PSEUDO-SPLINES, WAVELETS AND FRAMELETS

DONG BIN

(B.Sc., Peking University)

A THESIS SUBMITTED

FOR THE DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

NATIONAL UNIVERSITY OF SINGAPORE

2005

Acknowledgements

Firstly, I would like to express my sincere thanks to my advisor, Professor Shen Zuowei for his help and guidance all along. The topic, i.e. pseudo-splines, that Professor Shen suggested to me, is really interesting and inspiring. From the research work, I not only discovered many interesting properties of pseudo-splines, but also learnt a lot about basic and modern ideas of wavelet theory. Meanwhile, Professor Shen always encouraged me to solve problems and discover new things independently. It is quite good for me, because I can get maximal training of doing independent research, and this will also greatly benefit my future research work.

I would also like to thank Professor Sherman Riemenschneider for proofreading the whole draft of my first paper [20] (which constitutes the main part of this thesis), and giving many valuable suggestions.

My thank also goes to my fellow graduate students, Mr. Chai Anwei, Mr. Chen Libing, Mr. Pan Suqi, Mr. Xu Yuhong and Mr. Lim Zhi Yuan. During the past two years, we have discussed about wavelets frequently and we even organized many small seminars to exchange our knowledge in wavelets and to learn new ideas together. I would like to thank, especially, Mr. Pan Suqi for helping me check the proofs of some of the results in my papers [20, 21, 22], and Mr. Lim Zhi Yuan for providing me the matlab codes of drawing figures of refinable functions and wavelets.

Finally, I would like to thank Dr. Chew Soon Huat David for providing this wonderful Latex template of thesis.

Contents

Summary

This thesis collects all results about pseudo-splines given in the three papers [20, 21, 22] written by me and Professor Shen Zuowei.

The first type of pseudo-splines were introduced in [17] (see also [55]) to construct tight framelets with desired approximation orders via the *unitary extension principle* of [51]. In the spirit of the first type of pseudo-splines, we shall consider here the second type (see also [55]) of pseudo-splines to construct symmetric or antisymmetric tight framelets with desired approximation orders. Pseudo-splines provide a rich family of refinable functions. B-splines are one of the special classes of pseudo-splines; orthogonal refinable functions (whose shifts form an orthonormal system given in [16]) are another class of pseudo-splines; and so are the interpolatory refinable functions (which are the Lagrange interpolatory functions at $\mathbb{Z}$ and were first discussed in [23]). The other pseudo-splines with various orders fill in the gaps between the B-splines and orthogonal refinable functions for the first type, and between B-splines and interpolatory refinable functions for the second type. This gives a wide range of choices of refinable functions that meets various demands for balancing the approximation power, the length of the support, and the regularity in applications.

This thesis will give a regularity analysis, as well as detailed discussions of some other basic properties, of pseudo-splines of both types and provide various constructions of wavelets and framelets. It is easy to see that the regularity of the first type of pseudo-splines is between B-spline and orthogonal refinable function of the same order. However, there is no precise regularity estimate for pseudo-splines in general. In this thesis, an optimal estimate of the decay of the Fourier transform of all pseudo-splines is given. The regularity of pseudo-splines can then be deduced and hence, the regularity of the corresponding wavelets and framelets. The asymptotical regularity analysis, as the order of the pseudo-splines goes to infinity, is also provided.

In this thesis, we will also show that the shifts of a pseudo-spline are linearly independent. This is stronger than the (more obvious) statement that the shifts of a pseudo-spline form a Riesz system. In fact, the linear independence of a compactly supported (refinable) function and its shifts has been studied in several areas of approximation and wavelet theory (see e.g. [4, 18, 19, 32, 33, 38, 47]). Furthermore, the linear independence of the shifts of a pseudo-spline is a necessary and sufficient condition for the existence of a compactly supported function whose shifts form a biorthogonal dual system of the shifts of the pseudo-spline.

From a given pseudo-spline, a short support Riesz wavelet (that has the same length of support as that of the pseudo-spline) is constructed. The construction is rather simple and natural, however, the proof of the Riesz property of the corresponding wavelet system is highly nontrivial. Furthermore, this short support wavelet is one of the tight framelets constructed from the same pseudo-spline by a method provided both in [17] and this thesis. This reveals that in almost all pseudo-spline tight frame systems constructed so far, there is one framelet whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$.

Finally, we introduce a construction of smooth compactly supported biorthogonal Riesz wavelets using pseudo-splines. In fact, we give an implementable scheme to derive a dual refinable function from pseudo-splines, which satisfies any prescribed regularity. This automatically gives a construction of smooth biorthogonal Riesz wavelets with one of them being a linear combination of a pseudo-spline. As an example, an explicit formula of biorthogonal dual refinable functions of the interpolatory refinable function is given.

List of Tables

List of Figures

Chapter 1

Introduction

Pseudo-splines were first introduced in [17, 55] in order to construct tight framelets with required approximation order of the truncated frame series. Pseudo-splines are refinable and compactly supported. They give a wide variety of choices of refinable functions and provide numerous flexibilities in wavelet and framelet constructions and filter designs. Functions such as B-splines, interpolatory, or orthogonal refinable functions are special cases of them. An optimal regularity analysis of pseudo-splines does not come easily, as it has already been illustrated in a regularity estimate of the orthonormal refinable functions, which is a special case of pseudo-splines (see [5], and [15]). This thesis gives a complete optimal regularity analysis of both types of pseudo-splines. We also construct short Riesz wavelets from pseudo-splines in this thesis. (A short wavelet is the one whose support has the same length as that of the pseudo-spline from which the wavelet is derived.) Then, we connect the short Riesz wavelets with the tight framelets derived from pseudo-splines. It turns out that the short Riesz wavelet is one of the tight framelets derived from the same pseudo-spline. This reveals that the tight frame systems derived from the methods given both in [17] and this thesis have one framelet whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$, and this also leads to a new understanding of the structure of pseudo-spline tight frame systems.

A function $\phi \in L_2(\mathbb{R})$ is *refinable* if it satisfies the refinement equation

$$\phi = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2\cdot -k), \tag{1.1}$$

for some sequence $a \in \ell_2(\mathbb{Z})$, which is called refinement mask of ϕ.

By $L_p(\mathbb{R})$, for $1 \le p \le \infty$, we denote all the functions $f(x)$ satisfying

$$\|f(x)\|_{L_p(\mathbb{R})} := \Big(\int_{\mathbb{R}} |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty;$$

and $\ell_p(\mathbb{Z})$ the set of all sequences c defined on $\mathbb{Z}$ such that

$$\|c\|_{\ell_p(\mathbb{Z})} := \Big(\sum_{j\in\mathbb{Z}} |c(j)|^p\Big)^{\frac{1}{p}} < \infty.$$

The *Fourier-Laplace transform* of a compactly supported (measurable) function f is defined by

$$\widehat{f}(\zeta) := \int_{\mathbb{R}} f(t)e^{-i\zeta t}dt, \qquad \zeta \in \mathbb{C}.$$

When f is compactly supported and bounded, the Fourier-Laplace transform of f is analytic. When ζ is restricted to $\mathbb{R}$, $\widehat{f}$ becomes the *Fourier transform* of f. Note that Fourier transform can be defined for non-compactly supported measurable functions, such as $L_1(\mathbb{R})$ functions, which can be extended to more general function spaces (e.g. $L_2(\mathbb{R})$) naturally.

For a given finitely supported sequence c, its corresponding Laurent polynomial is defined by

$$\tilde{c}(z) := \sum_{j\in\mathbb{Z}} c(j)z^j, \quad \text{for } z \in \mathbb{C}\setminus\{0\}.$$

The corresponding trigonometric polynomial or Fourier series is

$$\widehat{c}(\xi) = \tilde{c}(e^{-i\xi}), \quad \xi \in \mathbb{R}.$$

With these, the refinement equation (1.1) can be written in terms of its Fourier transform as

$$\widehat{\phi}(\xi) = \widehat{a}(\xi/2)\widehat{\phi}(\xi/2), \qquad \xi \in \mathbb{R}.$$

We also call $\widehat{a}$ the *refinement mask* or just *mask* of ϕ for convenience.

The refinement equation (1.1) can also be written in terms of its Fourier-Laplace transform as

$$\widehat{\phi}(\zeta) = \tilde{a}(e^{-i\zeta/2})\widehat{\phi}(\zeta/2), \quad \text{for all } \zeta \in \mathbb{C}, \tag{1.2}$$

if ϕ is compactly supported and a finitely supported. We call $\tilde{a}$ a *symbol* of ϕ.

Pseudo-splines are defined in terms of their refinement masks. It starts with, for given nonnegative integers l and m with $l \le m-1$,

$$\left(\cos^2(\xi/2) + \sin^2(\xi/2)\right)^{m+l}. \tag{1.3}$$

The refinement masks of pseudo-splines are defined by the summation of the first $l+1$ terms of the binomial expansion of (1.3). In particular, the refinement mask of a *pseudo-spline of type I with order* (m, l) is given by, for $\xi \in [-\pi, \pi]$,

$$|{}_1\widehat{a}(\xi)|^2 := |{}_1\widehat{a}_{(m,l)}(\xi)|^2 := \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2) \tag{1.4}$$

and the refinement mask of a *pseudo-spline of type II with order* (m, l) (see also [55]) is given by, for $\xi \in [-\pi, \pi]$,

$${}_2\widehat{a}(\xi) := {}_2\widehat{a}_{(m,l)}(\xi) := \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2). \tag{1.5}$$

Except for some special circumstances, we always drop the subscript "(m, l)" in ${}_1\widehat{a}(\xi)_{(m,l)}$ and ${}_2\widehat{a}(\xi)_{(m,l)}$ for simplicity. We note that the mask of type I is obtained by taking the square root of the mask of type II using the Fejér-Riesz lemma (see e.g. [15] and [45]), i.e. ${}_2\widehat{a}(\xi) = |{}_1\widehat{a}(\xi)|^2$, and ${}_1\widehat{a}$ is a real-valued trigonometric polynomial. Pseudo-splines of type I was introduced and used in [17] in their constructions of tight framelets.

