# Construction of Biorthogonal Wavelets from Pseudo-splines

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## Abstract

Pseudo-splines constitute a new class of refinable functions with B-splines, interpolatory refinable functions and refinable functions with orthonormal shifts as special examples. Pseudo-splines were first introduced by Daubechies, Han, Ron and Shen in [10] and Selenick in [36], and their properties were extensively studied by Dong and Shen in [11]. It was further shown by Dong and Shen in [12] that the shifts of an arbitrarily given pseudo-spline are linearly independent. This implies the existence of biorthogonal dual refinable functions (of pseudo-splines) with an arbitrarily prescribed regularity. However, except for B-splines, there is no explicit construction of biorthogonal dual refinable functions with any given regularity. This paper focuses on an implementable scheme to derive a dual refinable function with a prescribed regularity. This automatically gives a construction of smooth biorthogonal Riesz wavelets with one of them being a pseudo-spline. As an example, an explicit formula of biorthogonal dual refinable functions of the interpolatory refinable function is given.

*Key words:* B-spline, biorthogonal Riesz wavelets, interpolatory, pseudo-spline, Riesz wavelets.

# 1  Introduction

Pseudo-splines were first introduced in [10] and [36] to obtain tight framelets via the unitary extension principle of [32] with better approximation orders. They were then extended and extensively studied in [11]. Pseudo-splines are compactly supported refinable functions in $L_2(\mathbb{R})$. Recall that a function $\phi \in L_2(\mathbb{R})$ is *refinable* if it satisfies the refinement equation

$$\phi = 2 \sum_{k \in \mathbb{Z}} a(k)\phi(2 \cdot -k), \qquad (1.1)$$

for some sequence $a \in \ell_2(\mathbb{Z})$. The sequence $a$ is the refinement mask of $\phi$.

By $L_2(\mathbb{R})$ we denote all the functions $f(x)$ satisfy

$$\|f(x)\|_{L_2(\mathbb{R})} := \left( \int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2} < \infty;$$

and $\ell_2(\mathbb{Z})$ the set of all sequences $u$ defined on $\mathbb{Z}$ such that

$$\|u\|_{\ell_2(\mathbb{Z})} := \left( \sum_{k \in \mathbb{Z}} |u(k)|^2 \right)^{\frac{1}{2}} < \infty.$$

The *Fourier-Laplace transform* of a compactly supported (measurable) function $f$ is defined by

$$\widehat{f}(\omega) := \int_{\mathbb{R}} f(x)e^{-i\omega x} dx, \qquad \omega \in \mathbb{C}.$$

When $f$ is compactly supported and bounded, the Fourier-Laplace transform of $f$ is analytic. When $\omega$ is restricted to $\mathbb{R}$, $\widehat{f}$ becomes the *Fourier transform* of $f$.

The *Fourier series* $\widehat{u}$ of a sequence $u$ in $\ell_2(\mathbb{Z})$ is defined by

$$\widehat{u}(\xi) := \sum_{k \in \mathbb{Z}} u(k)e^{-ik\xi}, \qquad \xi \in \mathbb{R}.$$

With these, the refinement equation (1.1) can be written in terms of its Fourier transform as

$$\widehat{\phi}(\xi) = \widehat{a}(\xi/2)\widehat{\phi}(\xi/2), \qquad \xi \in \mathbb{R}.$$

We also call $\widehat{a}$ a *refinement mask*, or just *mask* for convenience.

Pseudo-splines are defined in terms of their refinement masks. The refine-

ment mask of a *pseudo-spline of type I with order* $(m, l)$ is given by

$$|_1\widehat{a}(\xi)|^2 := |_1\widehat{a}_{(m,l)}(\xi)|^2 := \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2)$$

(1.2)

and the refinement mask of a *pseudo-spline of type II with order* $(m, l)$ is given by

$$_2\widehat{a}(\xi) := {}_2\widehat{a}_{(m,l)}(\xi) := \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2), \quad (1.3)$$

where $m \geq 1$ and $0 \leq l \leq m - 1$. We note that $|_1\widehat{a}_{(m,l)}(\xi)|^2 = {}_2\widehat{a}_{(m,l)}(\xi)$. Hence, $_1\widehat{a}_{(m,l)}$ is the square root of $_2\widehat{a}_{(m,l)}$, which is a $2\pi$-periodic trigonometric polynomial with real coefficients by Fejér-Riesz Lemma (see e.g. [8]). The corresponding pseudo-splines can be defined in terms of their Fourier transforms as

$$_k\widehat{\phi}_{(m,l)}(\xi) := \prod_{j=1}^{\infty} {}_k\widehat{a}_{(m,l)}(2^{-j}\xi), \; k = 1, 2,$$

with $_k\widehat{\phi}_{(m,l)}(0) = 1$. Unless it is necessary, we use $_k a$ and $_k\phi$ in stead of $_k a_{(m,l)}$ and $_k\phi_{(m,l)}$, $k = 1, 2$, i.e. we drop the subscript "$(m, l)$" or "$k$" in $_k a_{(m,l)}$ and $_k\phi_{(m,l)}$ for simplicity, whenever it is clear from the context.

The first type of *pseudo-splines* were introduced in [10] and [36] in their constructions of tight framelets derived from the unitary extension principle of [32] with desired approximation order for the truncated frame series. The second type of pseudo-splines were introduced in [11] and [36], where in [11] a detailed analysis of regularity and constructions of short Riesz wavelets and (anti)symmetric tight framelets were given. Pseudo-splines constitute a large class of refinable functions which includes B-splines, the orthogonal refinable functions (i.e. the refinable function with orthonormal shifts constructed by [9]) and the interpolatory refinable functions (which are the autocorrelations of the orthogonal refinable functions and were first studied by [13]) as its special cases. Recall that a B-spline (see e.g. [1]) with order $m$ and its refinement mask are defined by

$$\widehat{B}_m(\xi) = e^{-ij\frac{\xi}{2}} \left( \frac{\sin(\xi/2)}{\xi/2} \right)^m \quad \text{and} \quad \widehat{a}(\xi) = e^{-ij\frac{\xi}{2}} \cos^m(\xi/2),$$

where $j = 0$ when $m$ is even, $j = 1$ when $m$ is odd. A continuous function $\phi$ is said to be interpolatory if

$$\phi(j) = \delta(j), \quad j \in \mathbb{Z},$$

where $\delta(0) = 1$ and $\delta(j) = 0$, for $j \neq 0$. By definitions of refinement masks of pseudo-splines given in (1.2) and (1.3), one can see that when $l = 0$, pseudo-splines are B-splines; when $l = m - 1$, pseudo-splines are the orthogonal

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refinable functions for type I and the interpolatory refinable functions for type II. Pseudo-splines of the other orders fill in the gaps between B-splines and the orthogonal refinable functions for type I, and B-splines and the interpolatory refinable functions for type II.

For a given compactly supported $\phi \in L_2(\mathbb{R})$, a *shift* (integer translation) *invariant space* generated by $\phi$ is defined by

$$V_0(\phi) := \overline{\text{Span}\{\phi(\cdot - k), \ k \in \mathbb{Z}\}}. \tag{1.4}$$

We say that the generator $\phi$ is *stable*, if $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ forms a Riesz basis for $V_0(\phi)$. The stability of a function can be characterized by its *bracket product*. The bracket product of functions $f, \ g \in L_2(\mathbb{R})$ is defined by

$$[\widehat{f}, \widehat{g}](\xi) := \sum_{k \in \mathbb{Z}} \widehat{f}(\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi k)}.$$

A compactly supported distribution $\phi$ is said to be *pre-stable* if there exists $C_1 > 0$ such that

$$[\widehat{\phi}, \widehat{\phi}](\xi) \geq C_1,$$

for almost all $\xi \in \mathbb{R}$. It is known that (see e.g. [23]) the pre-stability of a compactly supported distribution $\phi$ is equivalent to that the Fourier transform of $\phi$ does not have $2\pi$-periodic zeros, i.e.

$$\left(\widehat{\phi}(\xi + 2\pi k)\right)_{k \in \mathbb{Z}} \neq \mathbf{0}, \quad \text{for all } \xi \in \mathbb{R}, \tag{1.5}$$

where $\mathbf{0}$ is zero sequence. A function $\phi \in L_2(\mathbb{R})$ is stable if and only if there exist $C_1, C_2 > 0$ s.t.

$$C_1 \leq [\widehat{\phi}, \widehat{\phi}](\xi) \leq C_2, \tag{1.6}$$

for almost all $\xi \in \mathbb{R}$. Since $\phi$ is compactly supported, the upper bound in (1.6) holds immediately (see e.g. [23] and [24]). Therefore, a compactly supported function $\phi \in L_2(\mathbb{R})$ is stable if and only if it is pre-stable.

