ \det$ (and we ignore det when there is no ambiguity).

Twisting by a certain character, we can and do assume $D$ has Sen weights $(0, \alpha)$ with $\alpha \in E \setminus \mathbb{Z}_{<0}$. For $k \in \mathbb{Z}_{\geq 1}$, denote by $V_k := \mathrm{Sym}^k E^2$ the $k$-th symmetric product of the standard representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. For a locally analytic representation $V$, we use $V^*$ to denote its strong continuous dual. Let $\mathfrak{c} \in \mathrm{U}(\mathfrak{gl}_2)$ be the Casimir operator.

**Theorem 1.1.** *(1) Assume* $\mathrm{End}(D) \cong E$*. Assume $\alpha \neq 0$, or $\alpha = 0$ and $D$ not de Rham, then $D$ admits a unique $(\varphi, \Gamma)$-submodule $D_{(0,\alpha+k)}$ of Sen weights $(0, \alpha + k)$ and we have ($[-]$ denoting the eigenspace)*

$$\pi(D_{(0,\alpha+k)})^* \cong (\pi(D)^* \otimes_E V_k)[\mathfrak{c} = (\alpha + k)^2 - 1].$$

*(2) Assume $\alpha = 0$ and $D$ is de Rham non-trianguline, then $(\pi(D)^* \otimes_E V_k)[\mathfrak{c} = k^2 - 1] \cong \pi(D, k)^*$ and we have an exact sequence ($\{-\}$ denoting the generalized eigenspace)*

$$0 \to \pi(D, k)^* \to (\pi(D)^* \otimes_E V_k)\{\mathfrak{c} = k^2 - 1\} \to \pi(D, -k)^* \to 0$$

*where $\pi(D, i)$ denotes Colmez's representations in [6] (for $D = \Delta$ of loc. cit.).*

**Remark 1.2.** *(1) Some parts of Theorem 1.1 (1) in trianguline case were obtained in [12, Thm. 5.2.11].*

*(2) A similar statement in Theorem (2) also holds in trianguline case, see Remark 3.7 (3).*

*(3) Suppose we are in the case (2), and let $\pi_\infty(D)$ be the smooth representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *associated to $D$ via the classical local Langlands correspondence. By [6, Thm. 0.6 (iii)], for any $(\varphi, \Gamma)$-submodule $D'$ of $D$ of Sen weights $(0, k)$, we have*

$$0 \to \pi(D')^* \to \pi(D, k)^* \to (\pi_\infty(D) \otimes_E V_k)^* \to 0.$$

*And the map $D' \to \pi(D')^*$ gives a one-to-one correspondence between the $(\varphi, \Gamma)$-submodules of $D$ of Sen weights $(0, k)$ and the subrepresentations of $\pi(D, k)^*$ of quotient $(\pi_\infty(D) \otimes_E V_k)^*$.*

*(4) Assume $D$ is not trianguline, the theorem allows to reconstruct Colmez's magical operator $\partial$ in [6, Thm. 0.8] and generalize it to the general (irreducible) setting. Indeed, when $D$ is as in Theorem 1.1 (2), the composition (where the second map is induced by the map $V_k \to E$, $\sum_{i=0}^k a_i e_1^i \otimes e_2^{k-i} \mapsto a_0$)*

$$(\pi(D, k)^* \otimes_E V_k)[\mathfrak{c} = k^2 - 1] \hookrightarrow \pi(D)^* \otimes_E V_k \twoheadrightarrow \pi(D)^*$$

*is an isomorphism of topological vector spaces. The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-action on $(\pi(D)^* \otimes_E V_k)[\mathfrak{c} = k^2 - 1]$ induces then a twisted* $\mathrm{GL}_2(\mathbb{Q}_p)$*-action on the space $\pi(D)^*$, and gives Colmez's formulas in the construction of $\pi(D, k)^*$. See § 3.4 for more details.*

Recall that a key ingredient in the construction of $\pi(D)$ is a delicate involution $w_D$ on $D^{\psi=0}$. When $D$ is irreducible, $w_D$ was obtained by continuously extending an involution on $(D^{\mathrm{int}})^{\psi=0}$, where $D^{\mathrm{int}}$ is the associated $(\varphi, \Gamma)$-module over $\mathcal{R}_E^{\mathrm{int}} := B_{\mathbb{Q}_p}^\dagger \otimes_{\mathbb{Q}_p} E$. Let $D' \subset D$ be a submodule of weight $(0, \alpha + k)$. Then $\nabla_k := (\nabla - k + 1) \cdots (\nabla - 1) \nabla D' \subset t^k D$, and we denote by $\frac{\nabla_k}{t^k} : D' \to D$ the map sending $x$ to $t^{-k} \nabla_k(x)$. The involutions $w_D$ and $w_{D'}$ have the following simple relation (though they are in general not comparable when restricted to $D^{\mathrm{int}}$ and $(D')^{\mathrm{int}}$):

**Corollary 1.3.** *We have $w_{D'} = w_D \circ \frac{\nabla_k}{t^k}$.*

We give a sketch of the proof of Theorem 1.1. The key ingredient is an operation, that we call *translation*, on $(\varphi,\Gamma)$-modules. The $\begin{pmatrix}\mathbb{Z}_p\setminus\{0\} & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}$-action on $V_k$ induces an $\mathcal{R}_E^+$-module structure on $V_k$ together with a semi-linear $(\varphi,\Gamma)$-action. In fact, we have $V_k\cong \mathcal{R}_E^+/X^{k+1}$. For a $(\varphi,\Gamma)$-module $D$ over $\mathcal{R}_E$, consider $D\otimes_E V_k$, equipped with the diagonal $\mathcal{R}_E^+$-action and $(\varphi,\Gamma)$-action (noting $\mathcal{R}_E^+$ has a natural coalgebra structure). One shows that the $\mathcal{R}_E^+$-action on $D\otimes_E V_k$ uniquely extends to an $\mathcal{R}_E$-action. In particular $D\otimes_E V_k$ is also a $(\varphi,\Gamma)$-module over $\mathcal{R}_E$.

Now let $D$ be as in Theorem 1.1, then $D$ is naturally equipped with a $\mathfrak{gl}_2$-action. We equip $D\otimes_E V_k$ with a diagonal $\mathfrak{gl}_2$-action. The Casimir $\mathfrak{c}$ turns out to be an endomorphism of $(\varphi,\Gamma)$-modules of $D\otimes_E V_k$. In particular, we can decompose $D\otimes_E V_k$ into generalized eigenspaces of $\mathfrak{c}$, which are $(\varphi,\Gamma)$-submodules over $\mathcal{R}_E$. We study the decomposition in § 2.2. For example, we show that if $D$ is as in Theorem 1.1 (1), then $D_{(0,\alpha+k)}\cong (D\otimes_E V_k)[\mathfrak{c}=(\alpha+k)^2-1]$.

By [5, Thm. 0.1], there is a unique $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf over $\mathbb{P}^1(\mathbb{Q}_p)$ of central character $\delta_D$ (which satisfies $\delta_D\varepsilon=\wedge^2 D$, $\varepsilon$ being the cyclotomic character), associated to $D$, whose global sections $D\boxtimes_{\delta_D}\mathbb{P}^1(\mathbb{Q}_p)$ sit in an exact sequence

$$0\to \pi(D)^*\otimes_E\delta_D\to D\boxtimes_{\delta_D}\mathbb{P}^1(\mathbb{Q}_p)\to \pi(D)\to 0. \tag{1}$$

It turns out that this construction is quite compatible with translations. Namely, to $D\otimes_E V_k$, one can naturally associate a $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf over $\mathbb{P}^1(\mathbb{Q}_p)$ (of central character $\delta_D z^k$) whose global sections $(D\otimes_E V_k)\boxtimes_{\delta_D z^k}\mathbb{P}^1(\mathbb{Q}_p)$ are $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariantly isomorphic to $(D\boxtimes_{\delta_D}\mathbb{P}^1(\mathbb{Q}_p))\otimes_E V_k$. Suppose $D$ is as in Theorem 1.1 (1), we then have (noting $\delta_D z^k=\delta_{D_{(0,\alpha+k)}}$)

$$\begin{aligned}((D\boxtimes_{\delta_D}\mathbb{P}^1(\mathbb{Q}_p))\otimes_E V_k)[\mathfrak{c}=(\alpha+k)^2-1]&\cong((D\otimes_E V_k)[\mathfrak{c}=(\alpha+k)^2-1])\boxtimes_{\delta_D z^k}\mathbb{P}^1(\mathbb{Q}_p)\\ &\cong D_{(0,\alpha+k)}\boxtimes_{\delta_{D_{(0,\alpha+k)}}}\mathbb{P}^1(\mathbb{Q}_p).\end{aligned} \tag{2}$$

Using the isomorphism and (1), one can deduce Theorem 1.1 (1). Theorem 1.1 (2) follows by similar arguments. A main difference is that in this case, the translations can only produce $(\varphi,\Gamma)$-submodules $t^iD$. For example, $(D\otimes_E V_k)[\mathfrak{c}=k^2-1]\cong D$ (noting the $\mathfrak{gl}_2$-actions are however different), and $(D\otimes_E V_k)\{\mathfrak{c}=k^2-1\}\cong D\oplus t^kD$ (again, just as $(\varphi,\Gamma)$-module). Similarly as in (2), we deduce

$$((D\boxtimes_{\delta_D}\mathbb{P}^1(\mathbb{Q}_p))\otimes_E V_k)[\mathfrak{c}=k^2-1]\cong D\boxtimes_{\delta_D z^k}\mathbb{P}^1(\mathbb{Q}_p)$$

and an exact sequence

$$0\to D\boxtimes_{\delta_D z^k}\mathbb{P}^1(\mathbb{Q}_p)\to((D\boxtimes_{\delta_D}\mathbb{P}^1(\mathbb{Q}_p))\otimes_E V_k)\{\mathfrak{c}=k^2-1\}\to t^kD\boxtimes_{\delta_D z^k}\mathbb{P}^1(\mathbb{Q}_p)\to 0.$$

Theorem 1.1 (2) follows then from these together with results in [6]. We refer to the context for details.

## Notation

Let $\varepsilon$ be the cyclotomic character of $\mathrm{Gal}_{\mathbb{Q}_p}$ and of $\mathbb{Q}_p^\times$.

We use the following notation for the Lie algebra $\mathfrak{gl}_2$ of $\mathrm{GL}_2(\mathbb{Q}_p)$: $\mathfrak{h}:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $a^+:=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $a^-:=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $u^+:=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $u^-:=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $\mathfrak{z}:=a^++a^-$, and $\mathfrak{c}:=\mathfrak{h}^2-2\mathfrak{h}+4u^+u^-=\mathfrak{h}^2+2\mathfrak{h}+4u^-u^+\in\mathrm{U}(\mathfrak{gl}_2)$ be the Casimir element.

Let $\mathcal{R}_E$ be the $E$-coefficient Robba ring of $\mathbb{Q}_p$, and $\mathcal{R}_E^+ := \{f = \sum_{n=0}^{+\infty} a_n X^n \mid f \in \mathcal{R}_E\}$. Note $\mathcal{R}_E^+$ is naturally isomorphic to the distribution algebra $\mathcal{D}(\mathbb{Z}_p, E)$ on $\mathbb{Z}_p$. Let $t = \log(1+X) \in \mathcal{R}_E^+ \subset \mathcal{R}_E$.

We use $\bullet - - \bullet$ (resp. $\bullet - \bullet$) to denote a possibly split extension (resp. a non-split extension), with the left object the sub and the right object the quotient.


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## 2 Translations of $(\varphi, \Gamma)$-modules

We discuss some properties of translations on $(\varphi, \Gamma)$-modules.