The corresponding pseudo-splines can be defined in terms of their Fourier transforms, i.e.

$$ {}_k\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} {}_k\widehat{a}(2^{-j}\xi), \ \ k = 1, 2. \tag{1.6}$$

The pseudo-splines with order $(m, 0)$ for both types are B-splines. Recall that a B-spline with order m and its refinement mask are defined by

$$\widehat{B}_m(\xi) = e^{-ij\frac{\xi}{2}} \left(\frac{\sin(\xi/2)}{\xi/2} \right)^m \quad \text{and} \quad \widehat{a}(\xi) = e^{-ij\frac{\xi}{2}} \cos^m(\xi/2),$$

where $j = 0$ when m is even, $j = 1$ when m is odd (for detailed discussions about B-splines, one may refer to [1]). The pseudo-splines of type I with order $(m, m-1)$ are the refinable functions with orthonormal shifts (called orthogonal refinable functions) given in [16]. The pseudo-splines of type II with order $(m, m-1)$ are the interpolatory refinable functions (which were first introduced in [23] and a systematic construction was given in [16]). Recall that a continuous function $\phi \in L_2(\mathbb{R})$ is interpolatory if $\phi(j) = \delta(j)$, $j \in \mathbb{Z}$, i.e. $\phi(0) = 1$, and $\phi(j) = 0$, for $j \neq 0$ (see e.g. [23]). The other pseudo-splines fill in the gap between the B-splines and orthogonal or interpolatory refinable functions.

For fixed m, since the value of the mask $|{}_k\widehat{a}(\xi)|$, for $k = 1, 2$ and $\xi \in \mathbb{R}$, increases with l (by part 1 of Lemma 2.2 in Chapter 2), and the length of the mask ${}_ka$ also increases with l, we conclude that the decay rate of the Fourier transform of a pseudo-spline decreases with l and the support of the corresponding pseudo-spline increases with l. In particular, for fixed m, the pseudo-spline with order $(m, 0)$ has the highest order of smoothness with the shortest support, the pseudo-spline with order $(m, m-1)$ has the lowest order of the smoothness with the largest support in the family. When we move from B-splines to orthogonal or interpolatory refinable functions, we sacrifice the smoothness and short support of the B-splines to gain some other desirable properties, such as orthogonality or interpolatory property. What do we get for the pseudo-splines of the other orders? When we move from B-splines to pseudo-splines, we gain the approximation power of the truncated tight frame systems derived from them, as we will discuss below.

For a given $\phi \in L_2(\mathbb{R})$, a *shift* (integer translation) *invariant space* generated by $\phi \in L_2(\mathbb{R})$ is defined by

$$V_0(\phi) := \overline{\mathrm{Span}\{\phi(\cdot - k),\ k \in \mathbb{Z}\}}. \tag{1.7}$$

Let

$$V_n(\phi) := \{f(2^n \cdot) : f \in V_0(\phi),\ n \in \mathbb{Z}\}. \tag{1.8}$$

The function ϕ is the generator of V_0, hence the generator of $V_n(\phi)$, $n \in \mathbb{Z}$. It is easy to see that for fixed m and type, pseudo-splines of all orders (m, l), $0 \le l \le m-1$, satisfy the same order of the Strang-Fix (SF) condition. (The type I pseudo-splines are of order m and type II are of order $2m$) Recall that a function ϕ satisfies the SF condition of order m if

$$\widehat{\phi}(0) \neq 0, \quad \widehat{\phi}^{(j)}(2\pi k) = 0, \quad j = 0, 1, 2, ..., m-1, \quad k \in \mathbb{Z}\backslash\{0\}.$$

Assume that ϕ satisfies the SF condition of order m_0. Then, the best approximation of a sufficiently smooth function f from $(V_n)_{n\in\mathbb{Z}}$ is m_0. Recall that $(V_n(\phi))_{n\in\mathbb{Z}}$ *provides approximation order* m_0 (or we can say that *the refinable function* ϕ *provides approximation order* m_0), if for all the f in the Sobolev space $W_2^{m_0}(\mathbb{R})$,

$$\mathrm{dist}(f, V_n) := \min\{\|f - g\|_{L_2(\mathbb{R})} : g \in V_n\} = O(2^{-nm_0}).$$

Therefore, even though the $(V_n)_{n\in\mathbb{Z}}$ may be generated by a different pseudo-spline for the fixed type with order (m, l), $0 \le l \le m-1$, the corresponding spaces $(V_n)_{n\in\mathbb{Z}}$ provide the same approximation order. However, in many applications of wavelets and framelets, we normally use

$$\mathcal{P}_n : f \mapsto \sum_{k\in\mathbb{Z}} \langle f, \phi_{n,k}\rangle\ \phi_{n,k} \tag{1.9}$$

to approximate f, where $\phi_{n,k} := 2^{\frac{n}{2}}\phi(2^n \cdot - k)$ and $\phi \in L_2(\mathbb{R})$ is a refinable function with mask a. The operation $\mathcal{P}_n f$ may not provide the best approximation of f

from V_n. We say that the *operator* $\mathcal{P}_n$ *provides approximation order* m_1, if for all f in the Sobolev space $W_2^{m_1}(\mathbb{R})$

$$\|f - \mathcal{P}_n f\|_{L_2(\mathbb{R})} = O(2^{-nm_1}).$$

As shown in [17], the approximation order of $\mathcal{P}_n f$ depends on the order of the zero of

$$1 - |\widehat{a}(\xi)|^2$$

at the origin. In fact, if $1 - |\widehat{a}|^2 = O(|\cdot|^{m_2})$ at the origin, then $m_1 = \min\{m_0, m_2\}$. For B-splines, m_2 never exceeds 2. This indicates that the approximation order of $\mathcal{P}_n$ can never exceed 2 even if a high order B-spline is used. On the other hand, for the pseudo-spline of either type with order (m, l), $0 \le l \le m-1$, the corresponding $m_2 = 2l + 2$ (see Theorem 3.10). Therefore, the approximation order of $\mathcal{P}_n$, with a pseudo-spline with order (m, l), $0 \le l \le m - 1$, as the underlying refinable function, is $\min\{m, 2l + 2\}$ for type I and $2l + 2$ for type II. More importantly, the approximation order of $\mathcal{P}_n$ determines the approximation order of the truncated series of a tight frame system. For given $\Psi := \{\psi_1, \psi_2, \ldots, \psi_r\}$, the system

$$X(\Psi) := \{\psi_{n,k} = 2^{\frac{n}{2}}\psi(2^n \cdot -k),\ \psi \in \Psi,\ n, k \in \mathbb{Z}\}$$

is a *tight frame* for $L_2(\mathbb{R})$ if

$$\sum_{g \in X(\Psi)} |\langle f, g\rangle|^2 = \|f\|^2_{L_2(\mathbb{R})}, \qquad \forall f \in L_2(\mathbb{R}).$$

For $X(\Psi)$, define the *truncated operator* as

$$\mathcal{Q}_n : f \mapsto \sum_{\psi \in \Psi, k \in \mathbb{Z}, j < n} \langle f, \psi_{j,k}\rangle\ \psi_{j,k}. \tag{1.10}$$

When the tight framelets Ψ are obtained via the unitary extension principle (see e.g. Section 5.2) from the multiresolution analysis generated by the same ϕ, then Lemma 2.4 in [17] shows that $\mathcal{P}_n f = \mathcal{Q}_n f$, for all $f \in L_2(\mathbb{R})$. Recall that for a compactly supported refinable function $\phi \in L_2(\mathbb{R})$, we define V_0 and V_n as in (1.7) and (1.8). Then, the sequence of spaces $(V_n)_{n\in\mathbb{Z}}$ forms a *multiresolution analysis* (MRA) generated by ϕ, i.e. (i) $V_n \subset V_{n+1}, \forall n \in \mathbb{Z}$; (ii) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L_2(\mathbb{R})$, $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$ (see e.g. [3] and [36]). The wavelet system $X(\Psi)$ is said to be MRA-based if there exists a MRA $(V_n)_{n\in\mathbb{Z}}$, such that $\Psi \in V_1$. If, in addition, the system $X(\Psi)$ is a (*tight*) *frame system*, we refer to the elements of Ψ as (*tight*) *framelets*.

Therefore, the tight frame system derived from a pseudo-spline normally gives better approximation order when the truncated series is used to approximate the underlying functions than that derived from B-splines. For fixed m, the choice of l depends entirely on applications. One needs to balance among the approximation order, the length of support of the wavelet, and regularity according to the practical problems in hand.

For a compactly supported function $\phi \in L_2(\mathbb{R})$ and some sequence $b \in \ell(\mathbb{Z})$, where $\ell(\mathbb{Z})$ denotes the space of all complex valued sequences defined on $\mathbb{Z}$, the *semi-convolution* of ϕ and b is defined by

$$\phi *' b := \sum_{j\in\mathbb{Z}} b(j)\phi(\cdot - j).$$

Note that for any $b \in \ell(\mathbb{Z})$ and a compactly supported function $\phi \in L_2(\mathbb{R})$, $\phi *' b$ converges uniformly on any compact set (see e.g. [3]).

In order to introduce the concept of the linear independence of the shifts of a compactly supported function ϕ, we first recall the notion of stability of ϕ which is related to, somehow weaker than, the linear independence. A function $\phi \in L_2(\mathbb{R})$ is *stable* if there exist $0 < C_1, C_2 < \infty$, such that for any sequence $b \in \ell_2(\mathbb{Z})$,

$$C_1\|b\|_{\ell_2(\mathbb{Z})} \leq \left\|\phi *' b\right\|_{L_2(\mathbb{R})} \leq C_2\|b\|_{\ell_2(\mathbb{Z})}. \tag{1.11}$$

Note that the stability of ϕ is equivalent to that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $V_0(\phi)$, where $V_0(\phi)$ is defined by (1.7). The stability of function $\phi \in L_2(\mathbb{R})$ can also be characterized by its *bracket product* (see e.g. [3] and [35]). Recall that the bracket product of $L_2(\mathbb{R})$ functions f and g is defined by

$$[\widehat{f}, \widehat{g}](\xi) := \sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi k)}.$$

It is also well known that (see e.g. [3, 15, 35, 50]) a function $\phi \in L_2(\mathbb{R})$ is stable if and only if there exist two constants $0 < C_1, C_2 < \infty$ such that

$$C_1 \le [\widehat{\phi}, \widehat{\phi}](\xi) \le C_2 \tag{1.12}$$

holds for almost every $\xi \in \mathbb{R}$.

When ϕ is compactly supported in $L_2(\mathbb{R})$, it was shown by Jia and Micchelli in Theorem 2.1 of [35] that the upper bound of (1.11) always holds. Furthermore, Theorem 3.5 of [35] asserts that the lower bound of (1.11) is equivalent to

$$\big(\widehat{\phi}(\xi + 2\pi k)\big)_{k\in\mathbb{Z}} \neq \mathbf{0} \quad \text{for all } \xi \in \mathbb{R}, \tag{1.13}$$

where $\mathbf{0}$ denotes the zero sequence in $\ell(\mathbb{Z})$. Hence, Jia and Micchelli proved that the stability of a compactly supported function $\phi \in L_2(\mathbb{R})$ is equivalent to (1.13).