Another related, but stronger, concept used here is the linear independence of $\phi$ and its shifts. The shifts of a compactly supported distribution $\phi$ is *linearly independent*, if

$$\sum_{j \in \mathbb{Z}} b(j)\phi(\cdot - j) = 0 \quad \text{implies} \quad b(j) = 0, \text{ for all } j \in \mathbb{Z} \text{ and } b \in \ell(\mathbb{Z}),$$

where $\ell(\mathbb{Z})$ denotes the space of all complex valued sequences defined on $\mathbb{Z}$. For a finitely supported sequence $a$, we define the Laurent polynomial $\tilde{a}(z)$ as

$$\tilde{a}(z) := \sum_{j \in \mathbb{Z}} a(j)z^j, \quad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

If $a$ is the refinement mask of a compactly supported refinable function $\phi \in L_2(\mathbb{R})$, the Laurent polynomial $\tilde{a}$ is called the *symbol* of $\phi$, and the refinement

equation (1.1) can be written in terms of its Fourier-Laplace transform as

$$\widehat{\phi}(\omega) = \tilde{a}(e^{-i\omega/2})\widehat{\phi}(\omega/2), \quad \text{for all } \omega \in \mathbb{C}.$$

It was shown in [30] that the shifts of a compactly supported distribution are linearly independent if and only if the Fourier-Laplace transform of $\phi$ satisfies

$$\left(\widehat{\phi}(\omega + 2\pi k)\right)_{k \in \mathbb{Z}} \neq \mathbf{0}, \quad \text{for all } \omega \in \mathbb{C}. \tag{1.7}$$

Comparing (1.5) and (1.7), we can see immediately that for a compactly supported function $\phi \in L_2(\mathbb{R})$, linear independence of the shifts of $\phi$ implies the stability of $\phi$. Actually (see e.g. [12] and [24]), the linear independence of the shifts of a compactly supported refinable function $\phi \in L_2(\mathbb{R})$ is equivalent to that $\phi$ is stable and the symbol $\tilde{a}(z)$ does not have symmetric zeros on $\mathbb{C} \setminus \{0\}$, i.e. $\tilde{a}(z)$ and $\tilde{a}(-z)$ do not have common zeros on $\mathbb{C} \setminus \{0\}$. Based on this, it was proved in [12] that all pseudo-splines have linearly independent shifts, i.e. all pseudo-splines are stable and their symbols do not have symmetric zeros on $\mathbb{C} \setminus \{0\}$. One should consult [12] for more details.

We shall also introduce the concept of *multiresolution analysis* (MRA), since all constructions considered in this paper is based on MRA. Define

$$V_j(\phi) := \{f(2^j \cdot) : f \in V_0(\phi), \ j \in \mathbb{Z}\},$$

where $V_0(\phi)$ is defined in (1.4) with $\phi \in L_2(\mathbb{R})$ being a compactly supported refinable function. Then, the sequence of spaces $(V_j)_{j \in \mathbb{Z}}$ forms an MRA generated by $\phi$, i.e. (i) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$, (ii) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L_2(\mathbb{R})$, $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ (see e.g. [2] and [22]).

For a given wavelet $\psi \in L_2(\mathbb{R})$, define the *wavelet system* by

$$X(\psi) := \{\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k), \ j, k \in \mathbb{Z}\}.$$

We call the system $X(\psi)$ a Bessel system if for some $C_1 > 0$, and for every $f \in L_2(\mathbb{R})$,

$$\sum_{g \in X(\psi)} |\langle f, g \rangle|^2 \leq C_1 \|f\|_{L_2(\mathbb{R})}^2.$$

A Bessel system $X(\psi)$ is a *Riesz basis* for $L_2(\mathbb{R})$ if there exists $C_2 > 0$ such that

$$C_2 \|\{c_{j,k}\}\|_{\ell_2(\mathbb{Z}^2)} \leq \left\| \sum_{(j,k) \in \mathbb{Z}^2} c_{j,k}\psi_{j,k} \right\|_{L_2(\mathbb{R})}, \quad \text{for all } \{c_{j,k}\} \in \ell_2(\mathbb{Z}^2),$$

and the span of $X(\psi)$ is dense in $L_2(\mathbb{R})$. We call the function $\psi$ *Riesz wavelet* and $X(\psi)$ *Riesz wavelet system*, if $X(\psi)$ forms a Riesz basis for $L_2(\mathbb{R})$. Two

wavelet systems $X(\psi)$ and $X(\psi^d)$ are said to be *biorthogonal Riesz wavelet bases*, if they are Riesz wavelet systems and for all $f \in L_2(\mathbb{R})$,

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}^d.$$

Moreover, we call $\psi$ and $\psi^d$ *biorthogonal (Riesz) wavelets*. The main goal of this paper is to construct a pair of compactly supported biorthogonal Riesz wavelets $X(\psi)$ and $X(\psi^d)$, such that $\psi$ is a linear combination of a pseudo-spline and the *dual wavelet $\psi^d$* satisfies any prescribed regularity.

Now we give a general framework of the MRA-based construction of biorthogonal wavelets starting from a given refinable function. Constructions of biorthogonal wavelets have been extensively studied in the literature. The interested reader can find general discussions in [3], [4], [7], [8], [16] and [19], and the references there.

Let $\phi \in L_2(\mathbb{R})$ be a compactly supported stable refinable function with finitely supported refinement mask $a$. The first step of the construction of a pair of compactly supported biorthogonal wavelets is to find a compactly supported stable refinable function $\phi^d \in L_2(\mathbb{R})$ with finitely supported refinement mask $a^d$ satisfying

$$\langle \phi, \phi^d(\cdot - k) \rangle = \delta(k), \quad k \in \mathbb{Z}. \tag{1.8}$$

If a stable refinable function $\phi^d \in L_2(\mathbb{R})$ satisfies (1.8), we call it the *(biorthogonal) dual refinable function* of $\phi$, or just *dual* of $\phi$ for simplicity. A necessary condition for $\phi$ and $\phi^d$ to satisfy (1.8) is

$$\widehat{a}\overline{\widehat{a^d}} + \widehat{a}(\cdot + \pi)\overline{\widehat{a^d}(\cdot + \pi)} = 1. \tag{1.9}$$

We call $a^d$ a *dual refinement mask*, or just *dual mask* for convenience. Most constructions starts with finding $a^d$ to satisfy (1.9). Suppose we have a dual mask $a^d$ in hand. We then need to check whether the corresponding refinable function $\phi^d$ is in $L_2(\mathbb{R})$ and stable, which can be done through the transition operator (see e.g. [4], [27] or [34]). With the stable dual pair $\phi$ and $\phi^d$ and their refinement masks $a$ and $a^d$ satisfying (1.9), the dual pair of wavelets can be constructed (see e.g. [4] and [8]) as

$$\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi), \quad \text{and} \quad \widehat{\psi}^d(2\xi) = \widehat{b}^d(\xi)\widehat{\phi}^d(\xi), \tag{1.10}$$

where

$$\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a^d}(\xi + \pi)} \quad \text{and} \quad \widehat{b}^d(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi + \pi)}. \tag{1.11}$$

Then the corresponding wavelet systems $X(\psi)$ and $X(\psi^d)$ form biorthogonal Riesz wavelet bases for $L_2(\mathbb{R})$ (see e.g. [3], [4] and [19]). Since the mask $a$ is assumed through out this paper to be finitely supported, the wavelet mask $b^d$ is also finitely supported. Therefore, $\psi^d$ can be written as a linear combination of $\phi^d$, which means that $\psi^d$ has the same regularity as $\phi^d$.