### 2.1 Generalities

Let $k \in \mathbb{Z}_{\geq 0}$, let $V_k := \mathrm{Sym}^k E^2$ be the algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ of highest weight $(0, k)$ (with respect to $B(\mathbb{Q}_p)$). On $V_k$, we have $\mathfrak{z} = k$ and $\mathfrak{c} = k(k+2)$. The $P^+ := \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$-action induces an $\mathcal{R}_E^+$-structure on $V_k$ together with a semi-linear $(\varphi, \Gamma)$-action given by $(1+X)v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v$, $\varphi(v) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v$, $\gamma(v) = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} v$. Let $e$ be the lowest weight vector of $V_k$, then we have a $(\varphi, \Gamma)$-equivariant isomorphism of $\mathcal{R}_E^+$-modules: $\mathcal{R}_E^+/X^{k+1} \xrightarrow{\sim} V_k$, $\alpha \mapsto \alpha e$.

Let $D$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_E$. Consider $D \otimes_E V_k$. We equip $D \otimes_E V_k$ with a diagonal $\mathcal{R}_E^+$-action (using the coalgebra structure $\mathcal{R}_E^+ \to \mathcal{R}_E^+ \otimes_E \mathcal{R}_E^+$, $(1+X) \mapsto (1+X) \otimes (1+X)$), and with a diagonal $(\varphi, \Gamma)$-action. It is clear that the resulting $(\varphi, \Gamma)$-action is $\mathcal{R}_E^+$-semi-linear. We also equip $D \otimes_E V_k$ with the natural topology so that $D \otimes_E V_k \cong D^{\oplus k+1}$ as topological $E$-vector space.

**Proposition 2.1.** *The $\mathcal{R}_E^+$-action on $D \otimes_E V_k$ uniquely extends to a continuous $\mathcal{R}_E$-action. With this action, $D \otimes_E V_k$ is a $(\varphi, \Gamma)$-module over $\mathcal{R}_E$ and is isomorphic to a successive extension: $t^k D - - t^{k-1} D - - \cdots - - tD - - D$.*

*Proof.* We first prove the proposition for the case $k = 1$. Let $e_0$ be the lowest weight vector in $V_1$, and $e_1 := Xe_0$ (so $Xe_1 = 0$). For $v = v_0 \otimes e_0 + v_1 \otimes e_1$, $Xv = Xv_0 \otimes e_0 + (v_0 + Xv_0 + Xv_1) \otimes e_1$. It is clear that $X$ is invertible on $D \otimes_E V_1$, and $X^{-1}(v_0 \otimes e_0 + v_1 \otimes e_1) = (X^{-1}v_0) \otimes e_0 + (-X^{-1}v_0 - X^{-2}v_0 + X^{-1}v_1) \otimes e_1$. For $f(T) \in E[T]$, $f(X^{-1})(v_0 \otimes e_0 + v_1 \otimes e_1) = (f(X^{-1})v_0) \otimes e_0 + (-(X^{-2} + X^{-1})f'(X^{-1})v_0 + f(X^{-1})v_1) \otimes e_1$. As $\mathcal{R}_E$ acts on $D$, by the formula we see $\mathcal{R}_E^+[1/X]$-action uniquely extends to a continuous $\mathcal{R}_E$-action. The proposition in this case follows. Using induction, we see the $\mathcal{R}_E^+$-action on $D \otimes_E V_1^{\otimes k}$ uniquely extends to a continuous $\mathcal{R}_E$-action. As $D \otimes_E V_k$ is a (closed) direct summand of $D \otimes_E V_1^{\otimes k}$ stable by $\mathcal{R}_E^+$, it is also stabilized by $\mathcal{R}_E^+[1/X]$ hence by $\mathcal{R}_E$.

Each $D \otimes_E (X^i \mathcal{R}_E^+/X^{k+1})$, for $i = 0, \cdots, k-1$, is clearly a $(\varphi, \Gamma)$-equivariant $\mathcal{R}_E^+$-submodule. Using induction and the easy fact that $X$ is invertible on the graded pieces $D \otimes_E (X^j \mathcal{R}_E^+/X^{j+1})$

(noting that the $\mathcal{R}_E^+$-action on the graded pieces is the same as acting only on $D$), one easily sees that $X$ is invertible on $D \otimes_E (X^i\mathcal{R}_E^+/X^{k+1})$. Hence $D \otimes_E (X^i\mathcal{R}_E^+/X^{k+1})$ is a $(\varphi,\Gamma)$-submodule of $D \otimes_E V_k$ over $\mathcal{R}_E$. On the graded piece, the induced $\mathcal{R}_E$-action is the unique one that extends the $\mathcal{R}_E^+$-action, hence coincides with the $\mathcal{R}_E$-action on $D$. We then easily see that $D \otimes_E (X^i\mathcal{R}_E^+/X^{i+1})$ isomorphic to $t^iD$. This concludes the proof. □

**Remark 2.2.** *In particular, we have the following morphisms of $(\varphi,\Gamma)$-modules over $\mathcal{R}_E$:*

$$D \otimes_E V_k \twoheadrightarrow D, \qquad \sum_{i=0}^{k} v_i \otimes t^i e \mapsto v_0, \tag{3}$$

$$D \hookrightarrow D \otimes_E V_k, \qquad v \mapsto v \otimes t^k e. \tag{4}$$

**Example 2.3.** *We have $\mathcal{R}_E \otimes_E V_1 \cong \mathcal{R}_E \oplus t\mathcal{R}_E$. Indeed, the element $1 \otimes e \in H^0_{(\varphi,\Gamma)}(\mathcal{R}_E \otimes_E V_1)$. This induces a morphism $\mathcal{R}_E \hookrightarrow \mathcal{R}_E \otimes_E V_1$, whose composition with (3) (for $D = \mathcal{R}_E$) is clearly an isomorphism. We see the extension in the proposition for $D = \mathcal{R}_E$ and $k = 1$ splits. See Remark 2.18 (1) for a non-split case.*

**Remark 2.4.** *Suppose $D$ is de Rham, then $D \otimes_E V_k$ is also de Rham. This easily follows from Proposition 2.21 (1) below (which is obtained by using certain $\mathfrak{gl}_2$-action). One can also directly prove it as follows. Indeed, by induction, it is sufficient to show $D \otimes_E V_1$ is de Rham. Let $\Delta$ be the p-adic differential equation associated to $D$ (of constant Hodge-Tate weight 0), and $n \in \mathbb{Z}_{\geq 0}$ such that $t^n\Delta \subset D$. We see $t^n\Delta \otimes_E V_1$ is a $(\varphi,\Gamma)$-submodule of $D \otimes_E V_1$, and the both have the same rank. It suffices to show $\Delta \otimes_E V_1$ is de Rham. But we have (e.g. by [7, Lem. 1.11]) $H^1_g(t\Delta \otimes_{\mathcal{R}_E} \Delta^\vee) \xrightarrow{\sim} H^1(t\Delta \otimes_{\mathcal{R}_E} \Delta^\vee)$, hence any extension of $\Delta$ by $t\Delta$ is de Rham.*

**Lemma 2.5.** *For $v \otimes w \in D \otimes_E V_k$, we have $\psi(v \otimes w) = \psi(v) \otimes \varphi^{-1}(w)$*

*Proof.* Write $v = \sum_{i=0}^{p-1}(1+X)^i\varphi(v_i)$ (so $\psi(v) = v_0$). We have (using $\varphi$ is invertible on $V_k$):

$$v \otimes w = \sum_{i=0}^{p-1}(1+X)^i(\varphi(v_i) \otimes (1+X)^{-i}w) = \sum_{i=0}^{p-1}(1+X)^i\varphi(v_i \otimes \varphi^{-1}((1+X)^{-i}w)).$$

The lemma follows. □

A $(\varphi,\Gamma)$-module $D$ is naturally equipped with a locally analytic action of $P^+$, where $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts via $(1+X) \in \mathcal{R}_E$, $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ via $\varphi$, and $\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$ via $\Gamma$. Moreover, by [5, § 1.3], $D$ corresponds to a $P^+$-sheaf $\mathscr{F}_D$ of analytic type over $\mathbb{Z}_p$, with the sections $\mathscr{F}_D(i+p^n\mathbb{Z}_p)$ over $i+p^n\mathbb{Z}_p$, which we also denote by $D \boxtimes (i+p^n\mathbb{Z}_p)$ as in *loc. cit.*, given by

$$(1+X)^i\varphi^n\psi^n((1+X)^{-i}v) =: \mathrm{Res}_{i+p^n\mathbb{Z}_p}(v) \tag{5}$$

for $v \in D$. In particular, we have $D \boxtimes \mathbb{Z}_p^\times \cong D^{\psi=0}$.

For a $P^+$-sheaf $\mathscr{F}$ of analytic type over $\mathbb{Z}_p$, it is direct to check the following data defines a $P^+$-sheaf of analytic type $\mathscr{F} \otimes_E V_k$ over $\mathbb{Z}_p$:

- $(\mathscr{F} \otimes_E V_k)(U) := \mathscr{F}(U) \otimes_E V_k$,

- $\mathrm{Res}_U^V|_{\mathscr{F}\otimes_E V_k} := \mathrm{Res}_U^V|_{\mathscr{F}} \otimes \mathrm{id}$,
- $g_U|_{(\mathscr{F}\otimes_E V_k)(U)} := g_U|_{\mathscr{F}(U)} \otimes g : (\mathscr{F} \otimes_E V_k)(U) \to (\mathscr{F} \otimes_E V_k)(g(U)) \cong \mathscr{F}(g(U)) \otimes_E V_k$ for $g \in P^+$.

**Lemma 2.6.** *The identity map on $D\otimes_E V_k$ induces a natural $P^+$-equivariant isomorphism $\mathscr{F}_D \otimes_E V_k \cong \mathscr{F}_{D\otimes_E V_k}$ .*

*Proof.* Let $i \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 0}$. For $x \otimes w \in D \otimes_E V_k$, using Lemma 2.5 and the formula in (5), we have

$$\mathrm{Res}_{i+p^n\mathbb{Z}_p}(x \otimes w) = \mathrm{Res}_{i+p^n\mathbb{Z}_p}(x) \otimes w.$$

The identity map induces then an isomorphism of sheaves on $\mathbb{Z}_p$: $\mathscr{F}_D \otimes_E V_k \cong \mathscr{F}_{D\otimes_E V_k}$. It is straightforward to check the isomorphism is $P^+$-equivariant, as the both are equipped with the diagonal $P^+$-action. □

The following lemma is a direct consequence of Lemma 2.6.

**Lemma 2.7.** *We have $(D \otimes_E V_k)^{\psi=0} = D^{\psi=0} \otimes_E V_k$ (as subspace of $D \otimes_E V_k$). Moreover, $\mathrm{Res}_{\mathbb{Z}_p^*}|_{D\otimes_E V_k} = \mathrm{Res}_{\mathbb{Z}_p^*}|_D \otimes \mathrm{id}$.*

## 2.2 Translations of $(\varphi, \Gamma)$-modules of rank 2

For a $(\varphi, \Gamma)$-module $D$ with an extra $\mathfrak{gl}_2$-action, we study the $(\varphi, \Gamma)$-module structure together with the (diagonal) $\mathfrak{gl}_2$-action of the translation $D \otimes_E V_k$.

### 2.2.1 A digression on $(\varphi, \Gamma)$-submodules

Let $D$ be a $(\varphi, \Gamma)$-module of rank 2 over $\mathcal{R}_E$. We briefly discuss the $(\varphi, \Gamma)$-submodules of $D$ and introduce some notation. Twisting $D$ by a rank one $(\varphi, \Gamma)$-module, we can and do assume that the Sen weights of $D$ are given by 0 and $\alpha \in E \setminus \mathbb{Z}_{<0}$. Let $D'$ be a $(\varphi, \Gamma)$-submodule of $D$, by [13, Prop. 4.1], there exists $n$ such that $D' \supset t^n D$. We are led to study the torsion $(\varphi, \Gamma)$-module $D/t^n D$.