A compactly supported function $\phi \in L_2(\mathbb{R})$ and its shifts are *linearly independent* if, for $b \in \ell(\mathbb{Z})$,

$$\phi *' b = 0 \quad \text{implies} \quad b(j) = 0, \text{ for all } j \in \mathbb{Z}.$$

The linear independence of a compactly supported function was first studied by Dahmen and Micchelli in [18] and [19], and Jia in [32] and [33] in the context of box splines. Ron in [47] (also see [4]) studied the linear independence of compactly supported distributions in terms of their Fourier-Laplace transforms. Applying Proposition 2.1 of [47] (also see [38]), one obtains that for an arbitrary compactly supported single variable distribution, which is not identically zero, there are at most finitely many $\zeta \in \mathbb{C}$ such that

$$\big(\widehat{\phi}(\zeta + 2\pi k)\big)_{k\in\mathbb{Z}} = \mathbf{0}.$$

Furthermore, Ron proved in Theorem 1.1 of [47] that the shifts of a compactly supported distribution are linearly independent if and only if the Fourier-Laplace transform of ϕ satisfies

$$\left(\widehat{\phi}(\zeta + 2\pi k)\right)_{k\in\mathbb{Z}} \neq \mathbf{0} \quad \text{for all } \zeta \in \mathbb{C}. \tag{1.14}$$

Comparing (1.13) and (1.14), we can see immediately that for a compactly supported function $\phi \in L_2(\mathbb{R})$, linear independence of the shifts of ϕ implies the stability of ϕ. More recently, Jia and Wang characterized the linear independence of single variable refinable functions in terms of their masks (see [38]).

The rest of the thesis is organized as follows: Chapter 2 gives some technical lemmata used in other chapters. Chapter 3 focuses on the analysis of regularity of pseudo-splines and the analysis of approximation order. In particular, the exact decay of the Fourier transforms of the pseudo-splines of both types with all orders are given. The asymptotical analysis is also provided. Chapter 4 proves that the shifts of an arbitrary pseudo-spline are linearly independent. In Chapter 5, a short Riesz wavelet, which has the same length of support as that of the corresponding pseudo-spline, is derived. Furthermore, (anti)symmetric tight framelets, which have the short Riesz wavelet as one of the framelets, are designed. In the last chapter, we introduce an implementable scheme of deriving compactly supported biorthogonal dual refinable functions from pseudo-splines, which can satisfy any prescribed regularity. This directly leads to a construction of compactly supported smooth biorthogonal Riesz wavelet bases for $L_2(\mathbb{R})$.

Chapter 2

Two Lemmas

This chapter gives two key technical lemmata that will be used to prove several key results of this thesis. We start with the following lemma on binomial coefficients, where part 1 is well known (see e.g. [12]) and the proof of part 3 is rather technical but needed in Chapter 5.

Lemma 2.1. *For given nonnegative integers* $m,\ j,\ l$, *we have:*

1. $\binom{m+1}{j} = \binom{m}{j} + \binom{m}{j-1}$ *for* $j \geq 1$ *and* $(j+1)\binom{m+j}{j+1} = (m+j)\binom{m-1+j}{j}$.
2. $2(m+1)\sum_{j=0}^{l-1}\binom{m+l}{j} - l\sum_{j=0}^{l}\binom{m+l}{j} \geq 0,$ *for* $m \geq 1$ *and* $1 \leq l \leq m-1$.
3. $\frac{2^l\binom{m+l}{l}^{\frac{1}{2}}}{\sum_{j=0}^{l}\binom{m+l}{j}} \leq 1,$ *for all* $m \geq 1$ *and* $0 \leq l \leq m-1$.

Proof. The identities in part 1 are well known and can be proven directly by the definition of the binomial coefficients.

For part 2, since $m > l$, we have

$$(m+1)\sum_{j=0}^{l-1}\binom{m+l}{j} - l\sum_{j=0}^{l-1}\binom{m+l}{j} \geq 0.$$

Subtracting this inequality from part 2, we conclude that it remains to check if

$$(m+1)\sum_{j=0}^{l-1}\binom{m+l}{j} - l\binom{m+l}{l} \geq 0$$

holds, in order to verify part 2. Since $(m+1)\binom{m+l}{l-1} = l\binom{m+l}{l}$, we have

$$(m+1)\sum_{j=0}^{l-1}\binom{m+l}{j} > (m+1)\binom{m+l}{l-1} = l\binom{m+l}{l}.$$

This gives part 2 immediately.

Finally, we prove part 3 by induction with respect to m. Since part 3 is obviously true for $l=0$, we now focus on $1 \leq l \leq m-1$. When $m=1$, the inequality trivially holds. Assume part 3 holds when $m = m_0$, i.e

$$2^{2l}\binom{m_0+l}{l} \leq \left(\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2,$$

for all $1 \leq l \leq m_0 - 1$. Consider the case $m = m_0+1$. We first show that part 3 holds for all l, where $1 \leq l \leq m_0 - 1$. For $1 \leq l \leq m_0 - 1$, we have

$$\begin{aligned}
2^{2l}\binom{m_0+l+1}{l} &= \frac{m_0+l+1}{m_0+1}2^{2l}\binom{m_0+l}{l} \\
&\leq \frac{m_0+l+1}{m_0+1}\left(\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2 \qquad \text{(by induction hypothesis)} \\
&= \left(\sum_{j=0}^{l}\binom{m_0+l}{j} + \Big(\sqrt{\frac{m_0+l+1}{m_0+1}}-1\Big)\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2 \\
&= \left(\sum_{j=0}^{l}\binom{m_0+l}{j} + \frac{l}{m_0+1+\sqrt{(m_0+l+1)(m_0+1)}}\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2 \\
&< \left(\sum_{j=0}^{l}\binom{m_0+l}{j} + \frac{l}{2m_0+2}\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2 \\
&\leq \left(\sum_{j=0}^{l}\binom{m_0+l}{j} + \sum_{j=0}^{l-1}\binom{m_0+l}{j}\right)^2 \qquad \text{(from part 2)} \\
&= \left(1+\sum_{j=1}^{l}\binom{m_0+l}{j} + \sum_{j=1}^{l}\binom{m_0+l}{j-1}\right)^2 \\
&= \left(\sum_{j=0}^{l}\binom{m_0+l+1}{j}\right)^2 \qquad \text{(from part 1).}
\end{aligned}$$

This shows that part 3 holds for all $1 \leq l \leq m_0 - 1$, it remains to show part 3 holds for $l = m_0$, i.e. to show

$$2^{2m_0}\binom{2m_0+1}{m_0} \leq \left(\sum_{j=0}^{m_0}\binom{2m_0+1}{j}\right)^2. \tag{2.1}$$

Observe that

$$\begin{aligned}\sum_{j=0}^{m_0}\binom{2m_0+1}{j} &= \frac{1}{2}\left(\sum_{j=0}^{m_0}\binom{2m_0+1}{j}+\sum_{j=0}^{m_0}\binom{2m_0+1}{j}\right)\\ &= \frac{1}{2}\left(\sum_{j=0}^{m_0}\binom{2m_0+1}{j}+\sum_{j=m_0+1}^{2m_0+1}\binom{2m_0+1}{j}\right) \quad (\text{by } \binom{n}{j}=\binom{n}{n-j})\\ &= \frac{1}{2}\sum_{j=0}^{2m_0+1}\binom{2m_0+1}{j} = 2^{2m_0}.\end{aligned}$$

Then (2.1) is equivalent to

$$\binom{2m_0+1}{m_0} \leq \sum_{j=0}^{m_0}\binom{2m_0+1}{j},$$

which is obviously true. This concludes the proof of part 3. $\square$

Define

$$P_{m,l}(y) := \sum_{j=0}^{l}\binom{m+l}{j}y^j(1-y)^{l-j} \tag{2.2}$$

and

$$R_{m,l}(y) := (1-y)^m P_{m,l}(y), \tag{2.3}$$

where $y = \sin^2(\xi/2)$ and m, l are nonnegative integers with $l \leq m-1$. Then, it is obvious that

$$R_{m,l}(\sin^2(\xi/2)) = {}_2\widehat{a}(\xi).$$

Next, we give several basic properties of the polynomials $P_{m,l}(y)$ and $R_{m,l}(y)$. Part 2-4 of the following lemma are mainly used in Chapter 3.3 and Chapter 5.

Lemma 2.2. *For nonnegative integers m and l with $l \leq m-1$, let $P_{m,l}(y)$ and $R_{m,l}(y)$ be the polynomials defined in (2.2) and (2.3). Then:*

1. $P_{m,l}(y) = \sum_{j=0}^{l} \binom{m-1+j}{j} y^j$.

2. $R'_{m,l}(y) = -(m+l)\binom{m+l-1}{l} y^l (1-y)^{m-1}$.

3. Define $Q(y) := R_{m,l}(y) + R_{m,l}(1-y)$. *Then,*

$$\min_{y\in[0,1]} Q(y) = Q(\frac{1}{2}) = 2^{1-m-l} \sum_{j=0}^{l} \binom{m+l}{j}.$$

4. Define $S(y) := R^2_{m,l}(y) + R^2_{m,l}(1-y)$. *Then,*

$$\min_{y\in[0,1]} S(y) = S(\frac{1}{2}) = 2^{1-2m-2l} (\sum_{j=0}^{l} \binom{m+l}{j})^2.$$

Proof. For fixed m, we prove part 1 by induction with respect to l. It is obviously true for $l = 0$. Now suppose part 1 holds for l_0. Consider $l = l_0 + 1$,

$$\begin{aligned} P_{m,l}(y) &= \sum_{j=0}^{l_0+1} \binom{m+l_0+1}{j} y^j (1-y)^{l_0-j+1} \\ &= (1-y)^{l_0+1} + \sum_{j=1}^{l_0+1} \binom{m+l_0+1}{j} y^j (1-y)^{l_0-j+1}. \end{aligned}$$