As we see from this framework, the key step in the construction is to design a pair of stable refinable functions satisfying (1.8). In the rest of this paper, we shall focus on the constructions of dual refinable functions $\phi^d$ from pseudo-splines with prescribed regularity.

This paper is organized as follows. Section 2 is devoted to constructions of dual refinable functions from pseudo-splines of both types and provide a regularity analysis. We shall give an implementable construction to obtain a class of dual refinable functions satisfying any prescribed regularity from an arbitrarily given pseudo-spline. In section 3, a rather explicit formula of dual refinable functions from pseudo-splines of type II with order $(m, m - 1)$ is provided. Two examples of biorthogonal wavelets constructed in Section 3 are given in the last section.

## 2   Duals of Pseudo-splines

In this section, we construct biorthogonal dual refinable functions from pseudo-splines, which can satisfy arbitrarily high order of regularity.

The regularity is defined as the followings: Recall that for $\alpha = n + \beta$, $n \in \mathbb{N}$, $0 \leq \beta < 1$, the Hölder space $C^\alpha$ (see e.g. [8]) is defined to be the set of functions which are $n$ times continuously differentiable and such that the $n$-th derivative $f^{(n)}$ satisfies the following condition,

$$|f^{(n)}(x + h) - f^{(n)}(x)| \leq C|h|^\beta, \ \forall x, h.$$

The number $\alpha$ is called the *regularity (exponent)* of $f$. It is well known (see e.g. [8]) that if

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|(1 + |\xi|)^\alpha < \infty,$$

then $f \in C^\alpha$. In particular, if $|\widehat{f}(\xi)| \leq C(1 + |\xi|)^{-1-\alpha-\varepsilon}$ holds for an arbitrary small $\varepsilon > 0$, $f \in C^\alpha$, which means that the regularity of $f$ can be estimated via the decay of its Fourier transform.

We first give the existence of dual refinable functions with the prescribed regularity which immediately follows from the result of [25].

**Theorem 2.1** *[25]. Let $\phi \in L_2(\mathbb{R})$ be compactly supported refinable function whose shifts are linearly independent. Then, for an arbitrary $\alpha > 0$, there exists a compactly supported refinable function $\phi^d \in L_2(\mathbb{R})$ with regularity $\alpha$, such that $\phi^d$ is the biorthogonal dual refinable function of $\phi$.*

Applying this theorem together with the fact that the shifts of pseudo-splines are linearly independent, we have:

**Corollary 2.2** *Let $\phi$ be a pseudo-spline. Then, for an arbitrary $\alpha > 0$, there exists a compactly supported refinable function $\phi^d \in L_2(\mathbb{R})$ with regularity $\alpha$, such that $\phi^d$ is the biorthogonal dual refinable function of $\phi$.*

**Remark 2.3**

(1) *The original theorem of [25] is stated in a different way. The compactly supported refinable function $\phi$ is assumed in [25] to be stable and have a minimal support. (A stable refinable function $\phi$ having a minimal support means, according to [25], that its symbol does not have symmetric zeros on $\mathbb{C} \setminus \{0\}$.) This is equivalent to that $\phi$ has linearly independent shifts by Lemma 2.1 of [12] (see also [24]).*

(2) *In the approach taken by [25], for a given compactly supported refinable function $\phi \in L_2(\mathbb{R})$ with linearly independent shifts, the existence of a compactly supported dual refinable function satisfying any desired regularity is reduced to the existence of a compactly supported dual refinable function in $L_2(\mathbb{R})$. The proof of existence of a compactly supported dual refinable function in $L_2(\mathbb{R})$ for a given $\phi$ starts with a finitely supported dual mask of some refinable distribution, which is derived by solving (2.2) numerically and may not even be pre-stable. Then use this mask and another sequence obtained by truncating the standard infinite dual mask of $a$ to derive a finitely supported dual mask of $a$ whose corresponding refinable function is in $L_2(\mathbb{R})$ and stable. To obtain a dual refinable function with higher regularity, it repeats the above processing by constructing an $L_2$ dual of function $B_m * \phi$ instead of $\phi$. To see this (see also [25]), let us consider $B_m * \phi$, with any given $m \geq 1$, where $B_m$ is B-spline of order $m$ whose Fourier transform is*

$$\widehat{B_m} := \left( \frac{1 - e^{-i\xi}}{i\xi} \right)^m.$$

*It can be easily verified that $B_m * \phi$ has linearly independent shifts. If there is a compactly supported dual refinable function $g \in L_2(\mathbb{R})$ of $B_m * \phi$, then $\phi^d := B_m(-\cdot) * g$ is a compactly supported dual of $\phi$ with regularity at least $m - 1 - \varepsilon$. Indeed, since for compactly supported functions $\phi, \phi^d \in L_2(\mathbb{R})$,*

$$\langle \phi, \phi^d(\cdot - k) \rangle = \delta(k), \quad k \in \mathbb{Z},$$

*is equivalent to*

$$[\widehat{\phi}, \widehat{\phi^d}] = 1,$$

*(see e.g. [4] and [8]), we have*

$$[\widehat{\phi}, \overline{\widehat{B_m}}\widehat{g}] = [\widehat{B_m}\widehat{\phi}, \widehat{g}] = 1.$$

Next, we explore a constructive way to get duals of pseudo-splines with prescribed regularities. For this, we first note that if the pseudo-spline of type

II with order $(m, l)$ has a compactly supported dual refinable function with regularity $\alpha$, then we can obtain a compactly supported refinable function with regularity at least $\alpha$ that is dual to the pseudo-spline of type I with the same order. Indeed, for the pseudo-spline $_2\phi_{(m,l)}$ of type II with order $(m, l)$, let $_2\phi^d \in L_2(\mathbb{R})$ be its compactly supported dual refinable function with regularity $\alpha$. Since $_2\widehat{\phi}_{(m,l)} = |_1\widehat{\phi}_{(m,l)}|^2 = {}_1\widehat{\phi}_{(m,l)} \cdot \overline{{}_1\widehat{\phi}_{(m,l)}}$, we have

$$1 = [{}_2\widehat{\phi}_{(m,l)}, {}_2\widehat{\phi}^d] = [{}_1\widehat{\phi}_{(m,l)} \cdot \overline{{}_1\widehat{\phi}_{(m,l)}}, {}_2\widehat{\phi}^d] = [{}_1\widehat{\phi}_{(m,l)}, {}_1\widehat{\phi}_{(m,l)} \cdot {}_2\widehat{\phi}^d].$$

Therefore,

$$_1\widehat{\phi}^d := {}_1\widehat{\phi}_{(m,l)} \cdot {}_2\widehat{\phi}^d \tag{2.1}$$

is a compactly supported dual refinable function with the regularity at least $\alpha$ by the fact that $_1\widehat{\phi}_{(m,l)} \in L_\infty(\mathbb{R})$. Hence, we only need to construct dual refinable functions of pseudo-splines of type II. In the rest of this section, we focus on discussions of dual refinable functions of pseudo-splines of type II with any prescribed regularity.