**Lemma 2.8.** *(1) If $\alpha \notin \mathbb{Z}$, then there exists a locally analytic character of $\mathbb{Q}_p^\times$ of weight $\alpha$ such that $D/t^n D \cong \mathcal{R}_E/t^n \oplus \mathcal{R}_E(\chi_\alpha)/t^n$.*

*(2) If $\alpha \in \mathbb{Z}_{\geq 0}$ and $D$ is not de Rham, then $D/t^n D$ is isomorphic to a non-split extension of $\mathcal{R}_E/t^n$ by $\mathcal{R}_E(z^\alpha)/t^n$.*

*(3) If $\alpha \in \mathbb{Z}_{\geq 0}$ and $D$ is de Rham, then $D/t^n D \cong \mathcal{R}_E(z^\alpha)/t^n \oplus \mathcal{R}_E/t^n$.*

*Proof.* The lemma follows from Fontaine's classification of $B_{\mathrm{dR}}$-representations [11, Thm. 3.19], and [3, Lem. 5.1.1]. □

The following two propositions follow easily from the lemma.

**Proposition 2.9** (Non-de Rham case)**.** *(1) If $\alpha \notin \mathbb{Z}$, for any $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, there exists a unique $(\varphi, \Gamma)$-submodule of $D$ of Sen weights $(n_1, \alpha + n_2)$, denoted by $D_{(n_1,\alpha+n_2)}$. Moreover, any $(\varphi, \Gamma)$-submodule of $D$ of rank 2 has this form.*

*(2) If $\alpha \in \mathbb{Z}_{\geq 0}$ and $D$ is not de Rham, for $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, $n_1 \leq n_2 + \alpha$, there exists a unique $(\varphi, \Gamma)$-submodule of $D$ of Sen weights $(n_1, \alpha + n_2)$, denoted by $D_{(n_1, \alpha+n_2)}$. Moreover, any $(\varphi, \Gamma)$-submodule of $D$ of rank 2 has this form.*

**Remark 2.10.** *When $n_1 = n_2 = n$, then the $(\varphi, \Gamma)$-submodule of $D$ of weights $(n, \alpha + n)$ is just $t^n D$.*

**Proposition 2.11** (De Rham case)**.** *Assume $D$ is de Rham.*

*(1) For each $n \in \mathbb{Z}$, $n \geq \alpha$, there exists a unique $(\varphi, \Gamma)$-submodule of $D$ of Sen weights $(n, n)$.*

*(2) If $\alpha \in \mathbb{Z}_{\geq 1}$, for $n_1 \in \mathbb{Z}_{\geq 0}$, $n_1 \leq \alpha$ (resp. $n_2 \in \mathbb{Z}_{\geq 0}$, there exist a unique $(\varphi, \Gamma)$-submodule of $D$ of Sen weights $(n_1, \alpha)$ (resp. $(0, \alpha + n_2)$), which we denote by $D_{(n_1, \alpha)}$ (resp. $D_{(0, \alpha+n_2)}$).*

*(3) If $\alpha = 0$, let $n \geq 1$, the $(\varphi, \Gamma)$-submodules of $D$ of Sen weights $(0, n)$ are parametrized by lines $\mathscr{L} \subset D_{\mathrm{dR}}(D)$, each denoted by $D_{n, \mathscr{L}}$.*

**Remark 2.12.** *(1) By the proposition, one easily gets a full description of $(\varphi, \Gamma)$-submodules of $D$ in de Rham case. Note also that in case (3), $D_{n,\mathscr{L}}$ can be isomorphic for different $\mathscr{L}$.*

*(2) Assume $D$ is de Rham and $\alpha \geq 1$. For $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, $n_1 \leq n_2 + \alpha$, by (2) there exists a unique $(\varphi, \Gamma)$-submodule, denoted by $D_{(n_1, \alpha+n_2)}$, of $D$ of Sen weights $(n_1, \alpha + n_2)$ such that $D_{(n_1, \alpha+n_2)} \subset D_{(0, \alpha+n_2)}$. We have $D_{(n, \alpha+n)} \cong t^n D$. Recall there is an equivalence of categories between de Rham $(\varphi, \Gamma)$-modules and filtered Deligne-Fontaine modules (cf. [1, Thm. A]). The associated Deligne-Fontaine modules (igonring the Hodge filtration) of $D_{(n_1, \alpha+n_2)}$ are all the same and isomorphic to that of $D$. For the Hodge filtration, to go from $D$ to $D_{(n_1, \alpha+n_2)}$ with $n_1 < \alpha + n_2$, one just shifts the degree of the filtration respectively.*

### 2.2.2 Translations

For a $(\varphi, \Gamma)$-module $D$ over $\mathcal{R}_E$, by differentiating the $P^+$-action, we obtain an action of its Lie algebra $\mathfrak{p}^+$ on $D$, where $a^+$ acts via the operator $\nabla$ (given by differentiating the $\Gamma$-action), and $u^+$ acts via $t$. We assume $D$ is of rank 2 and has Sen weights $(0, \alpha_D)$, with $\alpha_D \in E \setminus \mathbb{Z}_{<0}$. Let $P(T) \in E[T]$ be the monic Sen polynomial of $D$, hence $P(\nabla)(D) \subset tD$ (e.g. see [6, Lem. 1.6]). For an operator $\nabla'$ such that $\nabla'(D) \subset tD$, we use $\frac{\nabla'}{t}$ to denote the operator mapping $x$ to $\frac{1}{t}\nabla'(x)$. In particular, we have the operator $\frac{P(\nabla)}{t}$ on $D$. We recall the $\mathfrak{gl}_2$-actions on $D$, and we refer to [5, § 3.2.1] for details. We restrict to the case with infinitesimal character for our applications.

**Proposition 2.13.** *(1) If $\deg P(T) = 2$ (so is equal to $T(T - \alpha_D)$), then there exists a unique $\mathfrak{gl}_2$-action on $D$ extending the $\mathfrak{p}^+$-action satisfying that $D$ has infinitesimal character. The action is given by $u^- = -\frac{P(\nabla)}{t}$ and $\mathfrak{z} = \alpha_D - 1$. Consequently, $\mathfrak{c}$ acts via $\alpha_D^2 - 1$.*

*(2) If $\deg P = 1$ (so $\alpha_D = 0$ and $P(T) = T$), for $\alpha \in E$, there exists a unique $\mathfrak{gl}_2$-action on $D$ extending the $\mathfrak{p}^+$-action satifying that $D$ has infinitesimal character and $\mathfrak{z}$ acts via $\alpha - 1$. The action is given by $u^- = -\frac{\nabla(\nabla - \alpha)}{t}$. Consequently, $\mathfrak{c} = \alpha^2 - 1$.*

**Remark 2.14.** *(1) By [5, Prop. 3.4], if $D$ does not contain a pathological $(\varphi, \Gamma)$-submodule (cf. [5, Rem. 3.5]), then the uniqueness in the proposition already holds with the condition having infinitesimal character replaced by having central character (for $\mathfrak{z}$). If $D$ contains a pathological $(\varphi, \Gamma)$-submodule, then the $\mathfrak{p}^+$-action on $D$ can extend to a $\mathfrak{gl}_2$-action without infinitesimal character (but with central character).*

*(2) For a general rank two $(\varphi,\Gamma)$-module $D'$, there exist $D$ as above and a continuous character $\chi$ such that $D' \cong D \otimes_{\mathcal{R}_E} \mathcal{R}_E(\chi)$. The $\mathfrak{gl}_2$-action on $D'$ is then given by twisting the one on $D$ by $d\chi \circ \det$.*

In the sequel, we let $\alpha \in E$ such that $D$ is equipped with the $\mathfrak{gl}_2$-action with $u^- = -\frac{\nabla(\nabla-\alpha)}{t}$, $\mathfrak{z} = \alpha - 1$. For example, $\alpha = \alpha_D$ (hence $\alpha \notin \mathbb{Z}_{<0}$ in this case) if $\deg P(T) = 2$. If $P(T) = T$, $\alpha$ can be arbitrary. We equip $D \otimes_E V_k$ with a natural (diagonal) $\mathfrak{gl}_2$-action. Note that on $D \otimes_E V_k$, $\mathfrak{z} = \alpha + k - 1$.

**Lemma 2.15.** *The Casimir operator $\mathfrak{c}$ on $D \otimes_E V_k$ defines an endomorphism of $(\varphi,\Gamma)$-modules over $\mathcal{R}_E$.*

*Proof.* The operator $\mathfrak{c}$ commutes with the adjoint action of $P^+$. Hence $\mathfrak{c}$ is $\mathcal{R}_E^+$-linear and commutes with $\varphi$ and $\Gamma$. The lemma follows from $\mathrm{End}_{\mathcal{R}_E,(\varphi,\Gamma)}(D \otimes_E V_k) \xrightarrow{\sim} \mathrm{End}_{\mathcal{R}_E^+,(\varphi,\Gamma)}(D \otimes_E V_k)$. □

By the lemma, we can decompose $D \otimes_E V_k$ into a direct sum of generalized eigenspaces $(D \otimes_E V_k)\{\mathfrak{c} = \mu\}$, each being a saturated $(\varphi,\Gamma)$-submodule of $D \otimes_E V_k$.

**Lemma 2.16.** *For any $\mu \in E$, the composition $(D \otimes_E V_k)[\mathfrak{c} = \mu] \hookrightarrow D \otimes_E V_k \xrightarrow{(3)} D$ is injective.*

*Proof.* Let $e_i = t^i e$ for $i = 0, \cdots, k$, and $v = \sum_{i=0}^k v_i \otimes e_i$. We have (letting $v_{-1} = v_{k+1} = 0$)

$$\mathfrak{c}v = \sum_{i=0}^{k} (\mathfrak{c}v_i) \otimes e_i + \sum_{i=0}^{k+1} v_i \otimes (\mathfrak{c}e_i) + \sum_{i=0}^{k} (4u^- v_{i-1} + 4(i+1)(k-i)u^+ v_{i+1} + 2(2i-k)\mathfrak{h}v_i) \otimes e_i. \quad (6)$$

If $\mathfrak{c}v = \mu v$ and $v_0 = 0$, comparing the $e_0$ terms on both sides and using $u^+$ is injective on $D$, we easily see $v_1 = 0$. Using induction and similar argument (comparing the $e_{i-1}$ term), we see $v_i = 0$ for all $i$ hence $v = 0$. The lemma follows. □

Now consider the $k = 1$ case.