Applying the first identity in part 1 of Lemma 2.1, we have,

$$\begin{aligned} P_{m,l}(y) &= (1-y)^{l_0+1} + \sum_{j=1}^{l_0+1} \binom{m+l_0}{j} y^j (1-y)^{l_0-j+1} \\ &\quad + \sum_{j=1}^{l_0+1} \binom{m+l_0}{j-1} y^j (1-y)^{l_0-j+1} \\ &= \sum_{j=0}^{l_0+1} \binom{m+l_0}{j} y^j (1-y)^{l_0-j+1} + \sum_{j=1}^{l_0+1} \binom{m+l_0}{j-1} y^j (1-y)^{l_0-j+1} \\ &= \sum_{j=0}^{l_0} \binom{m+l_0}{j} y^j (1-y)^{l_0-j+1} + \binom{m+l_0}{l_0+1} y^{l_0+1} \\ &\quad + \sum_{j=0}^{l_0} \binom{m+l_0}{j} y^{j+1} (1-y)^{l_0-j} \end{aligned}$$

$$
\begin{aligned}
&= (1-y)P_{m,l_0}(y) + \binom{m+l_0}{l_0+1} y^{l_0+1} + yP_{m,l_0}(y) \\
&= P_{m,l_0}(y) + \binom{m+l_0}{l_0+1} y^{l_0+1} \\
&= \sum_{j=0}^{l_0} \binom{m-1+j}{j} y^j + \binom{m+l_0}{l_0+1} y^{l_0+1} \quad \text{(by induction hypothesis)} \\
&= \sum_{j=0}^{l_0+1} \binom{m-1+j}{j} y^j.
\end{aligned}
$$

We prove part 2 by induction with respect to l for given m. It is obviously true when $l = 0$. Suppose part 2 holds for l_0, i.e.

$$R'_{m,l_0}(y) = -(m+l_0)\binom{m+l_0-1}{l_0} y^{l_0}(1-y)^{m-1},$$

and consider the case $l = l_0 + 1 \leq m - 1$. Using part 1 and definition of $R_{m,l}(y)$ in (2.3), we have

$$
\begin{aligned}
R_{m,l_0+1}(y) &= (1-y)^m P_{m,l_0+1}(y) \\
&= (1-y)^m \left(P_{m,l_0}(y) + \binom{m+l_0}{l_0+1} y^{l_0+1} \right).
\end{aligned}
$$

Since $R_{m,l_0}(y) = (1-y)^m P_{m,l_0}(y)$, we have

$$R_{m,l_0+1}(y) = \binom{m+l_0}{l_0+1} y^{l_0+1}(1-y)^m + R_{m,l_0}(y).$$

Then,

$$
\begin{aligned}
R'_{m,l_0+1}(y) &= (l_0+1)\binom{m+l_0}{l_0+1} y^{l_0}(1-y)^m - m\binom{m+l_0}{l_0+1} y^{l_0+1}(1-y)^{m-1} \\
&\quad + R'_{m,l_0}(y) \\
&= (l_0+1)\binom{m+l_0}{l_0+1} y^{l_0}(1-y)^m - m\binom{m+l_0}{l_0+1} y^{l_0+1}(1-y)^{m-1} \\
&\quad -(m+l_0)\binom{m+l_0-1}{l_0} y^{l_0}(1-y)^{m-1}.
\end{aligned}
$$

Pulling the common factor $y^{l_0}(1-y)^{m-1}$ out, one obtains

$$\begin{aligned} R'_{m,l_0+1}(y) &= y^{l_0}(1-y)^{m-1}\Bigg((l_0+1)\binom{m+l_0}{l_0+1}(1-y) - m\binom{m+l_0}{l_0+1}y \\ &\quad -(m+l_0)\binom{m+l_0-1}{l_0}\Bigg) \\ &= y^{l_0}(1-y)^{m-1}\Bigg((l_0+1)\binom{m+l_0}{l_0+1} - (l_0+1)\binom{m+l_0}{l_0+1}y \\ &\quad -m\binom{m+l_0}{l_0+1}y - (m+l_0)\binom{m+l_0-1}{l_0}\Bigg) \end{aligned}$$

Combining the second and the third term, one obtains

$$\begin{aligned} R'_{m,l_0+1}(y) &= y^{l_0}(1-y)^{m-1}\Bigg((l_0+1)\binom{m+l_0}{l_0+1} - (m+l_0+1)\binom{m+l_0}{l_0+1}y \\ &\quad -(m+l_0)\binom{m+l_0-1}{l_0}\Bigg). \end{aligned}$$

By the second identity in part 1 of Lemma 2.1, one obtains $(l_0+1)\binom{m+l_0}{l_0+1} = (m+l_0)\binom{m+l_0-1}{l_0}$. Hence

$$R'_{m,l_0+1}(y) = -(m+l_0+1)\binom{m+l_0}{l_0+1}y^{l_0+1}(1-y)^{m-1}.$$

This concludes the proof of part 2.

For part 3, we compute $Q'(y)$, i.e.

$$Q'(y) = R'_{m,l}(y) + (R_{m,l}(1-y))' = R'_{m,l}(y) - R'_{m,l}(1-y).$$

Applying part 2, one obtains

$$Q'(y) = (m+l)\binom{m+l-1}{l}\Big(y^{m-1}(1-y)^l - (1-y)^{m-1}y^l\Big).$$

Now we show that $Q'(y) \le 0$ on $[0, \frac{1}{2}]$, $Q'(y) \ge 0$ on $[\frac{1}{2}, 1]$. Note that

$$y^{m-l-1} \le (1-y)^{m-l-1}, \qquad \text{for all } y \in [0, \frac{1}{2}].$$

Multiplying both sides by $y^l(1-y)^l$,

$$y^{m-1}(1-y)^l \le (1-y)^{m-1}y^l, \qquad \text{for all } y \in [0, \frac{1}{2}].$$

Similarly we have

$$y^{m-1}(1-y)^l \geq (1-y)^{m-1}y^l, \qquad \text{for all } y \in [\frac{1}{2}, 1].$$

We conclude that

$$Q'(y) \begin{cases} \leq 0, & y \in [0, \frac{1}{2}] \\ \geq 0, & y \in [\frac{1}{2}, 1]. \end{cases}$$

This means that $Q(y)$ reaches its minimum value at point $y = \frac{1}{2}$. Now we compute $Q(\frac{1}{2})$. Note that $Q(\frac{1}{2}) = 2R_{m,l}(\frac{1}{2}) = 2^{1-m}P_{m,l}(\frac{1}{2})$. Recall that $P_{m,l}(y)$ is defined in (2.2), i.e. $P_{m,l}(y) = \sum_{j=0}^{l}\binom{m+l}{j}y^j(1-y)^{l-j}$. Then

$$\min_{y\in[0,1]} Q(y) = Q(\frac{1}{2}) = 2^{1-m}2^{-l}\sum_{j=0}^{l}\binom{m+l}{j} = 2^{1-m-l}\sum_{j=0}^{l}\binom{m+l}{j}.$$

With part 3, the proof of part 4 is simpler. Since

$$S'(y) = 2R_{m,l}(y)R'_{m,l}(y) + 2R_{m,l}(1-y)\big(R_{m,l}(1-y)\big)',$$

using the identities

$$R_{m,l}(y) = (1-y)^m P_{m,l}(y),$$
$$R'_{m,l}(y) = -(m+l)\binom{m+l-1}{l}y^l(1-y)^{m-1}$$

and

$$\big(R_{m,l}(1-y)\big)' = (m+l)\binom{m+l-1}{l}y^{m-1}(1-y)^l,$$

we obtain

$$\begin{aligned}\frac{S'(y)}{2(m+l)\binom{m+l-1}{l}} &= y^m P_{m,l}(1-y)y^{m-1}(1-y)^l - (1-y)^m P_{m,l}(y)y^l(1-y)^{m-1} \\ &= y^{2m-1}\sum_{j=0}^{l}\binom{m-1+j}{j}(1-y)^{l+j} - (1-y)^{2m-1} \\ &\quad \times \sum_{j=0}^{l}\binom{m-1+j}{j}y^{l+j} \\ &= \sum_{j=0}^{l}\binom{m-1+j}{j}\Big((1-y)^{l+j}y^{2m-1} - y^{l+j}(1-y)^{2m-1}\Big).\end{aligned}$$

Since, for each $0 \le j \le l$, when $y \in [0, \frac{1}{2}]$, $y^{2m-l-j-1} \le (1-y)^{2m-l-j-1}$, and when $y \in [\frac{1}{2}, 1]$, $y^{2m-l-j-1} \ge (1-y)^{2m-l-j-1}$, then by similar arguments in part 2 we conclude,

$$S'(y) \begin{cases} \le 0, & y \in [0, \frac{1}{2}] \\ \ge 0, & y \in [\frac{1}{2}, 1]. \end{cases}$$

Thus $\min_{y\in[0,1]} S(y) = S(\frac{1}{2})$. Since $R_{m,l}(\frac{1}{2}) = 2^{-m-l} \sum_{j=0}^{l} \binom{m+l}{j}$, we have

$$\min_{y\in[0,1]} S(y) = S(\frac{1}{2}) = 2R_{m,l}^2(\frac{1}{2}) = 2^{1-2m-2l}\left(\sum_{j=0}^{l} \binom{m+l}{j}\right)^2.$$

□

Remark 2.3. From part 1 of Lemma 2.2 we know that the refinement mask of pseudo-spline of type I in (1.4) can be written as

$$|_1\widehat{a}(\xi)|^2 = \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m-1+j}{j} \sin^{2j}(\xi/2).$$

Hence, the pseudo-spline of type I with order $(m, m-1)$ is indeed the refinable function whose shifts form an orthonormal system constructed in [16] and the pseudo-spline of Type II with order $(m, m-1)$ is indeed the autocorrelation of the orthogonal refinable function, which is interpolatory.

Chapter 3

Basics of Pseudo-splines

This chapter is devoted to a systematic analysis of the regularity of pseudo-splines and approximation order of quasi-interpolatory operator $\mathcal{P}_n$ (see (1.9)) defined by pseudo-splines. These two are basic and essential properties of pseudo-splines. Indeed, the regularity of pseudo-splines determines the regularity of the corresponding wavelets and framelets; and the approximation order of $\mathcal{P}_n$ determines that of the truncated wavelet and framelet series. These two properties, together with the length of support, are the key criteria in selecting wavelets or framelets in various applications.

3.1 Regularity

In this section the regularity of the pseudo-splines is analyzed. For $\alpha = n + \beta,\ n \in \mathbb{N},\ 0 \leq \beta < 1$, the Hölder space C^α (see e.g. [15]) is defined to be the set of functions which are n times continuously differentiable and such that the n^{th} derivative $f^{(n)}$ satisfies the condition,

$$|f^{(n)}(x+h) - f^{(n)}(x)| \leq C|h|^\beta,\ \forall x, h.$$

It is well known (see [15]) that if

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|(1+|\xi|)^{\alpha} < \infty,$$

then $f \in C^{\alpha}$. In particular, if $|\widehat{f}(\xi)| \leq C(1+|\xi|)^{-1-\alpha-\varepsilon}$, for any $\varepsilon > 0$, then $f \in C^{\alpha}$. We shall call α the *regularity (exponent)* of f.