Construction of compactly supported dual refinable function $\phi^d$ always starts from constructing a dual mask $a^d$ from $a$ such that (1.9) is satisfied. This can be done whenever the symbol $\tilde{a}(z)$ does not have symmetric zeros on $\mathbb{C} \setminus \{0\}$. In fact, it is well known that in this case (see e.g. [4], [8] and [21]) one can always find $\tilde{a}^d(z)$ such that

$$\tilde{a}(z)\tilde{a}^d(z^{-1}) + \tilde{a}(-z)\tilde{a}^d(-z^{-1}) = 1, \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.2}$$

Indeed, let

$$\tilde{a}_e(z^2) := \sum_{j \in \mathbb{Z}} a(2j)z^{2j} \quad \text{and} \quad \tilde{a}_o(z^2) := \sum_{j \in \mathbb{Z}} a(2j+1)z^{2j}.$$

Then,

$$\tilde{a}(z) = \tilde{a}_e(z^2) + z\tilde{a}_o(z^2) \quad \text{and} \quad \tilde{a}(-z) = \tilde{a}_e(z^2) - z\tilde{a}_o(z^2). \tag{2.3}$$

Since $\tilde{a}(z)$ does not have symmetric zeros on $\mathbb{C} \setminus \{0\}$, $\tilde{a}_e(z^2)$ and $\tilde{a}_o(z^2)$ do not have common zeros on $\mathbb{C} \setminus \{0\}$ by (2.3). Then the Hilbert's Nullstellensatz assures the existence of Laurent polynomials $\tilde{q}_e$ and $\tilde{q}_o$ such that

$$\tilde{a}_e(z^2)\tilde{q}_e(z^2) + \tilde{a}_o(z^2)\tilde{q}_o(z^2) = \frac{1}{2}z^{2k}, \quad \text{for all } z \in \mathbb{C} \setminus \{0\} \text{ and } k \in \mathbb{N}. \tag{2.4}$$

Let

$$\tilde{q}(z) := \tilde{q}_e(z^2) + z^{-1}\tilde{q}_o(z^2),$$

and define

$$\tilde{a}^d(z) := z^{2k}\tilde{q}(z^{-1}).$$

Then, $\tilde{a}$ and $\tilde{a}^d$ satisfy (2.2) by applying (2.3) and (2.4). Let $a^d$ be the coefficients of $\tilde{a}^d(z)$. We conclude that $\widehat{a}$ and $\widehat{a}^d$ satisfy (1.9).

The solutions to (2.2) can be obtained by solving a polynomial equation utilizing **Maple** and **Singular** [15], which is an Ad-hoc construction, although sometimes it can be very efficient in both univariate and multivariate constructions (see e.g. [31]). The more efficient and systematic way of solving equation (2.2) is the method called *construction by cosets* (CBC), which was suggested in [7] and [16]. The method starts with a dual mask of a given refinement mask, then lifts the dual mask to a new dual mask whose underlying refinable function satisfies a desired order of the Stang-Fix condition. It should also be pointed out that the CBC algorithm gives the minimal support of the dual refinable functions for a given order of the Strang-Fix condition. All approaches of solving equation (2.2) normally derive dual refinable functions that satisfy some given order of Strang-Fix condition. The regularity has to be checked one by one numerically using methods given in [6], [8], [17], [20] and [33], although the regularity of a refinable function seems to increase as the order of Strang-Fix condition increases by numerical tests. Furthermore, since (2.2) is only a necessary condition for the underlying refinable functions $\phi$ and $\phi^d$ to be a dual pair for any given solution of equation (2.2), one needs to further check the stability of $\phi^d$, which can also be done numerically by methods given in [8] and [27].

Our method for pseudo-splines is similar to the both methods above in the aspect that we also start with a dual refinement mask satisfying very mild conditions, then create new dual masks from it. The difference is that we obtain new dual masks from this initial mask, whose underlying refinable functions are stable and have prescribed regularities. Since the regularity of a compactly supported refinable function implies its order of the Strang-Fix condition (see e.g. [5], [28] and [29]), and since once the prescribed regularity is given, the method gives a dual with the given regularity by choosing a proper parameter, our approach gains more than what the above methods may offer to pseudo-splines.

We start from an arbitrary pseudo-spline $\phi$ of type II with order $(m, l)$, $m \geq 2$, $0 \leq l \leq m - 1$, whose refinement mask is $a$. The first step is to find an initial finitely supported dual mask $b$. As we will see that for the case $m = 1$, the construction and the regularity analysis have already been considered in [4] (also see [8]).

**Condition 2.4** *Let $b$ be a finitely supported mask satisfying:*

*(1) $b$ is a (real-valued) dual mask of $a$, i.e.*

$$\widehat{a}(\xi)\overline{\widehat{b}(\xi)} + \widehat{a}(\xi + \pi)\overline{\widehat{b}(\xi + \pi)} = 1;$$

*(2) $\widehat{b}$ is real-valued and nonnegative;*
*(3) The refinable distribution $\vartheta$, corresponding to the refinement mask $b$, is pre-stable.*

**Remark 2.5** *Note that we did not require $\vartheta$ to be a function, and just require that it is pre-stable. Actually, by Corollary 2.2, there always exists a mask $b$ such that $\vartheta$ is a compactly supported stable refinable function in $L_2(\mathbb{R})$, which is a much more strong condition than part (3) above. For a given refinement mask $a$, it is not difficult to find such an initial dual mask $b$ by CBC method of [7] and [16]. Once we have this $b$, the prescribed regularity dual refinable function can be built up.*

The idea here is to use the mask $\widehat{c} := \widehat{a}\widehat{b}$. Let $\zeta$ be the corresponding refinable distribution of $c$. We will show that $c$ and $\zeta$ satisfy the following properties:

**Proposition 2.6** *Let $\phi$ be a pseudo-spline of type II with mask $a$ and $\vartheta$ be the refinable distribution corresponding to the mask $b$, which satisfies all the conditions in Condition 2.4. Let $\widehat{c} = \widehat{a}\widehat{b}$ and $\zeta$ be the corresponding refinable distribution. Then:*

*(1) $\widehat{c}$ is real-valued and nonnegative;*
*(2) $\zeta$ belongs to $L_2(\mathbb{R})$;*
*(3) $\zeta$ is stable.*

**Proof.** Part (1) is immediate by the fact that both $\widehat{a}$ and $\widehat{b}$ are real-valued and nonnegative.

Part (2) can be established by using Lemma 6.2.1 of [8]. Indeed, since the trigonometric polynomial $\widehat{c}$ is nonnegative and $\widehat{c}(\xi) + \widehat{c}(\xi + \pi) = 1$, there exists (by Fejér-Riesz Lemma) a trigonometric polynomial $\widehat{h}$ such that $|\widehat{h}|^2 = \widehat{c}$ and $|\widehat{h}(\xi)|^2 + |\widehat{h}(\xi + \pi)|^2 = 1$. Let $f$ be the corresponding refinable distribution to mask $h$. Lemma 6.2.1 of [8] gives that $\widehat{f} \in L_2(\mathbb{R})$. Since $|\widehat{f}|^2 = \widehat{\zeta}$, we conclude that $\widehat{\zeta} \in L_1(\mathbb{R})$. Hence, $\zeta$ is compactly supported and continuous, which gives that $\zeta \in L_2(\mathbb{R})$.