**Lemma 2.17.** *(1) Suppose $\alpha_D \notin \mathbb{Z}$ or $\alpha_D \in \mathbb{Z}_{\geq 1}$, then we have a $\mathfrak{gl}_2(\mathbb{Q}_p)$-equivariant isomorphism of $(\varphi,\Gamma)$-modules over $\mathcal{R}_E$:*

$$D \otimes_E V_1 \cong (D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1] \oplus (D \otimes_E V_1)[\mathfrak{c} = (\alpha-1)^2 - 1] = D_{(0,\alpha+1)} \oplus D_{(1,\alpha)}.$$

*(2) Suppose $\alpha_D = 0$ and $D$ is not de Rham, then $D \otimes_E V_1 = (D \otimes_E V_1)\{\mathfrak{c} = 0\}$ which, as $(\varphi,\Gamma)$-module or $\mathfrak{gl}_2$-module, is isomorphic to a non-split self-extension of $(D \otimes V_1)[\mathfrak{c} = 0] \cong D_{(0,1)}$.*

*(3) Suppose $\alpha_D = 0$ and $D$ is de Rham. Then $D \otimes_E V_1 \cong D \oplus tD$. And we have*

*(a) If $\alpha \neq 0$, then $D \otimes_E V_1 \cong (D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1] \oplus (D \otimes_E V_1)[\mathfrak{c} = (\alpha-1)^2 - 1] = D \oplus tD$.*

*(b) If $\alpha = 0$, then $D \otimes_E V_1 = (D \otimes_E V_1)\{\mathfrak{c} = 0\} = (D \otimes_E V_1)[\mathfrak{c}^2 = 0]$, and $(D \otimes V_1)[\mathfrak{c} = 0] \cong D$.*

*Proof.* Let $e_0 = e$ and $e_1 = te$. For $v = v_0 \otimes e_0 + v_1 \otimes e_1 \in D \otimes_E V_1$, by (6)

$$\mathfrak{c}(v) = (\alpha^2 + 2)v + (-2\mathfrak{h}v_0 + 4u^+ v_1) \otimes e_0 + (4u^- v_0 + 2\mathfrak{h}v_1) \otimes e_1.$$

Suppose $\mathfrak{c}(v) = (\alpha^2+2+\lambda)v$, then (noting $\mathfrak{h} = 2\nabla - \alpha + 1$):

$$\begin{cases} (4\nabla - 2\alpha + 2 + \lambda)v_0 = 4tv_1 \\ (4\nabla - 2\alpha + 2 - \lambda)v_1 = \frac{4\nabla(\nabla-\alpha)}{t}v_0 \end{cases}.$$

As $\nabla(tx) = t(\nabla+1)x$ for $x \in D$, we deduce

$$4\nabla(\nabla-\alpha)v_0 = (4\nabla - 2\alpha - 2 - \lambda)tv_1 = \frac{1}{4}(4\nabla - 2\alpha - 2 - \lambda)(4\nabla - 2\alpha + 2 + \lambda)v_0.$$

If $v \neq 0$ (hence $v_0 \neq 0$), $\lambda = \pm 2\alpha - 2$. We also see

$$(D \otimes_E V_1)[\mathfrak{c} = (\alpha \pm 1)^2 - 1] = \big\{v_0 \otimes e_0 + \frac{\nabla - \alpha/2\alpha \pm \alpha/2}{t} v_0 \otimes e_1 \mid v_0 \in D,\ (\nabla - \alpha/2 \pm \alpha/2)v_0 \subset tD\big\}. \tag{7}$$

It is clear that the image of $(D \otimes_E V_1)[\mathfrak{c} = (\alpha \pm 1)^2 - 1] \hookrightarrow D$ contains $tD$. In particular, $(D \otimes_E V_1)[\mathfrak{c} = (\alpha \pm 1)^2 - 1]$ is a $(\varphi, \Gamma)$-submodule of $D$ of rank 2. If $\alpha \neq 0$, we have then

$$(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1] \oplus (D \otimes_E V_1)[\mathfrak{c} = (\alpha-1)^2 - 1] \xrightarrow{\sim} D \otimes_E V_1. \tag{8}$$

If $\alpha = 0$, we have $D \otimes_E V_1 \cong (D \otimes_E V_1)\{\mathfrak{c} = 0\}$. Moreover, by direct calculation, we have $(D \otimes_E V_1)\{\mathfrak{c} = 0\} = (D \otimes_E V_1)[\mathfrak{c}^2 = 0]$.

Next we describe the $(\varphi, \Gamma)$-module $(D \otimes_E V_1)[\mathfrak{c} = (\alpha \pm 1)^2 - 1]$. We will use the $\mathfrak{gl}_2$-action (although one can directly describe it using (7)). For $x \in (D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1]$, using $\mathfrak{c} = \mathfrak{h}^2 - 2\mathfrak{h} + 4u^+u^-$, we have $\nabla(\nabla - \alpha - 1)(x) \in t((D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1])$. Hence the Sen polynomial of $(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1]$ divides $T(T - \alpha - 1)$. Similarly, the Sen polynomial of $(D \otimes_E V_1)[\mathfrak{c} = (\alpha-1)^2 - 1]$ divides $(T-1)(T-\alpha)$.

If $D$ is not de Rham, then $(D \otimes_E V_1)[\mathfrak{c} = (\alpha \pm 1)^2 - 1]$ twisted by any character is also not de Rham. Hence its Sen polynomial has to be of degree 2. Together with the above discussion, we easily deduce $(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1] = D_{(0,\alpha+1)}$, and if moreover $\alpha \neq 0$, $(D \otimes_E V_1)[\mathfrak{c} = (\alpha-1)^2 - 1] \cong D_{(1,\alpha)}$. If $\alpha = 0$, we see $(D \otimes_E V_1)/(D \otimes_E V_1)[\mathfrak{c} = 0]$ is also a $(\varphi, \Gamma)$-module of rank 2 of Sen weights $(0, 1)$. Consider the composition

$$tD \xrightarrow{(4)} D \otimes_E V_1 \twoheadrightarrow (D \otimes_E V_1)/(D \otimes_E V_1)[\mathfrak{c} = 0]. \tag{9}$$

By Lemma 2.16, $tD \cap (D \otimes_E V_1)[\mathfrak{c} = 0] = 0$ (as submodules of $D \otimes_E V_1$), hence (9) is also injective. We deduce $(D \otimes_E V_1)/(D \otimes_E V_1)[\mathfrak{c} = 0] \cong D_{(0,1)}$. If $D \otimes_E V_1 \cong D_{(0,1)} \oplus D_{(0,1)}$, then $D \otimes_E V_1$ is Hodge-Tate, which is impossible as its saturated $(\varphi, \Gamma)$-submodule $tD$ is not Hodge-Tate. So $D \otimes_E V_1$ is a non-split self-extension of $D_{(0,1)}$. This finishes (2) (and (1) for non-de Rham case).

Assume now $D$ is de Rham, and suppose first $D$ has distinct Sen weights. Then the $\mathfrak{gl}_2$-action on $D$ is unique and $\alpha = \alpha_D \in \mathbb{Z}_{\geq 1}$. The Sen weights of $(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1]$ are $(0, \alpha+1)$ or $(0, 0)$ or $(\alpha+1, \alpha+1)$. However, $(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1]$ is a submodule of $D$ (resp. a saturated submodule of $D \otimes_E V_1$), so it can not have Sen weights $(0, 0)$ (resp. $(\alpha+1, \alpha+1)$). We deduce hence $(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1] \cong D_{(0,\alpha+1)}$. As $(D \otimes_E V_1)[\mathfrak{c} = (\alpha-1)^2 - 1]$ has Sen weights $(1, \alpha)$ and contains $tD$, we see $(D \otimes_E V_1)[\mathfrak{c} = (\alpha-1)^2 - 1] \cong D_{(1,\alpha)}$.

Finally, suppose $D$ is de Rham of weights $(0, 0)$. By (7) and $\nabla D \subset tD$, the composition $(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1] \hookrightarrow D \otimes_E V_1 \twoheadrightarrow D$ is surjective. We have thus $D \otimes_E V_1 \cong D \oplus tD$. By comparing the Sen weights, the injection $tD \to (D \otimes_E V_1)/(D \otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2 - 1]$ (obtained as in (9)) is also an isomorphism. (3) follows. □

**Remark 2.18.** *(1) Assume $\alpha_D \neq 0$, then we obtain two filtrations on $D \otimes_E V_1$:*

$$D \otimes_E V_1 \cong D_{(0,\alpha+1)} \oplus D_{(1,\alpha)} \cong [tD - -D].$$

*If $D$ is not trianguline, then $\mathrm{Hom}_{(\varphi,\Gamma)}(D, D_{(0,\alpha+1)}) = \mathrm{Hom}_{(\varphi,\Gamma)}(D, D_{(1,\alpha)}) = 0$, so the extension $tD - -D$ is non-split.*

*(2) The induced $\mathfrak{gl}_2$-action on $(D\otimes_E V_1)[\mathfrak{c} = (\alpha\pm1)^2-1]$, and $(D\otimes_E V_1)\{\mathfrak{c} = 0\}/(D\otimes_E V_1)[\mathfrak{c} = 0]$ (when $\alpha = 0$) coincides with the one in Proposition 2.13 and Remark 2.2.2 (2).*

**Proposition 2.19.** *(1) Suppose $\alpha_D \notin \mathbb{Z}$ or $\alpha_D \in \mathbb{Z}$ and $\alpha_D \geq k$, then we have a $\mathfrak{gl}_2$-equivariant isomorphism of $(\varphi, \Gamma)$-modules over $\mathcal{R}_E$:*

$$D \otimes_E V_k \cong \oplus_{i=0}^k (D \otimes_E V_k)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1] = \oplus_{i=0}^k D_{(i,\alpha+k-i)}.$$

*(2) Suppose $\alpha_D \neq 0$, then $(D\otimes_E V_k)\{\mathfrak{c} = (\alpha+k)^2-1\} = (D\otimes_E V_k)[\mathfrak{c} = (\alpha+k)^2-1] \cong D_{(0,\alpha+k)}$.*

*(3) Suppose $\alpha_D = 0$ and $D$ is not de Rham, $(D\otimes_E V_k)\{\mathfrak{c} = k^2 - 1\}$ is a non-split self-extension of $(D \otimes_E V_k)[\mathfrak{c} = k^2 - 1] \cong D_{(0,k)}$.*

*(4) Suppose $\alpha_D = 0$ and $D$ is de Rham. If $\alpha \in E \setminus \mathbb{Z}_{<0}$, then $(D \otimes_E V_k)[\mathfrak{c} = (\alpha+k)^2 - 1] \cong D$, and if $\alpha = 0$, $(D \otimes_E V_k)\{\mathfrak{c} = k^2 - 1\} = (D \otimes_E V_k)[(\mathfrak{c} - k^2 + 1)^2 = 0] \cong D \oplus t^k D$.*

*Proof.* By Lemma 2.17 (and the proof) and an easy induction argument (see also Remark 2.18 (2)), we have

$$D \otimes_E V_1^{\otimes k} \cong \sum_{i=0}^{k} (D \otimes_E V_1^{\otimes k})\{\mathfrak{c} = (\alpha + k - 2i)^2 - 1\}. \tag{10}$$

And if $\alpha \notin \mathbb{Z}$ or $\alpha \in \mathbb{Z}$, $\alpha \geq k$, by Lemma 2.17 (1) and induction, we have (where the factors in the direct sum can have rank bigger than 2)

$$D \otimes_E V_1^{\otimes k} \cong \oplus_{i=0}^k (D \otimes_E V_1^{\otimes k})[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]. \tag{11}$$

The first isomorphism in (1) follows. By Lemma 6, $(D \otimes_E V_k)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]$ is a $(\varphi, \Gamma)$-submodule of $D$ of rank at most 2. By similar arguments as in the proof of Lemma 2.17, the Sen polynomial of $(D \otimes_E V_k)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]$ divides $(X - i)(X - (\alpha + k - i))$. As the Sen weights of $D \otimes_E V_k$ are given by $(0, \cdots, k, \alpha, \cdots, \alpha + k)$, by comparing the weights, we see the Sen weights of $(D \otimes_E V_k)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]$ are exactly $(i, \alpha + k - i)$. It rests to show

$$(D \otimes_E V_k)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1] \cong D_{(i,\alpha+k-i)}. \tag{12}$$

The $k = 1$ case was proved in Lemma 2.17 (1). Assume $k \geq 2$. We use induction and assume hence (1) holds for $k' < k$. For $i = 0$ (resp. $i = k$), (12) holds as $D_{(0,\alpha+k)}$ (resp. $D_{(k,\alpha)}$) is the unique submodule of $D$ of Sen weights $(0, \alpha + k)$ (resp. $(k, \alpha)$). Assume $1 \leq i \leq k - 1$, it is easy to see $(D \otimes_E V_k)[\mathfrak{c} = (\alpha+k-2i)^2-1]$ is a direct summand of $((D \otimes_E V_{k-1}) \otimes_E V_1)[\mathfrak{c} = (\alpha+k-2i)^2-1]$. As (1) holds fo $k - 1$, it is not difficult to see the latter is isomorphic to

$$(D_{(i,\alpha+k-1-i)} \otimes_E V_1)[\mathfrak{c} = (\alpha+k-2i)^2-1] \oplus (D_{(i-1,\alpha+k-i)} \otimes_E V_1)[\mathfrak{c} = (\alpha+k-2i)^2-1] \cong D_{(i,\alpha+k-i)}^{\oplus 2}. \tag{13}$$

By Clebsh-Gordan rule, we have $V_{k-2} \otimes_E (\wedge^2 V_1) \hookrightarrow V_{k-1} \otimes_E V_1 \twoheadrightarrow V_k$ and the composition is zero. By the induction hypothesis for $k - 2$, we have

$$(D \otimes_E V_{k-2} \otimes_E (\wedge^2 V_1))[\mathfrak{c} = (\alpha + k - 2i)^2 - 1] \cong tD_{(i-1,\alpha+k-1-i)} \cong D_{(i,\alpha+k-i)}.$$

Together with (13) and the fact $(D \otimes_E V_k)[\mathfrak{c} = (\alpha+k-2i)^2-1]$ has Sen weights $(i, \alpha+k-i)$, it is not difficult to deduce (12). This finishes the proof of (1).