The main idea here is to estimate the decay of the Fourier transform of pseudo-splines with order (m,l) in order to get the lower bound of the regularity of the pseudo-splines. It turns out that this lower bound coincides with the upper bound when m goes to infinity, as shown in section 3.2.

Since for any compactly supported refinable function ϕ in $L_2(\mathbb{R})$ with $\widehat{\phi}(0) = 1$, the refinement mask a must satisfy $\widehat{a}(0) = 1$ and $\widehat{a}(\pi) = 0$ (see e.g. [15] or [37]). Then it can be factorized as

$$\widehat{a}(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^n \mathcal{L}(\xi),$$

where n is the maximum multiplicity of zeros of $\widehat{a}$ at π and $\mathcal{L}(\xi)$ is a trigonometric polynomial with $\mathcal{L}(0) = 1$. Hence, we have

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi) = \prod_{j=1}^{\infty} \left(\frac{1+e^{-i(2^{-j}\xi)}}{2}\right)^n \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi) = \Big(\frac{1-e^{-i\xi}}{i\xi}\Big)^n \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi).$$

This shows that any compactly supported refinable function in $L_2(\mathbb{R})$ is the convolution of a B-spline of some order, say n, with a distribution (see [46]). Indeed, a B-spline of order n can also be defined via its Fourier transform by

$$\widehat{B}_n := \Big(\frac{1-e^{-i\xi}}{i\xi}\Big)^n.$$

The B-spline of order n is a piecewise polynomial of degree $n-1$ in $C^{n-1-\varepsilon}(\mathbb{R})$, is supported on $[0,n]$, and has refinement mask

$$\Big(\frac{1+e^{-i\xi}}{2}\Big)^n.$$

Since $\mathcal{L}(\xi)$ is bounded, $\mathcal{L}(\xi)$ is actually the refinement mask of a refinable distribution. Therefore, ϕ is the convolution of the B-spline B_n with the distribution. The regularity of ϕ comes from the B-spline factor while the distribution factor takes away the regularity. But the distribution component also provides some desirable properties for ϕ, such as interpolatory properties, orthogonality of its shifts and approximation order of certain quasi-interpolants.

The decay of $|\widehat{\phi}|$ can be characterized by $|\widehat{a}|$ as stated in the following theorem. The proof of this theorem can be found in [15]. Note that in the following theorem, we write $|\widehat{a}|$ in the form of

$$|\widehat{a}(\xi)| = \Big|\Big(\frac{1+e^{-i\xi}}{2}\Big)^n \mathcal{L}(\xi)\Big| = \cos^n(\xi/2)|\mathcal{L}(\xi)|, \quad \xi \in [-\pi, \pi].$$

Theorem 3.1. *Let a be the refinement mask of the refinable function ϕ of the form*

$$|\widehat{a}(\xi)| = \cos^n(\xi/2)|\mathcal{L}(\xi)|, \quad \xi \in [-\pi, \pi].$$

Suppose that

$$\begin{aligned} |\mathcal{L}(\xi)| &\le |\mathcal{L}(\frac{2\pi}{3})| \qquad \text{for } |\xi| \le \frac{2\pi}{3}, \\ |\mathcal{L}(\xi)\mathcal{L}(2\xi)| &\le |\mathcal{L}(\frac{2\pi}{3})|^2 \qquad \text{for } \frac{2\pi}{3} \le |\xi| \le \pi. \end{aligned} \tag{3.1}$$

Then $|\widehat{\phi}(\xi)| \le C(1+|\xi|)^{-n+\kappa}$, with $\kappa = \log(|\mathcal{L}(\frac{2\pi}{3})|)/\log 2$, and this decay is optimal.

This theorem allows us to estimate the decay of the Fourier transform of a refinable function via its refinement mask. Since $|{}_1\widehat{\phi}|^2 = |{}_2\widehat{\phi}|$, the decay rate of $|{}_1\widehat{\phi}|$ is half of that of $|{}_2\widehat{\phi}|$. Thus we can focus on the analysis of the decay of the Fourier transforms of pseudo-splines of type II. Based on part 1 of Lemma 2.2, we will show that $P_{m,l}(y)$, defined in (2.2), satisfies (3.1). This will lead directly to the estimate of the regularity of pseudo-splines. Note that the corresponding result for $l = m-1$ was proven in [5] which led to the optimal estimates of the regularity of the orthogonal and interpolatory refinable functions.

Proposition 3.2. *Let $P_{m,l}(y)$ be defined as in* (2.2)*, where l, m are nonnegative integers with $l \leq m-1$. Then*

$$P_{m,l}(y) \leq P_{m,l}(\frac{3}{4}), \qquad \text{for } y \in [0, \frac{3}{4}], \tag{3.2}$$

$$P_{m,l}(y)P_{m,l}(4y(1-y)) \leq \left(P_{m,l}(\frac{3}{4})\right)^2, \qquad \text{for } y \in [\frac{3}{4}, 1]. \tag{3.3}$$

Proof. It is clear that (3.2) is true. Indeed, by using part 1 of Lemma 2.2 we have that $P_{m,l}(y)$ is monotonically increasing on $[0, \frac{3}{4}]$ (in fact, it is monotonically increasing on $(0, \infty)$). Hence, we focus on the proof of (3.3).

Throughout this proof, we let m be fixed. Let

$$W_{m,l}(y) := P_{m,l}(y)P_{m,l}(4y(1-y)) - \left(P_{m,l}(\frac{3}{4})\right)^2.$$

Then, the inequality (3.3) is equivalent to

$$W_{m,l}(y) \leq 0 \quad \text{for all } y \in [\frac{3}{4}, 1]. \tag{3.4}$$

In order to show (3.4), we show, instead,

$$W_{m,l+1}(y) - W_{m,l}(y) \leq 0, \quad \text{for all } y \in [\frac{3}{4}, 1], \quad l = 0, 1, \ldots, m-2. \tag{3.5}$$

Note that since for $l = 0$, $P_{m,0}(y) = 1$ for all $y \in [0, 1]$, (3.4) is obviously true for $l = 0$. Hence, (3.4) follows from (3.5) and (3.3) follows from (3.4).

We now compute $W_{m,l+1}(y) - W_{m,l}(y)$. By part 1 of Lemma 2.2, one obtains

$$\begin{aligned} W_{m,l+1}(y) - W_{m,l}(y) \;=\; & \left(\sum_{j=0}^{l+1}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l+1}\binom{m-1+j}{j}(4y(1-y))^j\right) \\ & -\left(\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l}\binom{m-1+j}{j}(4y(1-y))^j\right) \\ & +P_{m,l}^2(\frac{3}{4}) - P_{m,l+1}^2(\frac{3}{4}). \end{aligned}$$

Splitting the sum $\sum_{j=0}^{l+1}\binom{m-1+j}{j}y^j$, one obtains

$$\begin{aligned}
W_{m,l+1}(y)-W_{m,l}(y) &= \left(\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l+1}\binom{m-1+j}{j}(4y(1-y))^j\right)\\
&\quad+\binom{m+l}{l+1}y^{l+1}\sum_{j=0}^{l+1}\binom{m-1+j}{j}(4y(1-y))^j\\
&\quad-\left(\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l}\binom{m-1+j}{j}(4y(1-y))^j\right)\\
&\quad+P_{m,l}^2\Big(\frac{3}{4}\Big)-P_{m,l+1}^2\Big(\frac{3}{4}\Big).
\end{aligned}$$

Combining the first and the third term, one obtains

$$\begin{aligned}
W_{m,l+1}(y)-W_{m,l}(y) &= \binom{m+l}{l+1}(4y(1-y))^{l+1}\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\\
&\quad+\binom{m+l}{l+1}y^{l+1}\sum_{j=0}^{l+1}\binom{m-1+j}{j}(4y(1-y))^j\\
&\quad+P_{m,l}^2\Big(\frac{3}{4}\Big)-P_{m,l+1}^2\Big(\frac{3}{4}\Big)\\
&= \binom{m+l}{l+1}\left((4y(1-y))^{l+1}\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right.\\
&\quad\left.+y^{l+1}\sum_{j=0}^{l+1}\binom{m-1+j}{j}(4y(1-y))^j\right) \qquad (3.6)\\
&\quad+P_{m,l}^2\Big(\frac{3}{4}\Big)-P_{m,l+1}^2\Big(\frac{3}{4}\Big).
\end{aligned}$$

Since

$$W_{m,l+1}\Big(\frac{3}{4}\Big)-W_{m,l}\Big(\frac{3}{4}\Big)=0-0=0,$$

it suffices to show that

$$W_{m,l+1}(y)-W_{m,l}(y)$$

monotonically decreases on $[\frac{3}{4},1]$, in order to show (3.5) ($W_{m,l+1}(y)-W_{m,l}(y)\leq 0,$

$y \in [\frac{3}{4}, 1])$. It is equivalent to show that

$$G(y) := (4y(1-y))^{l+1} \sum_{j=0}^{l} \binom{m-1+j}{j} y^j + y^{l+1} \sum_{j=0}^{l+1} \binom{m-1+j}{j} (4y(1-y))^j$$

monotonically decreases on $[\frac{3}{4}, 1]$ by (3.6), for which it suffices to show that

$$G'(y) \leq 0 \quad \text{for all } y \in [\frac{3}{4}, 1]. \tag{3.7}$$

Next, we derive G' as follows:

$$\begin{aligned} G'(y) &= (l+1)(4-8y)(4y(1-y))^l \sum_{j=0}^{l} \binom{m-1+j}{j} y^j \\ &\quad + (4y(1-y))^{l+1} \sum_{j=0}^{l} \binom{m-1+j}{j} j y^{j-1} \\ &\quad + (l+1) y^l \sum_{j=0}^{l+1} \binom{m-1+j}{j} (4y(1-y))^j \\ &\quad + y^{l+1}(4-8y) \sum_{j=0}^{l+1} \binom{m-1+j}{j} j (4y(1-y))^{j-1} \\ &= (l+1)(4-8y)(4y(1-y))^l \sum_{j=0}^{l} \binom{m-1+j}{j} y^j \\ &\quad + (4y(1-y))^{l+1} \sum_{j=1}^{l} \binom{m-1+j}{j} j y^{j-1} \\ &\quad + (l+1) y^l \sum_{j=0}^{l+1} \binom{m-1+j}{j} (4y(1-y))^j \\ &\quad + y^{l+1}(4-8y) \sum_{j=1}^{l+1} \binom{m-1+j}{j} j (4y(1-y))^{j-1}. \end{aligned}$$