For part (3), since $\zeta$ is compactly supported and belongs to $L_2(\mathbb{R})$, we only need to show that $\zeta$ is pre-stable by checking whether $\widehat{\zeta}$ has $2\pi$-periodic zeros or not (see e.g. [23]). We first prove that the set of all zeros of $\widehat{\phi}$ is $\{2\pi p\}_{p \in \mathbb{Z} \setminus \{0\}}$. Note that $\widehat{\phi}$ can be written as $\widehat{\phi} = \widehat{B}_{2m}\widehat{g}$ where $g$ is a refinable distribution with refinement mask $d$ defined by

$$\widehat{d}(\xi) := \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2).$$

Applying the following identity of Lemma 2.2 in [11] and letting $y = \sin^2(\xi/2)$,

$$\sum_{j=0}^{l} \binom{m+l}{j} y^j (1-y)^{l-j} = \sum_{j=0}^{l} \binom{m-1+j}{j} y^j, \quad y \in \mathbb{R}, \qquad (2.5)$$

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the mask $\widehat{d}$ can be rewritten as

$$\widehat{d}(\xi) = \sum_{j=0}^{l} \binom{m-1+j}{j} \sin^{2j}(\xi/2).$$

Then it is obvious that $\widehat{d} \geq 1$ on $\mathbb{R}$, which implies that $\widehat{g} > 0$ on $\mathbb{R}$. Therefore, the set of all zeros of $\widehat{\phi}$ is the same as that of $\widehat{B}_{2m}$ which is exactly $\{2\pi p\}_{p \in \mathbb{Z} \setminus \{0\}}$.

Now we shall prove the pre-stability of $\zeta$ by contradiction. Suppose that $\xi_0$ is a $2\pi$-periodic zero of $\widehat{\zeta}$, i.e.

$$\widehat{\zeta}(\xi_0 + 2\pi k) = \widehat{\phi}(\xi_0 + 2\pi k)\widehat{\vartheta}(\xi_0 + 2\pi k) = 0,$$

for all $k \in \mathbb{Z}$. Since by assumption, $\widehat{\vartheta}$ does not have $2\pi$-periodic zeros, there must be some $k_0 \in \mathbb{Z}$, such that $\widehat{\phi}(\xi_0 + 2\pi k_0) = 0$. Since the zero set of $\widehat{\phi}$ is $\{2\pi p\}_{p \in \mathbb{Z} \setminus \{0\}}$, there exists $p_0 \in \mathbb{Z} \setminus \{0\}$ such that $\xi_0 + 2\pi k_0 = 2\pi p_0$, i.e. $\xi_0 = 2\pi(p_0 - k_0) =: 2\pi m_0$. This gives that $\widehat{\zeta}(\xi_0 + 2\pi k) = \widehat{\zeta}(2\pi m_0 + 2\pi k) = 0$ for all $k \in \mathbb{Z}$. In particular, when $k = -m_0$, we have $\widehat{\zeta}(0) = 0$. Since $\widehat{\phi}(0) = 1$, we must have $\widehat{\vartheta}(0) = 0$. However, since $\widehat{b}(0) = 1$, we should have that $\widehat{\vartheta}(0) = 1$. This is a contradiction.

Having the mask $c$ in hand, we first note by the construction of $c$ and the fact that $b$ is a dual mask of $a$, we have

$$\widehat{c}(\xi) + \widehat{c}(\xi + \pi) = 1.$$

Thus

$$\left(\widehat{c} + \widehat{c}(\cdot + \pi)\right)^{2n-1} = 1, \quad \text{for } n \geq 2. \tag{2.6}$$

The first $n$ terms of the binomial expansion in (2.6) is

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{2n-1-j} \widehat{c}^j(\cdot + \pi) = \widehat{c}^n \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j} \widehat{c}^j(\cdot + \pi). \tag{2.7}$$

Since $\widehat{c} = \widehat{a}\widehat{b}$, we can factorize one $\widehat{a}$ out from the right hand side of (2.7) and the rest is denoted as $\widehat{a}^d$. As we shall see in a moment that the mask $a^d$ is indeed a dual mask of $a$ and the corresponding refinable function $\phi^d$ is indeed a dual of $\phi$. The detailed construction is given as the followings.

**Construction 2.7** *Let $\phi$ be pseudo-spline of type II with order $(m, l)$ and $a$ be its refinement mask. Let $b$ be the initial dual mask of $a$ satisfying all the conditions in Condition 2.4, and $\widehat{c} = \widehat{a}\widehat{b}$. Then define mask $a^d$ as*

$$\widehat{a}^d := \widehat{b} \cdot \widehat{c}^{n-1} \cdot \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j} \left(1 - \widehat{c}\right)^j. \tag{2.8}$$

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*The corresponding compactly supported refinable function is defined as*

$$\widehat{\phi}^d(\xi) := \prod_{j=1}^{\infty} \widehat{a}^d(2^{-j}\xi).$$

**Remark 2.8** *The idea here is not new. Similar idea as given in the above construction can also be found in [18], [35], [37] and [38]. Furthermore, this idea was used in [21] to construct multivariate biorthogonal wavelets via the multivariate interpolatory refinable functions, which also leads to the dual refinable functions of box splines with arbitrarily high regularity. The interested reader may consult these papers for details. Here, we not only give a construction, but also give a more precise regularity analysis for the construction. It is also worth to point out that one can choose the power $2n$ instead of $2n - 1$ in (2.6). The argument presented here still works after a proper adjustment of the last term in the summation of the definition of $\widehat{a}^d$ (see e.g. [21]). Finally, we note that all the dual refinable functions obtained by Construction 2.7 are symmetric, which is desirable in many applications.*

To ensure that the corresponding refinable functions $\phi^d$ is indeed a dual of $\phi$, we need to verify that (see e.g. [4] or [34]): (1), $a^d$ is a dual mask of $a$, i.e. $a$ and $a^d$ satisfy (1.9); (2), $\phi^d$ is stable. For the first condition, we note that the first $n$ terms of the expansion of (2.6) is exactly $\widehat{a}\overline{\widehat{a}^d}$ and the last $n$ terms of the expansion of (2.6) is exactly $\widehat{a}(\cdot + \pi)\overline{\widehat{a}^d(\cdot + \pi)}$ by applying the identity $\widehat{c}(\cdot + \pi) = 1 - \widehat{c}$. Thus, the first condition follows from identity (2.6). For the second condition, since $\phi^d$ is compactly supported, the stability of $\phi^d$ will follow from that: (1), $\phi^d$ is pre-stable; (2), $\phi^d \in L_2(\mathbb{R})$. We will prove the pre-stability of $\phi^d$ in Proposition 2.10 and $\phi^d \in L_2(\mathbb{R})$ in Theorem 2.11. In fact, Theorem 2.11 says more than $\phi^d \in L_2(\mathbb{R})$. It shows that the regularity exponent of $\phi^d$ increases as we choose larger $n$ in Construction 2.7.

The proof of the following proposition employs the following lemma of [21].

**Lemma 2.9** *[21]. Let $\phi_1$ and $\phi_2$ be two compactly supported refinable functions in $L_2(\mathbb{R})$ with refinement masks $a_1$ and $a_2$. Suppose the set of all zeros of $\widehat{a}_1$ contains that of the mask $\widehat{a}_2$. If $\phi_1$ is pre-stable, then $\phi_2$ is pre-stable.*

**Proposition 2.10** *Let $\phi^d$ be the compactly supported refinable distribution with refinement mask $a^d$ given in (2.8). Then $\phi^d$ is pre-stable.*

**Proof.** To show the the pre-stability of $\phi^d$, we prove that the set of all zeros of $\widehat{a}^d$ coincides with that of $\widehat{c}$. With this, the pre-stability of $\phi^d$ follows from the pre-stability of $\zeta$ by applying Lemma 2.9. In fact, since for $\xi \in \mathbb{R}$

$$\widehat{c}(\xi) \geq 0, \quad \text{and} \quad \widehat{c}(\xi) + \widehat{c}(\xi + \pi) = 1,$$

one obtains that $0 \leq \widehat{c} \leq 1$. Applying (2.5) with $m = n$, $l = n - 1$, $y = 1 - \widehat{c}$

and by the fact that $\widehat{c} \leq 1$, one obtains

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j}(1-\widehat{c})^j = \sum_{j=0}^{n-1} \binom{n-1+j}{j}(1-\widehat{c})^j \geq 1.$$

Since

$$\widehat{a}^d = \widehat{b} \cdot \widehat{c}^{n-1} \cdot \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j}(1-\widehat{c})^j,$$

we have that the set of all zeros of $\widehat{a}^d$ coincides with that of $\widehat{b}\widehat{c}^{n-1}$. Furthermore, since $\widehat{c} = \widehat{a}\widehat{b}$ and since

$$\widehat{b}\widehat{c}^{n-1} = \widehat{b}(\widehat{a}\widehat{b})^{n-1} = \widehat{a}^{n-1}\widehat{b}^n,$$

the set of all zeros of $\widehat{c}$ coincides with that of $\widehat{b}\widehat{c}^{n-1}$ and, hence, coincides with that of $\widehat{a}^d$.