(2) By (10), one can easily show that $(D \otimes_E V_1^{\otimes i})\{\mathfrak{c} = (\alpha+k)^2-1\} = 0$ for $i < k$ (note in this case $\alpha \notin \mathbb{Z}_{<0}$). Using Lemma 2.17 (1) and induction, we also have $(D \otimes_E V_1^{\otimes k})\{\mathfrak{c} = (\alpha+k)^2-1\} = (D \otimes_E V_1^{\otimes k})[\mathfrak{c} = (\alpha+k)^2-1] \cong D_{(0,\alpha+k)}$. (2) follows.

(3) By (10), $(D \otimes_E V_1^{\otimes i})\{\mathfrak{c} = k^2-1\} = 0$ for $i < k$. It suffices to show the same statement with $V_k$ replaced by $V_1^{\otimes k}$. By Lemma 2.16 and an induction argument using Lemma 2.17 (1), we get $(D \otimes_E V_1^{\otimes k})[\mathfrak{c} = k^2-1] \cong D_{(0,k)}$. By Lemma 2.17 (2) and induction, it is not difficult to see $(D \otimes_E V_1^{\otimes k})\{\mathfrak{c} = k^2-1\}$ is a self-extension of $(D \otimes_E V_1^{\otimes k})[\mathfrak{c} = k^2-1] \cong D_{(0,k)}$. We see the statement in (3) except the non-split property holds. If the extension splits, the multiplicity of $(T-k)$ in the Sen polynomial of $D \otimes_E V_k$ is one (noting $k$ is not a Sen weight of $(D \otimes_E V_k)/((D \otimes_E V_k)\{\mathfrak{c} = k^2-1\})$ by (10) and the discussion in the first paragraph), however the saturated $(\varphi, \Gamma)$-submodule $t^k D$ of $D \otimes_E V_k$ is not Hodge-Tate, a contradiction.

(4) Again by (10), if $\alpha \notin \mathbb{Z}_{<0}$, $(D \otimes_E V_1^{\otimes i})\{\mathfrak{c} = (\alpha+k)^2-1\} = 0$ for $i < k$, hence it suffices to prove the same statement for $V_1^{\otimes k}$. By Lemma 2.16 and Lemma 2.17 (3) with an induction argument, we have $(D \otimes_E V_1^{\otimes k})[\mathfrak{c} = (\alpha+k)^2-1] \cong D$. Assume now $\alpha = 0$, by Lemma 2.17 (3), we have an exact sequence (which splits as $(\varphi, \Gamma)$-module)

$$0 \to (D \otimes_E V_1^{\otimes(k-1)})\{\mathfrak{c} = k^2-1\} \to (D \otimes_E V_1^{\otimes k})\{\mathfrak{c} = k^2-1\} \to (tD \otimes_E V_1^{\otimes(k-1)})\{\mathfrak{c} = k^2-1\} \to 0,$$

where the $\mathfrak{gl}_2$-action on $D$ in the left term (resp. on $tD$ in the right term) fits into Lemma 2.17 (3)(a) for $\alpha = 1$ (resp. for $\alpha = -1$, after an appropriate twist). By (10) and an induction argument using (8), we have $(D \otimes_E V_1^{\otimes(k-1)})\{\mathfrak{c} = k^2-1\} \cong (D \otimes_E V_1^{\otimes(k-1)})[\mathfrak{c} = k^2-1] \cong D$, and $(tD \otimes_E V_1^{\otimes(k-1)})\{\mathfrak{c} = k^2-1\} \cong (tD \otimes_E V_1^{\otimes(k-1)})[\mathfrak{c} = (-k)^2-1]$ is a $(\varphi, \Gamma)$-submodule of rank 2 of $tD$, and has Sen polynomial dividing $T(T-k)$ (by similar arguments as in the proof of Lemma 2.17). As $(D \otimes_E V_1^{\otimes k})\{\mathfrak{c} = k^2-1\} \cong (D \otimes_E V_k)\{\mathfrak{c} = k^2-1\}$ is saturated in $D \otimes_E V_k$, we easily deduce $(tD \otimes_E V_1^{\otimes(k-1)})[\mathfrak{c} = k^2-1]$ has constant Sen weight $k$, hence is isomorphic to $t^k D$. This concludes the proof. □

**Remark 2.20.** *We will frequently use the following special case:*

$$(D \otimes_E V_k)[\mathfrak{c} = (\alpha+k)^2-1] \cong \begin{cases} D & \alpha = 0 \text{ and } D \text{ is de Rham} \\ D_{(0,\alpha+k)} & \text{otherwise} \end{cases}. \tag{14}$$

*Note that by induction, we also have*

$$\begin{aligned}(D \otimes_E V_k)[\mathfrak{c} = (\alpha+k)^2-1] &\cong (D \otimes_E V_1^{\otimes k})[\mathfrak{c} = (\alpha+k)^2-1] \\ &\cong \big((D \otimes_E V_1^{\otimes(k-1)})[\mathfrak{c} = (\alpha+k-1)^2-1] \otimes_E V_1\big)[\mathfrak{c} = (\alpha+k)^2-1]. \end{aligned} \tag{15}$$

*From this and (7), we have the following uniform description of $(D \otimes_E V_k)[\mathfrak{c} = (\alpha+k)^2-1]$ as submodule of $D$ (which can also be directly deduced from (14))*

$$(D \otimes_E V_k)[\mathfrak{c} = (\alpha+k)^2-1] = \{x \in D \mid \nabla_i(x) \in t^i D, \ \forall i = 1, \cdots, k\}$$

*where $\nabla_i := (\nabla - i + 1) \cdots (\nabla - 1)\nabla$.*

Finally we quickly discuss the translation on general $p$-adic differential equations, where everything is essentially the same as the rank two case. Let $\Delta$ be a de Rham $(\varphi,\Gamma)$-module of constant Hodge-Tate weight 0. For $\alpha\in E$, by [5, Prop. 3.6], we equip $\nabla$ with a $\mathfrak{gl}_2$-action extending the natural $\mathfrak{p}^+$-action such that $\mathfrak{z}=\alpha-1$ and $u^-=-\frac{\nabla(\nabla-\alpha)}{t}$ (so $\mathfrak{c}=\alpha^2-1$). Note that Lemma 2.16 still holds with $D$ replaced by $\nabla$.

**Proposition 2.21.** *(1)* $\Delta\otimes_E V_k\cong\oplus_{i=0}^k t^i\Delta$.

*(2) If* $\alpha\notin\mathbb{Z}_{<0}$, $(\Delta\otimes_E V_k)[\mathfrak{c}=(\alpha+k)^2-1]\cong\Delta$.

*(3) If* $\alpha=0$, $(\Delta\otimes_E V_k)\{\mathfrak{c}=k^2-1\}\cong\Delta\oplus t^k\Delta$.

*Proof.* (1) We consider the case where $\Delta$ is equipped with the above $\mathfrak{gl}_2$-action with $\alpha\notin\mathbb{Z}$. By similar argument in the proof of Lemma 2.17 and induction, we have a similar decomposition as in (11) for $\Delta$ (which holds with $V_1^{\otimes k}$ replaced by $V_k$). By considering the Sen weights, the Sen weights of $(\Delta\otimes_E V_k)[(\alpha+k-2i)^2-1]$ has to be the constant $i$. Similarly as in Lemma 2.16, $(\Delta\otimes_E V_k)[(\alpha+k-2i)^2-1]$ is a submodule of $\Delta$, hence is isomorphic to $t^i\Delta$.

(2) (3) follows by similar argument as for Lemma 2.17 (3) and Proposition 2.19 (4). □

**Remark 2.22.** *For a general de Rham $(\varphi,\Gamma)$-module $D$, let* $\mathrm{DF}:=D_{\mathrm{pst}}(D)$ *(ignoring the Hodge filtration) be the Deligne-Fontaine module associated to $D$. By Proposition 2.21 (1), the Deligne-Fontaine module associated to $D\otimes_E V_k$ is isomorphic to* $\mathrm{DF}^{\oplus k+1}$.

# 3 Locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

We show the compatibility of the translations on $(\varphi,\Gamma)$-modules and the translations on $\mathrm{GL}_2(\mathbb{Q}_p)$-representations.

## 3.1 Translations of $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaves on $\mathbb{P}^1(\mathbb{Q}_p)$

Let $\mathscr{F}$ be a $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf of analytic type over $\mathbb{P}^1(\mathbb{Q}_p)$ (cf. [5, § 1.3.1]). For $k\geq 1$, the following data defines a $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf, denoted by $\mathscr{F}\otimes_E V_k$, of analytic type over $\mathbb{P}^1(\mathbb{Q}_p)$: for compact opens $U$, $V$ of $\mathbb{P}^1(\mathbb{Q}_p)$,

- $(\mathscr{F}\otimes_E V_k)(U)=\mathscr{F}(U)\otimes_E V_k$,
- $\mathrm{Res}_U^V|_{\mathscr{F}\otimes_E V_k}\cong\mathrm{Res}_U^V|_{\mathscr{F}}\otimes\mathrm{id}$,
- $g_U|_{(\mathscr{F}\otimes_E V_k)(U)}=g_U|_{\mathscr{F}(U)}\otimes g$ for $g\in\mathrm{GL}_2(\mathbb{Q}_p)$.

Note that $w=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ induces an isomorphism $M_{\mathscr{F}}^+:=\mathscr{F}(\mathbb{Z}_p)\xrightarrow{\sim}\mathscr{F}(\mathbb{P}^1(\mathbb{Q}_p)\setminus\mathbb{Z}_p)$. It is clear that $M_{\mathscr{F}}^+$ gives rise to a $P^+$-sheaf of analytic type over $\mathbb{Z}_p$, and $w$ induces an involution on $\mathrm{Res}_{\mathbb{Z}_p^\times}(M_{\mathscr{F}}^+)$. We have $\mathscr{F}(\mathbb{P}^1(\mathbb{Q}_p))=\{(x,y)\in M_{\mathscr{F}}^+\times M_{\mathscr{F}}^+\mid w(\mathrm{Res}_{\mathbb{Z}_p^\times}(x))=\mathrm{Res}_{\mathbb{Z}_p^\times}(y)\}$. We refer to [5, § 3.1.1] for more discussion on the relation of $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaves and $P^+$-sheaves. Finally remark that the involution $w$ on $\mathrm{Res}_{\mathbb{Z}_p^\times}(M_{\mathscr{F}\otimes_E V_k}^+)\cong\mathrm{Res}_{\mathbb{Z}_p^\times}(M_{\mathscr{F}}^+)\otimes_E V_k$ is given by the diagonal action of $w$.

For $\mu \in E$, define $\mathscr{F}[\mathfrak{c}=\mu]$ (resp. $\mathscr{F}\{\mathfrak{c}=\mu\}$) to be the subsheaf of $(\mathfrak{c}=\mu)$-eigenspace (resp. generalized $(\mathfrak{c}=\mu)$-eigenspace). It is clear that these are $\mathrm{GL}_2(\mathbb{Q}_p)$-subsheaves of $\mathscr{F}$ over $\mathbb{P}_1(\mathbb{Q}_p)$.