Substituting j for $j-1$ in the second and the fourth term above, one obtains,

$$\begin{aligned} G'(y) &= (l+1)(4-8y)(4y(1-y))^l \sum_{j=0}^{l} \binom{m-1+j}{j} y^j \\ &\quad + (4y(1-y))^{l+1} \sum_{j=0}^{l-1} \binom{m+j}{j+1} (j+1) y^j \end{aligned}$$

$$+(l+1)y^l \sum_{j=0}^{l+1} \binom{m-1+j}{j} (4y(1-y))^j$$
$$+y^{l+1}(4-8y) \sum_{j=0}^{l} \binom{m+j}{j+1} (j+1)(4y(1-y))^j.$$

Applying part 1 of Lemma 2.1 to the second and the fourth term above, one obtains

$$\begin{aligned}
G'(y) &= (l+1)(4-8y)(4y(1-y))^l \sum_{j=0}^{l} \binom{m-1+j}{j} y^j \\
&\quad +(4y(1-y))^{l+1} \sum_{j=0}^{l-1} \binom{m-1+j}{j} (m+j) y^j \\
&\quad +(l+1)y^l \sum_{j=0}^{l+1} \binom{m-1+j}{j} (4y(1-y))^j \\
&\quad +y^{l+1}(4-8y) \sum_{j=0}^{l} \binom{m-1+j}{j} (m+j)(4y(1-y))^j \\
&= (l+1)(4-8y)(4y(1-y))^l \sum_{j=0}^{l} \binom{m-1+j}{j} y^j \\
&\quad +(4y(1-y))^{l+1} \sum_{j=0}^{l} \binom{m-1+j}{j} (m+j) y^j \\
&\quad -(m+l)\binom{m-1+l}{l} y^l (4y(1-y))^{l+1} \\
&\quad +(l+1)y^l \sum_{j=0}^{l} \binom{m-1+j}{j} (4y(1-y))^j \\
&\quad +(l+1)\binom{m+l}{l+1} y^l (4y(1-y))^{l+1} \\
&\quad +y^{l+1}(4-8y) \sum_{j=0}^{l} \binom{m-1+j}{j} (m+j)(4y(1-y))^j.
\end{aligned}$$

Since $(l+1)\binom{m+l}{l+1} = (m+l)\binom{m-1+l}{l}$ by part 1 of Lemma 2.1, we have

$$(l+1)\binom{m+l}{l+1} y^l (4y(1-y))^{l+1} - (m+l)\binom{m-1+l}{l} y^l (4y(1-y))^{l+1} = 0.$$

Hence,

$$\begin{aligned}G'(y) =&(l+1)(4-8y)\big(4y(1-y)\big)^l \sum_{j=0}^{l} \binom{m-1+j}{j} y^j \\ &+ \big(4y(1-y)\big)^{l+1} \sum_{j=0}^{l} \binom{m-1+j}{j} (m+j) y^j \\ &+ (l+1)y^l \sum_{j=0}^{l} \binom{m-1+j}{j} \big(4y(1-y)\big)^j \\ &+ y^{l+1}(4-8y) \sum_{j=0}^{l} \binom{m-1+j}{j} (m+j)\big(4y(1-y)\big)^j.\end{aligned}$$

Rewriting $G'(y)$, one obtains

$$\begin{aligned}G'(y) = \sum_{j=0}^{l} \binom{m-1+j}{j} \Big(&(l+1)(4-8y)(4y(1-y))^l y^j + (m+j)(4y(1-y))^{l+1} y^j \\ &+ (l+1)y^l(4y(1-y))^j + (m+j)(4-8y)y^{l+1}(4y(1-y))^j \Big).\end{aligned}$$

Pulling the common factor $y^j(4y(1-y))^j$ out from each term of the above summation, one obtains

$$\begin{aligned}G'(y) = \sum_{j=0}^{l} \binom{m-1+j}{j} y^j(4y(1-y))^j \Big(&(l+1)(4-8y)(4y(1-y))^{l-j} \\ &+ (m+j)(4y(1-y))^{l+1-j} + (l+1)y^{l-j} + (m+j)(4-8y)y^{l+1-j} \Big).\end{aligned} \tag{3.8}$$

For $0 \le j \le l \le m-2$, consider

$$\begin{aligned}g_{l,j}(y) :=&(l+1)(4-8y)\big(4y(1-y)\big)^{l-j} + (m+j)\big(4y(1-y)\big)^{l+1-j} \\ &+ (l+1)y^{l-j} + (m+j)(4-8y)y^{l+1-j}.\end{aligned} \tag{3.9}$$

The inequality (3.7) is verified if one can show that for all $0 \le j \le l \le m-2$

$$g_{l,j}(y) \le 0, \quad \text{for all } y \in [\frac{3}{4}, 1], \tag{3.10}$$

because the common factors $y^j(4y(1-y))^j$ in (3.8) are always nonnegative for all $y \in [\frac{3}{4}, 1]$.

Now consider the sum of the second and the fourth terms in (3.9). Since $2 \le 8y - 4$ and $4y(1-y) \le y$, $y \in [\frac{3}{4}, 1]$, we have

$$(4y(1-y))^{l+1-j} \le y^{l+1-j} \le 2y^{l+1-j} \le (8y-4)y^{l+1-j}.$$

This leads to the fact that

$$(4y(1-y))^{l+1-j} - (8y-4)y^{l+1-j} \le 0.$$

This in turn gives that

$$(m+j)(4y(1-y))^{l+1-j} + (m+j)(4-8y)y^{l+1-j} = (m+j)\big((4y(1-y))^{l+1-j} - (8y-4)y^{l+1-j}\big)$$

decreases as m increases. We further note that m is always $\ge l+2$. Hence,

$$\begin{aligned}
(m+j)(4y(1-y))^{l+1-j} &+ (m+j)(4-8y)y^{l+1-j} \\
&\le (l+2+j)\big((4y(1-y))^{l+1-j} - (8y-4)y^{l+1-j}\big) \\
&= (l+2+j)(4y(1-y))^{l+1-j} + (l+2+j)(4-8y)y^{l+1-j}
\end{aligned}$$

Putting this back to (3.9), one obtains

$$\begin{aligned}
g_{l,j}(y) &\le (l+1)(4-8y)(4y(1-y))^{l-j} + (l+2+j)(4y(1-y))^{l+1-j} + (l+1)y^{l-j} \\
&\quad +(l+2+j)(4-8y)y^{l+1-j} \\
&= (l+2+j)(4y(1-y))^{l+1-j} - (l+1)(8y-4)(4y(1-y))^{l-j} + (l+1)y^{l-j} \\
&\quad -(l+2+j)(8y-4)y^{l+1-j}.
\end{aligned}$$

Let

$$f_1(y) := (l+2+j)(4y(1-y))^{l+1-j} - (l+1)(8y-4)(4y(1-y))^{l-j}$$

and

$$f_2(y) := (l+1)y^{l-j} - (l+2+j)(8y-4)y^{l+1-j}.$$

Finally, we show that for $0 \le j \le l$

$$f_1(y) \le 0 \quad \text{and} \quad f_2(y) \le 0 \quad \text{for all } y \in [\frac{3}{4}, 1]. \tag{3.11}$$

This will lead to $g_{l,j}(y) \le 0$ for all $0 \le j \le l \le m-2$, which in turns implies (3.7), and that would conclude the proof.

For the first inequality of (3.11), since $4y(1-y) < 1$ for $y \in [\frac{3}{4}, 1]$ we have

$$\begin{aligned} f_1(y) &\le (l+2+j)(4y(1-y))^{l-j} - (l+1)(8y-4)(4y(1-y))^{l-j} \\ &= (4y(1-y))^{l-j}\Big((l+2+j) - (l+1)(8y-4)\Big). \end{aligned}$$

Since $8y - 4 \ge 2$ for $y \ge \frac{3}{4}$, we have

$$\begin{aligned} f_1(y) &\le (4y(1-y))^{l-j}\Big((l+2+j) - 2(l+1)\Big) \\ &= (4y(1-y))^{l-j}(j-l) \le 0 \end{aligned}$$

by $j \le l$.

For the second inequality of (3.11), note that

$$\begin{aligned} f_2(y) &= (l+1)y^{l-j} - (l+2+j)(8y^2-4y)y^{l-j} \\ &= (l+1)y^{l-j} - (l+2+j)(8y^2-4y-1)y^{l-j} - (l+2+j)y^{l-j} \\ &= -(j+1)y^{l-j} - (l+2+j)(8y^2-4y-1)y^{l-j} \\ &\le -(l+2+j)(8y^2-4y-1)y^{l-j}. \end{aligned}$$

Since $8y^2 - 4y - 1 \ge 0$, for $y \ge \frac{1+\sqrt{3}}{4}$ and since $\frac{1+\sqrt{3}}{4} < \frac{3}{4}$, $8y^2 - 4y - 1 \ge 0$, for $y \in [\frac{3}{4}, 1]$. Therefore, $f_2(y) \le 0$, for $y \in [\frac{3}{4}, 1]$. □

Remark 3.3. It is clear that $W_{m,0}(y) = 0$, $y \in [\frac{3}{4}, 1]$, because $P_{m,0} = 1$. It was also proven by [5] that $W_{m,m-1}(y) \le 0$, $y \in [\frac{3}{4}, 1]$, which is equivalent to (3.3). The decreasing of $W_{m,l}(y)$, for $y \in [\frac{3}{4}, 1]$, as l increases shown above indicates the difficulties when one tries to prove (3.3) directly for an arbitrary l, $0 < l < m-1$, since it has a smaller margin than the case when $l = m-1$ and (3.3) is already very difficult to prove for this special case. In fact, to some extent, the proof of (3.3) for the case when $l = m - 1$ relies on a numerical check for $m \leq 12$ (see [15] page 225). Inequality (3.3) for the case $l = m-1$ as proven in [5] (also see [15]) is one of the cornerstones of the wavelet theory, because it immediately leads to the optimal estimate of the decay of the Fourier transforms (hence, the regularity) of both interpolatory and orthogonal refinable functions. We take a different approach here by proving that $W_{m,l}(y)$, $y \in [\frac{3}{4}, 1]$, decreases as l increases. As a result, we obtain (3.3) for all $0 \leq l \leq m-1$ by the fact that $W_{m,0}(y) = 0$, $y \in [\frac{3}{4}, 1]$. This shows that introducing the concepts of the pseudo-splines gives a better understanding and a more complete picture of the proof of (3.3) and also, we hope, enriches the theory of wavelets. Note that the proof of (3.3) for all $0 \leq l \leq m-1$ given here does not rely on any numerical computation.