Now we shall analyze the regularity of $\phi^d$ by estimating the decay of $|\widehat{\phi}^d|$, and show that the regularity of $\phi^d$ increases as the parameter $n$ in Construction 2.7 increases.

Let

$$\mathcal{L} := \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j}(1-\widehat{c})^j. \tag{2.9}$$

Then,

$$\widehat{a}^d = \widehat{b}\widehat{c}^{n-1}\mathcal{L}.$$

This gives that

$$\widehat{\phi}^d(\xi) = \widehat{\vartheta}(\xi)\widehat{\zeta}^{n-1}(\xi) \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi). \tag{2.10}$$

Since $|\widehat{\vartheta}|$ is uniformly bounded and since $\widehat{\zeta} = \widehat{\vartheta}\widehat{\phi}$, we have

$$|\widehat{\vartheta}\widehat{\zeta}^{n-1}| = |\widehat{\vartheta}^n\widehat{\phi}^{n-1}| \leq C|\widehat{\phi}^{n-1}|.$$

Recall that the optimal decay of $|\widehat{\phi}|$ was given in Theorem 3.4 of [11], i.e.

$$|\widehat{\phi}(\xi)| \leq C(1+|\xi|)^{-s},$$

where

$$s := 2m - \frac{\log P_{m,l}(\frac{3}{4})}{\log 2} \tag{2.11}$$

and

$$P_{m,l}(y) = \sum_{j=0}^{l} \binom{m+l}{j} y^j(1-y)^{l-j}. \tag{2.12}$$

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Consequently we have

$$|\widehat{\vartheta}(\xi)\widehat{\zeta}^{n-1}(\xi)| \leq C(1+|\xi|)^{-s(n-1)}. \tag{2.13}$$

Since, by (2.5),

$$\mathcal{L} = \sum_{j=0}^{n-1} \binom{n-1+j}{j}(1-\widehat{c})^j,$$

and since $0 \leq \widehat{c} \leq 1$, one can see that $\mathcal{L}$ reaches its maximum value at $\widehat{c} = 0$ (note that $\widehat{c}(\pi) = 0$). Therefore

$$\max_{\xi \in [0,2\pi]} |\mathcal{L}(\xi)| = \binom{2n-1}{n}.$$

Then Lemma 7.1.1 of [8] gives that

$$\prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi) \leq C(1+|\xi|)^{\frac{\log\binom{2n-1}{n}}{\log 2}},$$

and hence, by (2.10), (2.13) and the above inequality, one obtains,

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\gamma}, \tag{2.14}$$

where

$$\gamma := s(n-1) - \frac{\log\binom{2n-1}{n}}{\log 2}. \tag{2.15}$$

Hence $\phi^d \in C^{\gamma-1-\varepsilon}$.

We note that the estimate given here is not optimal. It leads to a lower bound of the regularity of $\phi^d$. We remark that the optimal Sobolev regularity of a given refinable function can be obtained via its mask by applying transfer operator (see [8], [33] and references in there). Although the transfer operator approach is very efficient to compute the exact Sobolev regularity for each given refinable function, it cannot be used to analyze the regularity for a set of refinable functions obtained through a systematic construction.

In the following theorem we will show that for pseudo-splines of type II with order $m \geq 2$, the decay rate $\gamma$ of $|\widehat{\phi}^d|$ increases as $n$ increases. Moreover, an asymptotic analysis of the regularity of $\phi^d$ is provided.

**Theorem 2.11** *Let $\phi^d$ be the compactly supported refinable functions with refinement mask $a^d$ given in (2.8). The decays of $\widehat{\phi}^d$ is given by (2.14). Then:*

*(1) The decay rate $\gamma$ of $\widehat{\phi}^d$ given in (2.15) increases as $n$ increases. Consequently, $\phi^d$ is continuous for all $n \geq 2$ and its regularity exponent increases as $n$ increases, where $\phi^d \in C^{\gamma-1-\varepsilon}$ for all $\varepsilon > 0$. In particular, $\phi^d \in L_2(\mathbb{R})$ for all $n \geq 2$.*

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(2) *Asymptotically for large $n$ with fixed $m$, the decay rate $\gamma$ is $\mu n$, where $\mu = s - 2$ with $s$ defined in (2.11). Consequently we have,*

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\mu n}, \quad \phi^d \in C^{\mu n},$$

*asymptotically for large $n$*

**Proof.** For part (1), we first show that $\gamma$ increases as $n$ increases, which is equivalent to show that

$$M := sn - \frac{\log\binom{2n+1}{n+1}}{\log 2} - s(n-1) + \frac{\log\binom{2n-1}{n}}{\log 2} > 0.$$

Simplifying $M$, one obtains

$$
\begin{aligned}
M &= s - \frac{\log \frac{\binom{2n+1}{n+1}}{\binom{2n-1}{n}}}{\log 2} \\
&= s - \frac{\log \frac{4n+2}{n+1}}{\log 2} \\
&= s - 2 - \frac{\log \frac{n+\frac{1}{2}}{n+1}}{\log 2} \\
&> s - 2.
\end{aligned}
$$

Since the decay rate $s$ decreases as $l$ increases and increases as $m$ increases (see Proposition 3.5 of [11]), and $s > 2.678$ for $m = 2$, $l = 1$, we have that $s > 2.678$ for all $m \geq 2$ and $0 \leq l \leq m - 1$. Hence, we have $M > s - 2 > 0$. Consequently, the regularity exponent $\gamma - 1 - \varepsilon$ of $\phi^d$ increases as $n$ increases. Since for $n = 2$ we have that

$$\gamma = s - \frac{\log 3}{\log 2} > 2.678 - \frac{\log 3}{\log 2} > 1.09,$$

this proves that $\phi^d$ is continuous for all $n \geq 2$ and, hence, $\phi^d \in L_2(\mathbb{R})$ for all $n \geq 2$.

For part (2), we consider the asymptotic behavior of $\gamma$ when $n$ is large. Note that

$$\gamma = (n-1)s - \frac{\log\binom{2n-1}{n}}{\log 2} = n\left((1-\frac{1}{n})s - \frac{\frac{1}{n}\log\binom{2n-1}{n}}{\log 2}\right).$$

We now use Stirling approximation, i.e. $n! \sim \sqrt{2\pi}e^{(n+\frac{1}{2})\log n - n}$ (see e.g. [14]) to estimate $\frac{1}{n}\log\binom{2n-1}{n}$ for large $n$. We have

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$$\frac{1}{n}\log\binom{2n-1}{n} \sim \frac{1}{n}\Big(\log(2n-1)! - \log n! - \log(n-1)!\Big)$$

$$\sim \frac{1}{n}\Big(\log\big(\sqrt{2\pi}e^{(2n-\frac{1}{2})\log(2n-1)-(2n-1)}\big) - \log\big(\sqrt{2\pi}e^{(n+\frac{1}{2})\log n-n}\big)$$

$$- \log\big(\sqrt{2\pi}e^{(n-\frac{1}{2})\log(n-1)-(n-1)}\big)\Big)$$

$$\sim \frac{1}{n}\Big((2n-\frac{1}{2})\log(2n-1) - (2n-1) - (n+\frac{1}{2})\log n + n$$

$$-(n-\frac{1}{2})\log(n-1) + (n-1)\Big)$$

$$\sim \frac{1}{n}\Big((2n-\frac{1}{2})\log(2n-1) - (n+\frac{1}{2})\log n - (n-\frac{1}{2})\log(n-1)\Big)$$

$$\sim 2\log(2n-1) - \log n - \log(n-1)$$

$$\sim \log\Big(\frac{4n^2 - 4n + 1}{n(n-1)}\Big) \sim 2\log 2.$$

Applying the above approximation to the estimate of $\gamma$ one obtains

$$\gamma = n\Big((1-\frac{1}{n})s - \frac{\frac{1}{n}\log\binom{2n-1}{n}}{\log 2}\Big) \sim n\Big(s - \frac{2\log 2}{\log 2}\Big) = n(s-2).$$

Thus we have shown that $\gamma \sim (s-2)n$, asymptotically for large $n$. Consequently, one obtains that for large $n$,

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\mu n}, \quad \phi^d \in C^{\mu n},$$

with $\mu = s - 2$.