Let $D$ be a $(\varphi,\Gamma)$-module over $\mathcal{R}_E$. Assume there is an involution $w$ on $D^{\psi=0} = D \boxtimes \mathbb{Z}_p^*$. Let $\delta : \mathbb{Q}_p^\times \to E^\times$ be a continuous character. Assume that $(D,\delta,w)$ is compatible in the sense of [5, § 3.1.2]. Let $\mathscr{G}_{D,\delta,w}$ be the associated $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf of analytic type over $\mathbb{P}^1(\mathbb{Q}_p)$. We will frequently use Colmez's notation $D \boxtimes_{\delta,w} U := \mathscr{G}_{D,\delta,w}(U)$ for $U \subset \mathbb{P}^1(\mathbb{Q}_p)$.

Let $w_k := w \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is an involution on $(D \otimes_E V_k) \boxtimes \mathbb{Z}_p^\times \cong (D \boxtimes \mathbb{Z}_p^\times) \otimes_E V_k$.

**Proposition 3.1.** *If $(D,\delta,w)$ is compatible, then $(D \otimes_E V_k, z^k\delta, w_k)$ is compatible, and there is a natural isomorphism of $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaves over $\mathbb{P}^1(\mathbb{Q}_p)$:*

$$\mathscr{G}_{D\otimes_E V_k,\delta z^k,w_k} \xrightarrow{\sim} \mathscr{G}_{D,\delta,w} \otimes_E V_k.$$

*In particular, we have a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism $(D \otimes_E V_k) \boxtimes_{z^k\delta,w_k} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{\delta,w} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E V_k$.*

*Proof.* From the data $(D \otimes_E V_k, z^k\delta, w_k)$, we can construct a sheaf $\mathscr{G}'$ over $\mathbb{P}^1(\mathbb{Q}_p)$ as in [5, § 3.1.1] with $\mathscr{G}'(\mathbb{P}^1(\mathbb{Q}_p)) = \{(x,y) \in (D \otimes_E V_k) \times (D \otimes_E V_k) \mid w_k(\mathrm{Res}_{\mathbb{Z}_p^\times}(x)) = \mathrm{Res}_{\mathbb{Z}_p^\times}(y)\}$, which is equipped with an action of the group $\tilde{G}$ in [5, Rem. 3.1] using the formulas in [5, § 3.1.1]. It is then straightforward to check $(\mathscr{G}_{D,\delta} \otimes_E V_k)(\mathbb{P}^1(\mathbb{Q}_p)) \to \mathscr{G}'(\mathbb{P}^1(\mathbb{Q}_p))$, $(x,y) \mapsto (x,y)$ induces an isomorphism of sheaves over $\mathbb{P}^1(\mathbb{Q}_p)$, which is equivariant under the $\tilde{G}$-action. As $\mathscr{G}_{D,\delta} \otimes_E V_k$ is a $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf, the $\tilde{G}$-action on $\mathscr{G}'$ factors through $\mathrm{GL}_2(\mathbb{Q}_p)$. The proposition follows. □

**Corollary 3.2.** *Suppose $(D,\delta,w)$ is compatible. Let $\mu \in E$ such that $(D \otimes_E V_k)[\mathfrak{c}=\mu] \neq 0$. Then $((D \otimes_E V_k)[\mathfrak{c}=\mu], z^k\delta, w_k)$ and $((D \otimes_E V_k)\{\mathfrak{c}=\mu\}, z^k\delta, w_k)$ are compatible. And we have natural isomorphisms of $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaves over $\mathbb{P}^1(\mathbb{Q}_p)$:*

$$\mathscr{G}_{D\otimes_E V_k[\mathfrak{c}=\mu],\delta z^k,w_k} \xrightarrow{\sim} (\mathscr{G}_{D,\delta,w} \otimes_E V_k)[\mathfrak{c}=\mu], \quad \mathscr{G}_{D\otimes_E V_k\{\mathfrak{c}=\mu\},\delta z^k,w_k} \xrightarrow{\sim} (\mathscr{G}_{D,\delta,w} \otimes_E V_k)\{\mathfrak{c}=\mu\}.$$

*In particular, we have $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphisms*

$$(D \otimes_E V_k)[\mathfrak{c}=\mu] \boxtimes_{z^k\delta,w_k} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{\delta,w} \mathbb{P}^1(\mathbb{Q}_p))[\mathfrak{c}=\mu],$$
$$(D \otimes_E V_k)\{\mathfrak{c}=\mu\} \boxtimes_{z^k\delta,w_k} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{\delta,w} \mathbb{P}^1(\mathbb{Q}_p))\{\mathfrak{c}=\mu\}.$$

*Proof.* By the above proposition, the involution $w_k$ comes from the $w$-action on $\mathscr{G}_{D\otimes_E V_k,\delta z^k,w_k}(\mathbb{Z}_p^\times)$ hence commutes with $\mathfrak{c}$. We see in particular $w_k$ stabilizes $(D \otimes_E V_k)[\mathfrak{c}=\mu] \boxtimes \mathbb{Z}_p^\times$ and $(D \otimes_E V_k)\{\mathfrak{c}=\mu\} \boxtimes \mathbb{Z}_p^\times$. The restriction maps also commute with $\mathfrak{c}$, hence $(D \otimes_E V_k)[\mathfrak{c}=\mu] = (\mathscr{G}_{D\otimes_E V_k,\delta z^k,w_k}[\mathfrak{c}=\mu])(\mathbb{Z}_p)$ (resp. $(D \otimes_E V_k)\{\mathfrak{c}=\mu\} = (\mathscr{G}_{D\otimes_E V_k,\delta z^k,w_k}\{\mathfrak{c}=\mu\})(\mathbb{Z}_p)$). The corollary then follows by the same argument as in Proposition 3.1. □

### 3.2 $p$-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

Let $\delta_D : \mathbb{Q}_p^\times \to E^\times$ be the character such that $\wedge^2 D \cong \mathcal{R}_E(\delta_D\varepsilon)$. Recall that by [5, Thm. 0.1], if $D$ is indecomposable, there exists a unique involution $w_D$ such that $(D,\delta_D,w_D)$ is compatible. We briefly recall the construction and some properties of $w_D$.

(1) When $D$ is irreducible, then there exist a continuous character $\chi$ and an étale $(\varphi,\Gamma)$-module $D_0$ such that $D \cong D_0 \otimes_{\mathcal{R}_E} \mathcal{R}_E(\chi)$. Let $\mathcal{D}_0$ be the continuous étale $(\varphi,\Gamma)$-module over

$B_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} E$ associated to $D_0$. One defines first an involution $w_{\mathcal{D}_0}$ on $\mathcal{D}_0^{\psi=0}$ (see [4, Rem. II.1.3]). Then the restriction of $w_{\mathcal{D}_0}$ on $(\mathcal{D}_0^\dagger)^{\psi=0}$ extends uniquely to an involution $w_{D_0}$ on $D_0$ such that $(D_0, \delta_{D_0}, w_{D_0})$ is compatible.(cf. [4, § V.2]). Let $w_D := w_{D_0} \otimes \chi(-1)$. It is straightforward to check that $(D, \delta_D, w_D)$ is also compatible and $D \boxtimes_{\delta_D, w_D} \mathbb{P}^1(\mathbb{Q}_p) \cong (D_0 \boxtimes_{\delta_{D_0}, w_{D_0}} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E \chi \circ \det$.

(2) When $D$ is a non-split extension of $\mathcal{R}_E(\delta_2)$ by $\mathcal{R}_E(\delta_1)$. On each $\mathcal{R}_E(\delta_i)^{\psi=0}$, there is a unique involution $w_i$ such that $(\mathcal{R}_E(\delta_i), \delta_i, w_i)$ is compatible (cf. [5, Rem. 3.8 (i), § 4.3]), and there is an exact sequence (cf. [5, Thm. 6.8]):

$$0 \to \mathcal{R}_E(\delta_1) \boxtimes_{\delta_D, w_1} \mathbb{P}^1(\mathbb{Q}_p) \to D \boxtimes_{\delta_D, w_D} \mathbb{P}^1(\mathbb{Q}_p) \to \mathcal{R}_E(\delta_2) \boxtimes_{\delta_D, w_2} \mathbb{P}^1(\mathbb{Q}_p) \to 0.$$

**Remark 3.3.** *(1) If $D$ contains a pathological submodule, i.e. up to twist $D$ is isomorphic to a non-de Rham extension $\mathcal{R}_E - t^n \mathcal{R}_E$ with $n \in \mathbb{Z}_{\geq 0}$, then by [5, § 6.5.1, 6.5.2], the $\mathfrak{c}$-action on $D \boxtimes_{\delta_D, w_D} \mathbb{P}^1(\mathbb{Q}_p)$ is not scalar. While for other cases, $\mathfrak{c}$ is scalar.*

*(2) Suppose $D$ does not have pathological submodules, and assume $D$ has Sen weights $(0, \alpha_D)$ with $\alpha_D \in E \setminus \mathbb{Z}_{<0}$. The $\mathfrak{gl}_2$-action on $D$ induced from $D \boxtimes_{\delta_D, w_D} \mathbb{P}^1(\mathbb{Q}_p)$ coincides with the one given in § 2.2.2 with $\alpha = 0$ when $\alpha_D = 0$.*

We write $D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p) := D \boxtimes_{\delta_D, w_D} \mathbb{P}^1(\mathbb{Q}_p)$. Recall that we have a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant exact sequence (cf. [5, Thm. 0.1])

$$0 \to \pi(D)^* \otimes_E \delta_D \circ \det \to D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p) \to \pi(D) \to 0, \tag{16}$$

where $\pi(D)$ is the locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ (of central character $\delta_D$) corresponding to $D$ in the $p$-adic local Langlands correspondence. Note that if $D' \cong D \otimes_{\mathcal{R}_E} \mathcal{R}_E(\chi)$, then $D' \boxtimes_{\delta_{D'}} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E \chi \circ \det$, hence $\pi(D') \cong \pi(D) \otimes_E \chi \circ \det$.

## 3.3 Change of weights

Twisting $D$ by a continuous character, we assume $D$ has Sen weights $(0, \alpha_D)$ with $\alpha_D \in E \setminus \mathbb{Z}_{<0}$. Let $k \in \mathbb{Z}_{\geq 1}$. We deduce from (16) an exact sequence

$$0 \to \pi(D)^* \otimes_E V_k \otimes_E \delta_D \circ \det \to (D \otimes_E V_k) \boxtimes_{\delta_D z^k, w_{D,k}} \mathbb{P}^1 \to \pi(D) \otimes_E V_k \to 0.$$

Let $\mu \in E$, we have two exact sequences:

$$0 \to (\pi(D)^* \otimes_E V_k \otimes_E \delta_D)\{\mathfrak{c} = \mu\} \to (D \otimes_E V_k)\{\mathfrak{c} = \mu\} \boxtimes_{\delta_D z^k, w_{D,k}} \mathbb{P}^1 \to (\pi(D) \otimes_E V_k)\{\mathfrak{c} = \mu\} \to 0,$$

$$0 \to (\pi(D)^* \otimes_E V_k \otimes_E \delta_D)[\mathfrak{c} = \mu] \to (D \otimes_E V_k)[\mathfrak{c} = \mu] \boxtimes_{\delta_D z^k, w_{D,k}} \mathbb{P}^1 \to (\pi(D) \otimes_E V_k)[\mathfrak{c} = \mu]. \tag{17}$$

**Theorem 3.4.** *Assume $D$ is indecomposible and $D$ does not have pathological submodules.*

*(1) Assume $\alpha_D \notin \mathbb{Z}$ or $\alpha_D \in \mathbb{Z}$ and $\alpha \geq k$. For $i = 0, \cdots, k$, $D_{(i,\alpha+k-i)} \boxtimes_{\delta_{D_{(i,\alpha+k-i)}}} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p))[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]$ and $\pi(D_{(i,\alpha+k-i)}) \cong (\pi(D) \otimes_E V_k)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]$.*