With this proposition, one obtains the regularity of pseudo-splines by applying Theorem 3.1.

Theorem 3.4. *Let ${}_2\phi$ be the pseudo-spline of type II with order (m,l). Then*

$$|{}_2\widehat{\phi}(\xi)| \leq C\big(1 + |\xi|\big)^{-2m+\kappa},$$

where $\kappa = \log(P_{m,l}\big(\frac{3}{4}\big))/\log 2$. Consequently, ${}_2\phi \in C^{\alpha_2 - \varepsilon}$ with $\alpha_2 = 2m - \kappa - 1$. Furthermore, let ${}_1\phi$ be the pseudo-spline of type I with order (m,l). Then

$$|{}_1\widehat{\phi}(\xi)| \leq C\big(1 + |\xi|\big)^{-m+\frac{\kappa}{2}}.$$

Consequently, ${}_1\phi \in C^{\alpha_1 - \varepsilon}$ with $\alpha_1 = m - \frac{\kappa}{2} - 1$.

Proof. Since

$$P_{m,l}(\sin^2(\xi)) = \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2),$$

the refinement mask of pseudo-spline of type II with order (m, l) is

$$\begin{aligned} {}_2\widehat{a}(\xi) &= \cos^{2m}(\xi/2)\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2) \\ &= \big(\cos(\xi/2)\big)^{2m}P_{m,l}(\sin^2(\xi/2)). \end{aligned}$$

Hence, $|\mathcal{L}(\xi)|$ in Theorem 3.1 is exactly $P_{m,l}(\sin^2(\xi/2))$ here. Let $y = \sin^2(\xi/2)$. Applying (3.2) of Proposition 3.2

$$P_{m,l}(y) \le P_{m,l}\big(\frac{3}{4}\big), \quad y \in [0, \frac{3}{4}].$$

we have

$$\begin{aligned} |\mathcal{L}(\xi)| &= P_{m,l}(\sin^2(\xi/2)) \\ &= P_{m,l}(y) \le P_{m,l}\big(\frac{3}{4}\big) = P_{m,l}(\sin^2(\frac{\pi}{3})) \quad \text{for } |\xi| \le \frac{2\pi}{3}. \end{aligned}$$

Note that

$$|\mathcal{L}(2\xi)| = P_{m,l}(\sin^2(\xi)) = P_{m,l}(4\sin^2(\xi/2)(1-\sin^2(\xi/2))) = P_{m,l}(4y(1-y)).$$

Applying (3.3) of Proposition 3.2

$$P_{m,l}(y)P_{m,l}(4y(1-y)) \le \big(P_{m,l}\big(\frac{3}{4}\big)\big)^2, \quad y \in [\frac{3}{4}, 1],$$

we have

$$\begin{aligned} |\mathcal{L}(\xi)\mathcal{L}(2\xi)| &= P_{m,l}(\sin^2(\xi/2))P_{m,l}(4\sin^2(\xi/2)(1-\sin^2(\xi/2))) \\ &= P_{m,l}(y)P_{m,l}(4y(1-y)) \\ &\le \big(P_{m,l}\big(\frac{3}{4}\big)\big)^2 = \big(P_{m,l}(\sin^2(\frac{\pi}{3}))\big)^2, \quad \text{for } \frac{2\pi}{3} \le |\xi| \le \pi. \end{aligned}$$

Hence, by Theorem 3.1, ${}_2\widehat{\phi}$ satisfies

$$|{}_2\widehat{\phi}(\xi)| \le C\big(1+|\xi|\big)^{-2m+\kappa},$$

where $\kappa = \log(P_{m,l}\big(\frac{3}{4}\big))/\log 2$. This leads to ${}_2\phi \in C^{\alpha_2-\varepsilon}$, where $\alpha_2 = 2m-\kappa-1$.

Since the decay of $|{}_1\widehat{\phi}|$ is exactly half of $|{}_2\widehat{\phi}|$, we have

$$|{}_1\widehat{\phi}(\xi)| \leq C\big(1+|\xi|\big)^{-m+\frac{\kappa}{2}},$$

consequently, ${}_1\phi \in C^{\alpha_1-\varepsilon}$, where $\alpha_2 = m - \frac{\kappa}{2} - 1$. □

Table 3.1 gives the decay rates $\beta_{m,l}$ of the Fourier transform of pseudo-splines of type II with order (m,l), for $2 \leq m \leq 8$ and $0 \leq l \leq m-1$. The regularity exponent of the corresponding pseudo-spline is, at least, $\alpha_2 = \beta_{m,l} - 1 - \varepsilon$. The decay rate of the Fourier transform of pseudo-spline of type I with the same order is $\frac{\beta_{m,l}}{2}$ and its regularity exponent α_1 is $\frac{\alpha_2-1}{2}$. Therefore, the table shows that for either type of pseudo-spline and fixed order m, the decay rate of their Fourier transform decreases as l increases, while for fixed l, it increases as m increases. This is true indeed as shown in the following proposition.

Table 3.1: The followings are the decay rates $\beta_{m,l} = 2m - \kappa$ of pseudo-splines of type II with order (m,l), for $2 \leq m \leq 8$ and $0 \leq l \leq m-1$.

(m,l)	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$
$m=2$	2.67807						
$m=3$	4.29956	3.27208					
$m=4$	6.00000	4.73321	3.82507				
$m=5$	7.75207	6.27890	5.19506	4.35316			
$m=6$	9.54057	7.88626	6.64465	5.66363	4.86449		
$m=7$	11.35614	9.54057	8.15608	7.04717	6.13261	5.36349	
$m=8$	13.19265	11.23182	9.71691	8.48992	7.46770	6.59988	5.85310

Proposition 3.5. *Let* $\beta_{m,l} = 2m - \kappa$ *with* $\kappa = \log P_{m,l}\big(\frac{3}{4}\big)/\log 2$ *as given in Theorem 3.4. Then:*

1. *For fixed* m, $\beta_{m,l}$ *decreases as* l *increases.*

2. *For fixed l, $\beta_{m,l}$ increases as m increases.*

3. *When $l = m-1$, $\beta_{m,l}$ increases as m increases.*

Consequently, the pseudo-spline of type I with order $(2,1)$ *has the lowest regularity exponent, which is, at least,* 0.339*, among all the pseudo-splines (of either type) with order* (m,l)*,* $m \geq 2$ *and* $0 \leq l \leq m-1$*.*

Proof. Part 1 follows directly from (1) of Lemma 2.2, which shows that $P_{m,l}\left(\frac{3}{4}\right)$ increases as l increases for fixed m.

For part 2, note that

$$\beta_{m,l} = 2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}.$$

Consider

$$2^{\beta_{m,l}} = 2^{2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}} = \frac{4^m}{P_{m,l}\left(\frac{3}{4}\right)} = \frac{1}{4^{-m} P_{m,l}\left(\frac{3}{4}\right)}.$$

Hence, part 2 is equivalent to that

$$I_m := 4^{-m} P_{m,l}\left(\frac{3}{4}\right)$$

decreases as m increases for fixed l, which is equivalent to show that for fixed $0 \leq l \leq m-1$,

$$I_{m+1} - I_m < 0. \tag{3.12}$$

Note that

$$\begin{aligned}
I_{m+1} - I_m &= 4^{-m-1} P_{m+1,l}\left(\frac{3}{4}\right) - 4^{-m} P_{m,l}\left(\frac{3}{4}\right) \\
&= 4^{-m-1} \sum_{j=0}^{l} \binom{m+j}{j} \left(\frac{3}{4}\right)^j - 4^{-m} \sum_{j=0}^{l} \binom{m-1+j}{j} \left(\frac{3}{4}\right)^j \\
&= 4^{-m-1} \sum_{j=0}^{l} \left(\binom{m+j}{j} - 4 \binom{m-1+j}{j} \right) \left(\frac{3}{4}\right)^j.
\end{aligned}$$

Inequality (3.12) follows from the fact that for $0 \leq j \leq m-1$,

$$\binom{m+j}{j} = \frac{m+j}{m}\binom{m-1+j}{j} = (1+\frac{j}{m})\binom{m-1+j}{j} < 4\binom{m-1+j}{j}. \tag{3.13}$$

This concludes the proof of part 2.

For part 3, using a similar argument in the proof of part 2, one can derive that it is equivalent to show that

$$J_m := 4^{-m} P_{m,m-1}(\frac{3}{4})$$

decreases as m increases, which, in turn, is equivalent to show that

$$J_{m+1} - J_m < 0 \quad \text{for } m \geq 1. \tag{3.14}$$

Note that, similar to the proof of part 2, we have

$$J_{m+1} - J_m = 4^{-m-1}\Big(\sum_{j=0}^{m}\binom{m+j}{j}(\frac{3}{4})^j - 4\sum_{j=0}^{m-1}\binom{m-1+j}{j}(\frac{3}{4})^j\Big).$$

Let

$$M := \sum_{j=0}^{m}\binom{m+j}{j}(\frac{3}{4})^j - 4\sum_{j=0}^{m-1}\binom{m-1+j}{j}(\frac{3}{4})^j.$$

Then, (3.14) is equivalent to $M < 0$ for $m \geq 1$. It is easy to check that $M < 0$, when $m = 1$. We consider now the case when $m \geq 2$. First, we note that

$$\begin{aligned} M &= \sum_{j=0}^{m-1}\binom{m+j}{j}(\frac{3}{4})^j - 4\sum_{j=0}^{m-1}\binom{m-1+j}{j}(\frac{3}{4})^j + \binom{2m}{m}(\frac{3}{4})^m \\ &= \sum_{j=0}^{m-1}\Big(\binom{m+j}{j} - \binom{m-1+j}{j}\Big)(\frac{3}{4})^j - 3\sum_{j=0}^{m-1}\binom{m-1+j}{j}(\frac{3}{4})^j + \binom{2m}{m}(\frac{3}{4})^m \\ &= \sum_{j=1}^{m-1}\binom{m-1+j}{j-1}(\frac{3}{4})^j - 3\sum_{j=0}^{m-1}\binom{m-1+j}{j}(\frac{3}{4})^j + \binom{2m}{m}(\frac{3}{4})^m, \end{aligned}$$

where the last identity follows from part 1 of Lemma 2.1. Substituting j for $j-1$ in the first term, one obtains that

$$M = \frac{3}{4}\sum_{j=0}^{m-2}\binom{m+j}{j}(\frac{3}{4})^j - 3\sum_{j=0}^{m-1}\binom{m-1+j}{j}(\frac{3}{4})^j + \binom{2m}{m}(\frac{3}{4})^m, \tag{3.15}$$