So far we have shown in Proposition 2.10 that $\phi^d$ is pre-stable and proved in part (1) of Theorem 2.11 that $\phi^d \in L_2(\mathbb{R})$. Furthermore, $\phi^d$ is compactly supported as one can easily see from the Construction 2.7. Therefore, we conclude that $\phi^d$ is stable. Having the stability of $\phi^d$, together with $a$ and $a^d$ satisfying (1.9), Theorem 3.14 of [34] (also see [4]) leads to the conclusion that $\phi$ and $\phi^d$ is a pair of dual refinable functions, i.e.

$$\langle \phi, \phi^d(\cdot - k)\rangle = \delta(k).$$

Therefore the corresponding pair of biorthogonal Riesz wavelets $\psi$ and $\psi^d$ can be constructed by (1.10) and (1.11), and the systems $X(\psi)$ and $X(\psi^d)$ form a pair of biorthogonal Riesz wavelet bases for $L_2(\mathbb{R})$.

**Remark 2.12** *The pair of masks $\widehat{a}$, $\widehat{a}^d$ in Construction 2.7 can be viewed as*

*one of many possible factorizations of the trigonometric polynomial*

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{\,2n-1-j} \widehat{c}^{\,j}(\cdot + \pi)$$

*given by (2.7). In fact, we can choose factorization $\widehat{h}$ and $\widehat{h}^d$ arbitrarily such that*

$$\widehat{h}\widehat{h}^d = \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{\,2n-1-j} \widehat{c}^{\,j}(\cdot + \pi).$$

*When the compactly supported refinable functions corresponding to the masks $h$ and $h^d$ are in $L_2(\mathbb{R})$ and pre-stable, a dual pair of compactly supported biorthogonal wavelet systems can be derived from them. For example, let $n' > 0$ and define*

$$\widehat{h} := \widehat{c}^{\,n'} \quad and \quad \widehat{h}^d := \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{\,2n-2-n'-j}(1 - \widehat{c})^j, \ n' \geq 1.$$

*As long as $n$ and $n'$ are chosen properly, one can get a desired dual pair of refinement masks for a dual pair of compactly supported refinable functions. In particular, let $\widehat{c} = \cos^2(\xi/2)$ be the mask of piecewise linear B-spline which is interpolatory. Then, the construction here coincides with the biorthogonal wavelet construction given in [4].*

For the dual mask $a^d$ given in Construction 2.7, we cannot have an explicit form of it in general, because we need to find mask $b$ numerically first. For some special pseudo-splines, however, we do have an explicit form for all the dual masks constructed from Construction 2.7. In the next section we will give a detailed construction of dual refinable functions from pseudo-splines of type II with order $(m, m - 1)$.

## 3    Duals of a Special Case

Let $\phi$ be pseudo-spline of type II with order $(m, m - 1)$ with $m \geq 1$, i.e. an interpolatory refinable function, and let $a$ be its refinement mask. Since $\phi$ is interpolatory, the mask $\widehat{a}$ satisfies $\widehat{a} + \widehat{a}(\cdot + \pi) = 1$. Hence, $\widehat{b}$ in Condition 2.4 can be simply chosen to be 1, and the corresponding refinable distribution is $\widehat{\vartheta} = 1$. Then all the conditions in Condition 2.4 are satisfied. Following the construction given by (2.8), one can obtain the dual mask $\widehat{a}^d$ as

$$\widehat{a}^d := \widehat{a}^{n-1} \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{a}^{n-1-j}\left(1 - \widehat{a}\right)^j. \tag{3.1}$$

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The corresponding refinable function $\widehat{\phi}^d$ can be defined as

$$\widehat{\phi}^d(\xi) := \prod_{j=1}^{\infty} \widehat{a}^d(2^{-j}\xi).$$

Since $\widehat{b} = 1$, we have $\widehat{c} = \widehat{b}\widehat{a} = \widehat{a}$, where $c$ given in Proposition 2.6. Therefore, the trigonometric polynomial $\mathcal{L}$ defined in (2.9) can now be written as,

$$\mathcal{L} = \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{a}^{n-1-j} \left(1 - \widehat{a}\right)^{j}.$$

This gives that

$$\widehat{\phi}^d(\xi) = \widehat{\phi}^{n-1}(\xi) \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi).$$

Since $\widehat{a}$ satisfies $0 \leq \widehat{a} \leq 1$ and since $\widehat{\vartheta} = 1$, following a similar argument in Section 2 we have that

$$|\widehat{\phi}^d(\xi)| \leq C(1 + |\xi|)^{-\beta}, \tag{3.2}$$

where the decay rate $\beta$ satisfies

$$\beta = s(n-1) - \frac{\log \binom{2n-1}{n}}{\log 2} \tag{3.3}$$

with $s' = 2m - \frac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}$, and $P_{m,l}(y)$ defined in (2.12). Hence $\phi^d \in C^{\beta-1-\varepsilon}$.

The decay estimates for $|\widehat{\phi}^d|$ here are not accurate. However, for the simplest case when $m = 1$, i.e. $\widehat{a} = \cos^2(\xi/2)$, we do have optimal decay estimate for $|\widehat{\phi}^d|$. Indeed, in this case

$$\widehat{a}^d = \cos^{2n-2}(\xi/2) \sum_{j=0}^{n-1} \binom{2n-1}{j} \cos^{2(n-1-j)}(\xi/2) \sin^{2j}(\xi/2)$$

$$= \cos^{2n-2}(\xi/2) \sum_{j=0}^{n-1} \binom{2n-1}{j} \sin^{2j}(\xi/2)(1 - \sin^2(\xi/2))^{(n-1-j)}.$$

The optimal decay of $\widehat{\phi}^d$ is

$$|\widehat{\phi}^d(\xi)| \leq C(1 + |\xi|)^{-\rho}, \tag{3.4}$$

where

$$\rho := 2(n-1) - \frac{\log P_{n,n-1}(\frac{3}{4})}{\log 2}. \tag{3.5}$$

The complete construction and analysis for this special case have already been given by [4] (see also [8]). In fact, by applying the approach in Remark 2.12

this leads to their construction of a pair of biorthogonal compactly supported symmetric wavelets with any prescribed regularity.

The following table gives the decay rates of $|\widehat{\phi}^d|$ in (3.3) with some choices of $m$ and $n$.

| $\beta$ | n=2 | n=3 | n=4 | n=5 | n=6 |
|---------|-----|-----|-----|-----|-----|
| $m=2$ | 1.0931 | 2.0342 | 2.9049 | 3.7350 | 4.5386 |
| $m=3$ | 1.6871 | 3.2222 | 4.6870 | 6.1110 | 7.5086 |
| $m=4$ | 2.2411 | 4.3282 | 6.3459 | 8.3230 | 10.2736 |

Table 1
In the above estimates of $\beta$, $|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\beta}$.