*(2) Assume $\alpha_D \neq 0$ or $D$ not de Rham, then $D_{(0,\alpha+k)} \boxtimes_{\delta_{D_{(0,\alpha+k)}}} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p)[\mathfrak{c} = (\alpha + k)^2 - 1]$ and $\pi(D_{(0,\alpha+k)}) \cong (\pi(D) \otimes_E V_k)[\mathfrak{c} = (\alpha + k)^2 - 1]$.*

*Proof.* The first isomorphisms in (1) and (2) follow directly from Corollary 3.2, Proposition 2.19 and the uniqueness of the compatible involution (cf. [5, Prop. 3.17, Rem. 3.8], noting $\delta_{D_{(i,\alpha+k-i)}} =$

$z^k\delta_D$). For the second isomorphisms, we only prove (1), (2) following by similar arguments. We have an exact sequence

$$0 \to \pi(D_{(i,\alpha+k-i)})^* \otimes_E \delta_{D_{(i,\alpha+k-i)}} \to D_{(i,\alpha+k-i)} \boxtimes_{\delta_{D_{(i,\alpha+k-i)}}} \mathbb{P}^1(\mathbb{Q}_p) \to \pi(D_{(i,\alpha+k-i)}) \to 0.$$

By (16), the same argument as in [6, Lem. 3.21] and the fact $\pi(D_{(i,\alpha+k-i)})$ does not have finite dimensional subrepresentations, we see the injection

$$(\pi(D)^* \otimes_E V_k \otimes_E \delta_D)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1] \longrightarrow D_{(i,\alpha+k-i)} \boxtimes_{\delta_{D_{(i,\alpha+k-i)}}} \mathbb{P}^1(\mathbb{Q}_p)$$

factors through $\pi(D_{(i,\alpha+k-i)})^* \otimes_E \delta_{D_{(i,\alpha+k-i)}}$. The quotient of $\pi(D_{(i,\alpha+k-i)})^* \otimes_E \delta_{D_{(i,\alpha+k-i)}}$ by $(\pi(D)^* \otimes_E V_k \otimes_E \delta_D)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]$ injects into the $E$-space of compact type $(\pi(D) \otimes_E V_k)[\mathfrak{c} = (\alpha + k - 2i)^2 - 1]$, which, by the same argument as in [6, Lem. 3.21], has to be finite dimensional. As $\pi(D_{(i,\alpha+k-i)})$ does not have finite dimensional subrepresentations, we deduce the second isomorphism in (1). □

**Remark 3.5.** *(1) When $D$ is trianguline, certain cases (concerning $\pi(D)$) were also obtained in [12, Thm. 5.2.11].*

*(2) Suppose $\alpha_D = 0$ and $D$ is not de Rham. By Theorem 3.4 (2), one easily sees the right map in (17) for such $D$ and $\mu = k^2 - 1$ is surjective.*

We move to $\alpha = 0$ and de Rham case. This case is different, as the translation in this case does not directly give non-trivial $(\varphi, \Gamma)$-submodules (i.e. submodules other than $t^i D$). If $D$ is moreover non-trianguline, we let $\pi(D, i)$ for $i \in \mathbb{Z}$ be Colmez's representations in [6] (for $D = \nabla$ of *loc. cit.*).

**Theorem 3.6.** *Assume $D$ is indecomposable, de Rham of Hodge-Tate weights $(0, 0)$. Then $(D, z^k\delta_D, w_{D,k})$ and $(t^k D, z^k\delta_D, w_{D,k})$ are compatible. We have*

$$D \boxtimes_{z^k\delta_D, w_{D,k}} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p))[\mathfrak{c} = k^2 - 1],$$

*and an exact sequence*

$$0 \to D \boxtimes_{z^k\delta_D, w_{D,k}} \mathbb{P}^1(\mathbb{Q}_p) \to ((D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E V_k)\{\mathfrak{c} = k^2 - 1\} \to t^k D \boxtimes_{z^k\delta_D, w_{D,k}} \mathbb{P}^1(\mathbb{Q}_p) \to 0.$$

*If $D$ is moreover non-trianguline, then $(\pi(D)^* \otimes_E V_k)[\mathfrak{c} = k^2 - 1] \cong \pi(D, k)^*$, and we have an exact sequence*

$$0 \to \pi(D, k)^* \to (\pi(D)^* \otimes_E V_k)\{\mathfrak{c} = k^2 - 1\} \to \pi(D, -k)^* \to 0. \tag{18}$$

*Proof.* The first part of the theorem follows directly from Proposition 2.19 (4) and Corollary 3.2. Assume $D$ is non-trianguline, by the uniqueness of the involution ([5, Prop. 3.17]) $D \boxtimes_{z^k\delta_D, w_{D,k}} \mathbb{P}^1(\mathbb{Q}_p)$ (resp. $t^k D \boxtimes_{z^k\delta_D, w_{D,k}} \mathbb{P}^1(\mathbb{Q}_p)$) is just the representation $D \boxtimes_{z^k\delta_D} \mathbb{P}^1(\mathbb{Q}_p)$ (resp. $t^k D \boxtimes_{z^k\delta_D} \mathbb{P}^1(\mathbb{Q}_p)$) of [6, § 3.3]. Similarly as in the proof of Theorem 3.4, using the same argument as in [6, Lem. 3.21] by comparing (17) and the exact sequence in [6, Rem. 3.20], we deduce $(\pi(D)^* \otimes_E V_k)[\mathfrak{c} = k^2 - 1] \cong \pi(D, k)^*$. By [6, Prop. 3.23], we have

$$0 \to \pi(D, -k)^* \otimes_E \delta_D \to t^k D \boxtimes_{z^k\delta_D} \mathbb{P}^1(\mathbb{Q}_p) \to \pi(D, k) \to 0.$$

Again by similar arguments in [6, Lem. 3.21], the composition

$$(\pi(D)^* \otimes_E V_k \otimes_E \delta_D)\{\mathfrak{c}=k^2-1\} \to ((D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E V_k)\{\mathfrak{c}=k^2-1\} \twoheadrightarrow t^k D \boxtimes_{z^k\delta_D} \mathbb{P}^1(\mathbb{Q}_p)$$

factors through $\kappa : (\pi(D)^* \otimes_E V_k)\{\mathfrak{c}=k^2-1\} \to \pi(D,-k)^*$. Similarly, the composition

$$\begin{aligned}\pi(D,-k)^* \otimes_E \delta_D \longrightarrow ((D \boxtimes_{\delta_D} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E V_k)\{\mathfrak{c}=k^2-1\}/D \boxtimes_{z^k\delta_D} \mathbb{P}^1(\mathbb{Q}_p) \\ \longrightarrow (\pi(D) \otimes_E V_k)\{\mathfrak{c}=k^2-1\}/\pi(D,-k) \quad (19)\end{aligned}$$

has to be zero, so $\kappa$ is surjective. This concludes the proof. □

**Remark 3.7.** *(1) We have* $t^k D \boxtimes_{z^k\delta_D, w_{D,k}} \mathbb{P}^1(\mathbb{Q}_p) \cong (D \boxtimes_{z^{-k}\delta_D} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E z^k \circ \det \cong (\check{D} \boxtimes_{z^k\delta_{\check{D}}} \mathbb{P}^1(\mathbb{Q}_p))^\vee \otimes_E z^k \circ \det$*, where* $\check{D} := D^\vee \otimes_E \varepsilon$*, see [5, Prop. 3.2] for the last isomorphism.*

*(2) As* $\pi(D,-k)^* \subsetneq \pi(D,k)^*$*, we see that the right map in (17) is not surjective in the case (for* $\mu = k^2-1$*), in contrast to Remark 3.5 (2).*

*(3) The second part of the theorem also holds in the trianguline case. We discuss the representations* $\pi(D,i)$ *in the corresponding cases. Twisting by smooth characters, there are only two cases (noting* $D$ *is indecomposable):*

*(a)* $D \cong [\mathcal{R}_E(|\cdot|) - \mathcal{R}_E]$*,*

*(b)* $D \cong [\mathcal{R}_E - \mathcal{R}_E]$*, is the unique de Rham non-split extension.*

*For the case (a), we let* $\pi(D,-i) := \Pi_i$*,* $\pi(D,i) := \Pi_i'$ *be as in [5, Prop. 6.13] for* $i \in \mathbb{Z}_{>0}$*. The second part of Theorem 3.6 for such* $D$ *follows by similar arguments in the proof and [5, Prop. 6.13].*

*For the case (b), let* $\mathrm{val}_p := \mathbb{Q}_p^* \to E$ *be the smooth character sending* $p$ *to* $1$*, to which we associate a smooth character* $\eta : \mathbb{Q}_p^* \to E[\epsilon]/\epsilon^2$*,* $a \mapsto 1 + \mathrm{val}_p(a)\epsilon$*. Note* $\eta$ *is a two dimensional smooth representation of* $\mathbb{Q}_p^*$ *over* $E$*. For* $i \in \mathbb{Z}_{\geq 0}$ *(resp.* $i \in \mathbb{Z}_{<0}$*), let* $\pi(D,i) := (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} z^i \eta \otimes 1)^{\mathrm{an}}$ *(resp.* $\pi(D,i) := (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} z^i \eta \otimes 1)^{\mathrm{an}} \otimes_E z^{-i} \circ \det$*) (which has central character* $\varepsilon^{-1} z^{|i|}$*). By discussions in [5, § 6.5.1], the second part of Theorem 3.6 in this case similarly follows.*

*(4) In all cases, let* $\pi_\infty(D)$ *be the smooth representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *associated to* $D$ *via the classical local Langlands correspondence. Let* $D' \subset D$ *be a* $(\varphi,\Gamma)$*-submodule of Sen weights* $(0,k)$*, and assume* $D'$ *is indecomposable. By [6, Prop. 2.4, Rem. 2.5], we have (note* $\delta_{D'} = z^k\delta_D$*)*

$$D' \boxtimes_{\delta_{D'}} \mathbb{P}^1(\mathbb{Q}_p) \hookrightarrow D \boxtimes_{z^k\delta_D, w_{D,k}} \mathbb{P}^1(\mathbb{Q}_p),$$

*which induces* $\pi(D')^* \hookrightarrow \pi(D,k)^*$*. Moreover, we have*

$$\pi(D,-k)^* \subsetneq \pi(D')^* \subsetneq \pi(D,k)^*,$$

*with* $\pi(D,k)^*/\pi(D')^* \cong \pi(D')^*/\pi(D,-k)^* \cong (\pi_\infty(D) \otimes_E V_k)^*$*. When* $D$ *is irreducible or be as in case (3)(a),* $\pi(D,k)^*/\pi(D,-k)^* \cong ((\pi_\infty(D) \otimes_E V_k)^*)^{\oplus 2}$*. Moreover, by [5, Thm. 6.15] [6, Thm. 0.6(iii)], the map* $D' \mapsto \pi(D')^*$ *is a one-to-one correspondence between the* $(\varphi,\Gamma)$*-submodules of Sen weights* $(0,k)$ *to the subrepresentation of* $\pi(D,k)^*$ *of quotient isomorphic to* $(\pi_\infty(D) \otimes_E V_k)^*$ *(which also corresponds to non-split extensions of* $(\pi_\infty(D) \otimes_E V_k)^*$ *by* $\pi(D,-k)^*$*, cf. [2, Lem. 3.1.3][8, Thm. 2.5]). When* $D$ *is as in (3)(b), then* $D$ *admits a unique indecomposible* $(\varphi,\Gamma)$*-submodule of weights* $(0,k)$*, which has the form* $[t^k\mathcal{R}_E - \mathcal{R}_E]$*. Correspondingly, in this case* $\pi(D,k)^*/\pi(D,-k)^*$ *is a non-split self-extension of* $(\pi_\infty(D) \otimes_E V_k)^*$*, and* $\pi(D')^*$ *is the (unique) subrepresentation of* $\pi(D,k)^*$ *of quotient* $(\pi_\infty(D) \otimes_E V_k)^*$*.*

We finally discuss the translations on the $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaves associated to rank one $(\varphi,\Gamma)$-modules. Twisting by characters, it suffices to consider $\mathcal{R}_E \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ for a continuous character $\delta$ of $\mathbb{Q}_p^*$. Let $\alpha := \mathrm{wt}(\delta)+1$, the corresponding $\mathfrak{gl}_2$-action on $\mathcal{R}_E$ satisfies $\mathfrak{z} = \mathrm{wt}(\delta)$, and $u^- = \frac{\nabla(\nabla-\alpha)}{t}$. We have by [5, Prop. 4.14], $(\mathcal{R}_E \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p))^\vee \cong \mathcal{R}_E(\varepsilon)\boxtimes_{\delta^{-1}} \mathbb{P}^1(\mathbb{Q}_p) \cong (\mathcal{R}_E \boxtimes_{\delta^{-1}\varepsilon^{-2}} \mathbb{P}^1(\mathbb{Q}_p)) \otimes_E \varepsilon\circ\det$. So it suffices to consider the case $\alpha \in E\setminus \mathbb{Z}_{<0}$, and we assume this is so. The following theorem follows easily from Proposition 2.21 (applied to $\Delta = \mathcal{R}_E$) and Corollary 3.2.

**Theorem 3.8.** *We have $((\mathcal{R}_E \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p))\otimes_E V_k)[\mathfrak{c} = (\alpha+k)^2-1] \cong \mathcal{R}_E \boxtimes_{z^k\delta} \mathbb{P}^1(\mathbb{Q}_p)$, and an exact sequence*

$$0 \to \mathcal{R}_E \boxtimes_{z^k\delta} \mathbb{P}^1(\mathbb{Q}_p) \to ((\mathcal{R}_E \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p))\otimes_E V_k)\{\mathfrak{c} = (\alpha+k)^2-1\} \to t^k\mathcal{R}_E \boxtimes_{z^k\delta} \mathbb{P}^1(\mathbb{Q}_p) \to 0.$$

**Remark 3.9.** *By [5, Prop. 4.12], we have an exact sequence*

$$0 \to ((\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta\otimes 1)^{\mathrm{an}})^* \otimes_E \delta\circ\det \to \mathcal{R}_E \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) \to (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1}\otimes \delta\varepsilon)^{\mathrm{an}} \to 0.$$

*We refer to [12] for a detailed study of translations on locally analytic principal series.*

### 3.4 Some complements

In this section, let $D$ be an indecomposible $(\varphi,\Gamma)$-module of rank 2, equipped with the induced $\mathfrak{gl}_2$-action from $D\boxtimes_{\delta_D}\mathbb{P}^1(\mathbb{Q}_p)$. We assume moreover $D$ does not contain pathological submodules.

Suppose $D$ is not trianguline. We reveal and generalize Colmez's operator $\partial$ on $\pi(D)^*$ [6]. By [9, Cor. 2.7], $u^+$ is injective on $\pi(D)^*$. By the same argument as in Lemma 2.16, we have

**Lemma 3.10.** *Assume $D$ is not trianguline, the $P^+$-equivariant composition*

$$\jmath_k : (\pi(D)^*\otimes_E V_k)[\mathfrak{c}=\mu] \longhookrightarrow \pi(D)^*\otimes_E V_k \twoheadrightarrow \pi(D)^* \tag{20}$$

*is injective.*

The following lemma follows by direct calculation.

**Lemma 3.11.** *Let $M$ be an $E$-vector space equipped with a $\mathfrak{gl}_2$-action. Let $\alpha\in E$, assume on $M$, $\mathfrak{c} = \alpha^2-1$ and $\mathfrak{z} = \alpha - 1$. Then for $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$, we have on $M$:*

$$u^+ \,\mathrm{Ad}_g(u^+) = (-ca^+ + au^+)(-c(a^+-\alpha)+au^+).$$

*In particular, if $u^+$, and $\mathrm{Ad}_g(u^+)$ are injective on $M$, so are the operators $(-ca^+ + au^+)$ and $(-c(a^+-\alpha)+au^+)$.*

Consider the $k=1$ case. By the same argument as in the proof of Lemma 2.17 (with $D$ replaced by $\pi(D)^*$), $(\pi(D)^*\otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2-1]$ has the form $v = v_0\otimes e_0 + v_1\otimes e_1$ with $v_0\in\pi(D)^*$ satisfying $a^+(v_0)\in u^+\pi(D)^*$ and $v_1 = \frac{a^+}{u^+}v_0$ (well-defined as $u^+$ is injective). The map $\jmath_1 : (\pi(D)^*\otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2-1] \hookrightarrow \pi(D)^*\otimes_E V_1 \to \pi(D)^*$ sends $v$ to $v_0$. We let $\partial := \frac{a^+}{u^+} : \mathrm{Im}(\jmath_1)\to\pi(D)^*$.

**Lemma 3.12.** *For $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathbb{Q}_p)$, and $u\in\mathrm{Im}(\jmath_1)$, we have $g(u)\in(-c\partial + a)\,\mathrm{Im}(\jmath_1)$. Moreover, $\jmath_1(g(v)) = \det(g)(-c\partial+a)^{-1}g(\jmath_1(v))$ for $v\in(\pi(D)^*\otimes_E V_1)[\mathfrak{c} = (\alpha+1)^2-1]$.*

*Proof.* Write $v = v_0 \otimes e_0 + v_1 \otimes e_1$. Hence $\jmath_1(g(v)) = cg(v_1) + dg(v_0)$. As $a^+(v_0) = u^+(v_1)$, we have $\mathrm{Ad}_g(a^+)g(v_0) = \mathrm{Ad}_g(u^+)g(v_1)$. So $u^+ \mathrm{Ad}_g(u^+)\jmath_1(g(v)) = u^+ \mathrm{Ad}_g(ca^+ + du^+)g(v_0)$. By a direct calculation, $\mathrm{Ad}_g(ca^+ + du^+) = \det(g)(-c(a^+ - \alpha + 1) + au^+)$. Together with Lemma 3.11, $(-ca^+ + au^+)\jmath_1(g(v)) = u^+ g(\jmath(v))$. The lemma follows. □

Let $\jmath_1^i$ be the similar map with $(\pi(D)^* \otimes_E V_{i-1})[\mathfrak{c} = (\alpha + i - 1)^2 - 1]$ replacing $\pi(D)^*$. It is easy to see by induction that (15) holds with $D$ replaced by $\pi(D)^*$. We have $\jmath_k = \jmath_1^k \circ \jmath_1^{k-1} \cdots \circ \jmath_1^1$, and operators:

$$\mathrm{Im}(\jmath_1^k) \xrightarrow{\partial} \mathrm{Im}(\jmath_1^{k-1}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{Im}(\jmath_1^2) \xrightarrow{\partial} \pi(D)^*. \tag{21}$$

By Lemma 3.12 and induction (with an analogue of (15)), we have:

**Proposition 3.13.** *For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$, and $u \in \mathrm{Im}(\jmath_k)$, we have $g(u) \in (-c\partial + a)^k \mathrm{Im}(\jmath_k)$. Moreover, $\jmath_k(g(v)) = \det(g)^k(-c\partial + a)^{-k} g(\jmath_k(v))$.*

**Remark 3.14.** *(1) In particular, one can construct the representation $(\pi(D)^* \otimes_E V_k)[\mathfrak{c} = (\alpha + k)^2 - 1]$ from $\pi(D)^*$: Let $M$ be the subspace of $\pi(D)^*$ consisting of vectors $v$ such that $\nabla_i(v) \in (u^+)^i \pi(D)^*$ for $i = 1, \cdots, k-1$, where $\nabla_i := (a^+ - i + 1) \cdots a^+$. For $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $v \in M$, one can show that $g(v)$ lies in $(-c\partial + a)^k M$. The formula*

$$g *_k v := \det(g)^k (-c\partial + a)^k g(v)$$

*defines a $\mathrm{GL}_2(\mathbb{Q}_p)$-action on $M$. The topology on $M$ is a bit subtle. If $M$ is closed in $\pi(D)^*$ (for example when $D$ is de Rham, by [10, Prop. 9.1]), we equip $M$ with the induced topology. In general, using (6), from $v_0 := v \in M$, we inductively construct $\{v_i\}_{i=0,\cdots,k}$ with $v_i \in \pi(D)^*$, and obtain an injection $M \hookrightarrow \pi(D)^* \otimes_E V_k$, $v \mapsto \sum_{i=0}^k v_i \otimes e_i$. The image is closed with $\pi(D)^* \otimes_E V_k \cong (\pi(D)^*)^{\oplus k+1}$ as topological vector space (as it is exactly the $\mathfrak{c} = (\alpha + k)^2 - 1$ eigenspace), and we equip $M$ with the induced topology. It is then clear $M \cong (\pi(D)^* \otimes_E V_k)[\mathfrak{c} = (\alpha + k)^2 - 1]$. When $D$ is de Rham of constant Sen weights $(0, 0)$, $\mathrm{Im}(\jmath_k) = D$, this reveals Colmez's construction of $\pi(D, k)^*$ ($\cong (\pi(D)^* \otimes_E V_k)[\mathfrak{c} = k^2 - 1]$).*

*(2) If $u^+$ is not injective or equivalently $D$ is trianguline, the kernel of $\jmath_1$ consists exactly of $v_1 \otimes e_1$ with $v_1 \in \pi(D)^*[u^+ = 0]$, and is not stabilized by $\mathrm{GL}_2(\mathbb{Q}_p)$. So in this case, we can not directly construct $(\pi(D)^* \otimes_E V_1)[\mathfrak{c} = (\alpha + 1)^2 - 1]$ from certain subspaces of $\pi(D)^*$ using a twisted $\mathrm{GL}_2(\mathbb{Q}_p)$-action.*

Finally, we discuss the relation of involutions. We keep the assumption on $D$ in the first paragraph of the section (while $D$ can be trianguline). Let $D' \subset D$ be a $(\varphi, \Gamma)$-submodule of Sen weights $(0, \alpha + k)$. If $\alpha \neq 0$ or $\alpha = 0$ and $D$ is not de Rham, then $D' \cong D_{(0,\alpha+k)}$. Similarly, as in (21), we have operators

$$\partial^k : D_{(0,\alpha+k)} \xrightarrow{\partial} D_{(0,\alpha+k-1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} D_{(0,\alpha+1)} \xrightarrow{\partial} D.$$

If $\alpha = 0$ and $D$ is de Rham, then similarly as in (21) we have $\partial^k : D \to D$. In any case, we have $\partial^k = \frac{\nabla_k}{t^k}$. We have the following relation on the involutions.

**Proposition 3.15.** *We have $w_{D'} = w_D \circ \frac{\nabla_k}{t^k} = w_D \circ \partial^k$.*

*Proof.* We only prove the case for $k=1$, the general case following by an induction argument.

Assume first $\alpha \neq 0$ or $\alpha = 0$ and $D$ is not de Rham. We have $D' = D_{(0,\alpha+1)}$. By Theorem 3.4, we see $w_{D'} = w_D \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $(D')^{\psi=0} \hookrightarrow D^{\psi=0} \otimes_E V_1$. For $v = v_0 \otimes e_0 + v_1 \otimes e_1$, $\jmath_1(w_{D'}(v)) = w_D(v_1) = w_D(\partial(v_0)) = w_D(\partial(\jmath_1(v)))$.

Assume now $\alpha = 0$ and $D$ de Rham, by the same argument we have $w_{D,1} = w_D \circ \partial$ as operator on $D^{\psi=0}$. By [6, Prop. 2.4, Rem. 2.5], $w_{D'} = w_{D,1}|_{(D')^{\psi=0}}$. The proposition follows. □

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