Next, we will give an asymptotic analysis of the decay of $\widehat{\phi}^d$ given in (3.2) in terms of its refinement mask $\widehat{a}^d$ given in (3.1). For $m=1$, the asymptotic analysis of decay of $\widehat{\phi}^d$ given in (3.4) can be done by following the analysis in [4] or [8], which leads to the optimal decay rate $0.4150\cdots$.

**Proposition 3.1** *Let $\phi^d$ be the refinable function with the refinement mask $a^d$ given in (3.1). The decay of $\widehat{\phi}^d$ is given by (3.2). Then:*

*(1) For fixed $m \geq 2$ and asymptotically for large $n$, we have*

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\nu n} \quad and \quad \phi^d \in C^{\nu n},$$

*where $\nu = 2(m-1) - \dfrac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}$.*
*(2) For fixed $n \geq 2$ and asymptotically for large $m$, we have*

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\sigma m} \quad and \quad \phi^d \in C^{\sigma m},$$

*where $\sigma = \left(2 - \dfrac{\log 3}{\log 2}\right)(n-1)$.*

**Proof.** Part (1) is immediate from part (2) of Theorem 2.11 by letting $s = 2m - \dfrac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}$.

For part (2), let $m$ be asymptotically large and $n$ be fixed. Then,

$$\beta = (n-1)\left(2m - \frac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}\right) - \frac{\log \binom{2n-1}{n}}{\log 2}$$
$$\sim m(n-1)\left(2 - \frac{\frac{1}{m}\log P_{m,m-1}(\frac{3}{4})}{\log 2}\right).$$

Recall that we have already shown in Theorem 3.6 of [11] (see also [8], [39]

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and [26]) that

$$\frac{1}{m}P_{m,m-1}\left(\frac{3}{4}\right) \sim \log 3. \tag{3.6}$$

Applying (3.6) one obtains

$$\beta \sim m(n-1)\left(2 - \frac{\log 3}{\log 2}\right) =: \sigma m.$$

Thus we have shown that with fixed $n$,

$$|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\sigma m} \quad \text{and} \quad \phi^d \in C^{\sigma m},$$

with $\sigma = (n-1)\left(2 - \frac{\log 3}{\log 2}\right)$.

The following tables provide some numerical results for the asymptotic rates $\mu$ and $\sigma$ given by Proposition 3.1.

|       | m=1    | m=2    | m=3    | m=4    | m=5    |
|-------|--------|--------|--------|--------|--------|
| $\nu$ | 0.4150 | 0.6781 | 1.2721 | 1.8251 | 2.3532 |

Table 2
In the above estimates of $\nu$, $|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\nu n}$, asymptotically for large $n$.

|          | n=2    | n=3    | n=4    | n=5    | n=6    |
|----------|--------|--------|--------|--------|--------|
| $\sigma$ | 0.4150 | 0.8301 | 1.2451 | 1.6601 | 2.0752 |

Table 3
In the above estimates of $\sigma$, $|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\sigma m}$, asymptotically for large $m$.

## 4    Examples

In this section we give two examples of biorthogonal Riesz wavelets constructed in Section 3. In the first example, we start with pseudo-spline of type II with order $(2,1)$ and $n = 2$; in the second one, we start with pseudo-spline of type II with order $(3,2)$ and $n = 2$.

**Example 4.1** *We first choose $\widehat{a}$ to be the refinement mask of a pseudo-spline of type II with order $(2,1)$, i.e.*

$$\widehat{a} = \cos^4(\xi/2)(1 + 2\sin^2(\xi/2)).$$

*By Construction 2.7 with $n = 2$ we have that*

$$\widehat{a}^d := \widehat{a}\left(3 - 2 \cdot \widehat{a}\right).$$

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*Define wavelet masks and wavelets as*

$$\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}^d(\xi + \pi)} \quad and \quad \widehat{b}^d(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi + \pi)};$$

$$\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi) \quad and \quad \widehat{\psi}^d(2\xi) = \widehat{b}^d(\xi)\widehat{\phi}^d(\xi),$$

*where $\widehat{\phi}$ and $\widehat{\phi}^d$ are the refinable functions corresponding to the refinement masks $\widehat{a}$ and $\widehat{a}^d$. The systems $X(\psi)$ and $X(\psi^d)$ form a pair of biorthogonal wavelet bases for $L_2(\mathbb{R})$. The figures of $\phi$, $\phi^d$, $\psi$ and $\psi^d$ are given in Figure 1.*
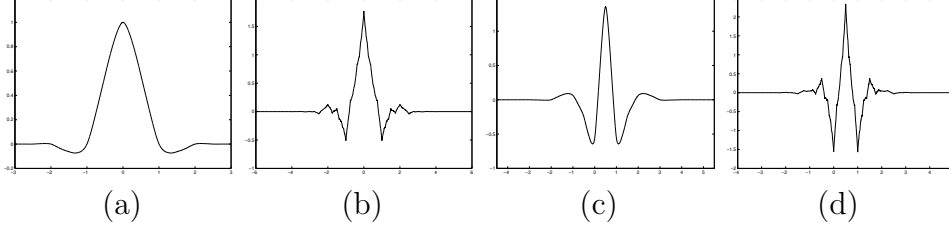


(a)       (b)       (c)       (d)

Fig. 1. The figures of $\phi$ and $\phi^d$ in Example 4.1 are given in graphs (a) and (b). Figures of the corresponding Riesz wavelets $\psi$ and $\psi^d$ are given in (c) and (d).

**Example 4.2** *We first choose $\widehat{a}$ to be the refinement mask of a pseudo-spline of type II with order $(3, 2)$, i.e.*

$$\widehat{a} = \cos^6(\xi/2)(1 + 3\sin^2(\xi/2) + 6\sin^4(\xi/2)).$$

*By Construction 2.7 with $n = 2$ we have that*

$$\widehat{a}^d := \widehat{a}\Big(3 - 2 \cdot \widehat{a}\Big).$$

*Define wavelet masks and wavelets as*

$$\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}^d(\xi + \pi)} \quad and \quad \widehat{b}^d(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi + \pi)};$$

$$\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi) \quad and \quad \widehat{\psi}^d(2\xi) = \widehat{b}^d(\xi)\widehat{\phi}^d(\xi),$$

*where $\widehat{\phi}$ and $\widehat{\phi}^d$ are the refinable functions corresponding to the refinement masks $\widehat{a}$ and $\widehat{a}^d$. The systems $X(\psi)$ and $X(\psi^d)$ form a pair of biorthogonal wavelet bases for $L_2(\mathbb{R})$. The figures of $\phi$, $\phi^d$, $\psi$ and $\psi^d$ are given in Figure 2.*
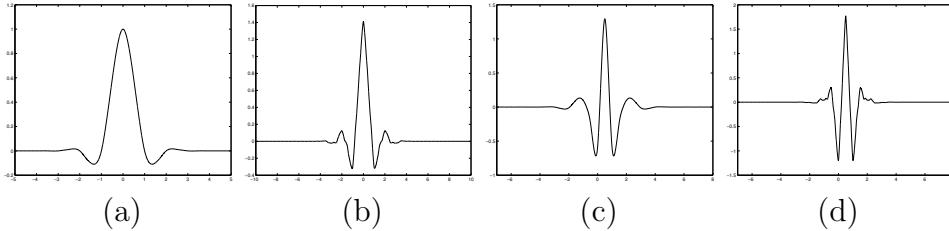


(a)       (b)       (c)       (d)

Fig. 2. The figures of $\phi$ and $\phi^d$ in Example 4.2 are given in graphs (a) and (b). Figures of the corresponding Riesz wavelets $\psi$ and $\psi^d$ are given in (c) and (d).

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