Splitting the second term in (3.15), one obtains

$$\begin{aligned} M = {} & \frac{3}{4}\sum_{j=0}^{m-2}\binom{m+j}{j}(\frac{3}{4})^j - 3\sum_{j=0}^{m-2}\binom{m-1+j}{j}(\frac{3}{4})^j \\ & + \binom{2m}{m}(\frac{3}{4})^m - 3\binom{2m-2}{m-1}(\frac{3}{4})^{m-1} \end{aligned} \tag{3.16}$$

For the last two terms of (3.16), we have

$$\begin{aligned} \binom{2m}{m}(\frac{3}{4})^m - 3\binom{2m-2}{m-1}(\frac{3}{4})^{m-1} &= (\frac{3}{4})^m\left(\binom{2m}{m} - 4\binom{2m-2}{m-1}\right) \\ &= (\frac{3}{4})^m\left((4-\frac{2}{m})\binom{2m-2}{m-1} - 4\binom{2m-2}{m-1}\right) \\ &< 0. \end{aligned}$$

Therefore,

$$\begin{aligned} M &< \frac{3}{4}\sum_{j=0}^{m-2}\binom{m+j}{j}(\frac{3}{4})^j - 3\sum_{j=0}^{m-2}\binom{m-1+j}{j}(\frac{3}{4})^j \\ &< \sum_{j=0}^{m-2}\binom{m+j}{j}(\frac{3}{4})^j - 3\sum_{j=0}^{m-2}\binom{m-1+j}{j}(\frac{3}{4})^j \\ &= \sum_{j=0}^{m-2}\left(\binom{m+j}{j} - 3\binom{m-1+j}{j}\right)(\frac{3}{4})^j. \end{aligned}$$

Applying (3.13), one obtains, for $0 \le j \le m-2$,

$$\binom{m+j}{j} = (1+\frac{j}{m})\binom{m-1+j}{j} < 3\binom{m-1+j}{j}.$$

Therefore, we conclude that $M < 0$ and part 3 follows.

Since the decay rate of the Fourier transform of the pseudo-spline of type I with order $(2,1)$ is $\frac{\beta_{m,l}}{2} \approx 1.33903$, it belongs to, at least $C^{0.339}$. Hence, it follows from parts 1-3 that an arbitrary pseudo-spline of either type with order (m,l), $m > 2$, $0 \le l \le m-1$ has higher regularity exponent. $\square$

3.2 Asymptotical Analysis

Proposition 3.5 reveals that the decay rates of the Fourier transforms of pseudo-splines of either type increase as m increases for fixed l and decrease as l increases for fixed m. In this section, we give an asymptotical analysis of the decay rate which, in turn, gives an asymptotical analysis of the regularity of ${}_1\phi$ and ${}_2\phi$ as the order $(m, l) \to \infty$.

Theorem 3.6. *Let ${}_1\phi$ and ${}_2\phi$ be the pseudo-splines of type I and II respectively with order (m, l). Fix $l = \lfloor \lambda m \rfloor$, $0 \le \lambda \le 1$, where $\lfloor \lambda m \rfloor$ denotes the largest integer which is smaller than or equal to λm. Then, we have*

$$|{}_1\widehat{\phi}(\xi)| \le C(1 + |\xi|)^{-\frac{\mu}{2}m} \quad \textit{and} \quad {}_1\phi \in C^{\frac{\mu}{2}m};$$

and

$$|{}_2\widehat{\phi}(\xi)| \le C(1 + |\xi|)^{-\mu m} \quad \textit{and} \quad {}_2\phi \in C^{\mu m},$$

where $\mu = \frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}$, asymptotically for large m. This means that the asymptotic rate of pseudo-spline of type I and type II are $\frac{\mu}{2}$ and μ respectively.

Proof. As the estimate of the type I follows immediately from that of type II, we only give the estimate for pseudo-splines of type II. We first prove the following fact:

$$x^{-l} P_{m,l}(x) \ge y^{-l} P_{m,l}(y), \quad \text{for } 0 < x \le y \le 1. \tag{3.17}$$

Indeed, assertion part 1 of Lemma 2.2 gives for $0 < x \le y \le 1$,

$$x^{-l} P_{m,l}(x) = \sum_{j=0}^{l} \binom{m-1+j}{j} x^{j-l} \ge \sum_{j=0}^{l} \binom{m-1+j}{j} y^{j-l} = y^{-l} P_{m,l}(y).$$

The key step to compute the asymptotic rate is to estimate the upper and lower bound of $P_{m,l}\left(\frac{3}{4}\right)$ in terms of m and l. For this, let $x = \frac{3}{4}$ and $y = 1$ in (3.17), we obtain

$$P_{m,l}\left(\frac{3}{4}\right) \ge \left(\frac{3}{4}\right)^l P_{m,l}(1) = \left(\frac{3}{4}\right)^l \binom{m+l}{l}. \tag{3.18}$$

Next, let $x = \frac{1}{2}$ and $y = \frac{3}{4}$ in (3.17), we obtain

$$P_{m,l}(\frac{3}{4}) \leq (\frac{3}{2})^l P_{m,l}(\frac{1}{2}).$$

Since

$$P_{m,l}(\frac{1}{2}) = \sum_{j=0}^{l} \binom{m+l}{j} 2^{-j} 2^{j-l} = 2^{-l} \sum_{j=0}^{l} \binom{m+l}{j},$$

one obtains

$$P_{m,l}(\frac{3}{4}) \leq (\frac{3}{4})^l \sum_{j=0}^{l} \binom{m+l}{j}. \tag{3.19}$$

Putting (3.18) and (3.19) together, we obtain the following estimates of $P_{m,l}\big(\frac{3}{4}\big)$,

$$(\frac{3}{4})^l \binom{m+l}{l} \leq P_{m,l}(\frac{3}{4}) \leq (\frac{3}{4})^l \sum_{j=0}^{l} \binom{m+l}{j}.$$

Since for $l \leq m-1$,

$$\sum_{j=0}^{l} \binom{m+l}{j} \leq m \binom{m+l}{l}.$$

Hence,

$$(\frac{3}{4})^l \binom{m+l}{l} \leq P_{m,l}(\frac{3}{4}) \leq m (\frac{3}{4})^l \binom{m+l}{l}. \tag{3.20}$$

Next, we will use this estimate to analyze the decay of ${}_2\widehat{\phi}$ with order (m, l) as m goes to infinity. The upper bound of $P_{m,l}\big(\frac{3}{4}\big)$ in (3.20) gives

$$2m - \frac{\log P_{m,l}\big(\frac{3}{4}\big)}{\log 2} \geq 2m - \frac{\log\big(m(\frac{3}{4})^l \binom{m+l}{l}\big)}{\log 2}.$$

We estimate the right hand side of the above inequality asymptotically for large (m, l) to obtain the asymptotical lower bound of $2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}$. For this, we first recall the Stirling approximation, i.e. $m! \sim \sqrt{2\pi} e^{(m+\frac{1}{2})\log m - m}$ (see e.g. [24]), where $a_m \sim b_m$ means that $\frac{a_m}{b_m} \to 1,\ m \to \infty$. By Stirling approximation,

$$
\begin{aligned}
\log m! &\sim \log \sqrt{2\pi} e^{(m+\frac{1}{2})\log m - m} \\
&\sim \log \sqrt{2\pi} + (m + \frac{1}{2}) \log m - m \\
&\sim \big(m \log m - m\big) \frac{\log \sqrt{2\pi} + (m+\frac{1}{2}) \log m - m}{m \log m - m} \\
&\sim \big(m \log m - m\big) \frac{(1 + \frac{1}{2m}) \log m - 1}{\log m - 1} \\
&\sim m \log m - m. \qquad (3.21)
\end{aligned}
$$

Applying (6.20), one obtains

$$
\begin{aligned}
\log \binom{m+l}{l} &= \log(m+l)! - \log m! - \log l! \\
&\sim (m+l)\log(m+l) - (m+l) - (m \log m - m) - (l \log l - l) \\
&\sim (m+l)\log(m+l) - m \log m - l \log l.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
2m - \frac{\log\big(m \frac{3}{4}^l \binom{m+l}{l}\big)}{\log 2} &= 2m - \frac{\log m + l \log \frac{3}{4} + \log \binom{m+l}{l}}{\log 2} \\
&\sim m\bigg(2 - \frac{\frac{l}{m} \log \frac{3}{4} + (1 + \frac{l}{m}) \log(m+l) - \log m - \frac{l}{m} \log l}{\log 2}\bigg).
\end{aligned}
$$

By the assumption, $l = \lfloor \lambda m \rfloor,\ 0 \leq \lambda \leq 1$. Hence, when m is sufficiently large, $\frac{l}{m} \sim \lambda$ and therefore,

$$
\begin{aligned}
2m - \frac{\log\big(m \frac{3}{4}^l \binom{m+l}{l}\big)}{\log 2} &\sim m\bigg(2 - \frac{\lambda \log \frac{3}{4} + (1+\lambda)\log(1+\lambda)m - \log m - \lambda \log \lambda m}{\log 2}\bigg) \\
&\sim m\bigg(2 - \frac{\log\,(1+\lambda)\big(\frac{3+3\lambda}{4\lambda}\big)^{\lambda}}{\log 2}\bigg) \\
&\sim m\bigg(\frac{\log\,(\frac{4}{1+\lambda})^{\lambda+1}(\frac{\lambda}{3})^{\lambda}}{\log 2}\bigg).
\end{aligned}
$$

Now we obtain the asymptotical lower bound of $2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}$, i.e. asymptotically, for large m with $l = \lfloor \lambda m \rfloor$,

$$
2m - \frac{\log |P_{m,l}\big(\frac{3}{4}\big)|}{\log 2} \geq m\bigg(\frac{\log\,(\frac{4}{1+\lambda})^{\lambda+1}(\frac{\lambda}{3})^{\lambda}}{\log 2}\bigg). \qquad (3.22)
$$

Next, we use the left hand side of (3.20) to obtain the asymptotical upper bound of $2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}$. First note that (3.20) gives

$$2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2} \leq 2m - \frac{l \log \frac{3}{4} + \log \binom{m+l}{l}}{\log 2}.$$

Applying arguments similar to the estimate of the lower bound by using (6.20), we will obtain the following

$$\begin{aligned} 2m - \frac{l \log \frac{3}{4} + \log \binom{m+l}{l}}{\log 2} &\sim m\left(2 - \frac{\frac{l}{m} \log \frac{3}{4} + (1 + \frac{l}{m}) \log(m+l) - \log m - \frac{l}{m} \log l}{\log 2}\right) \\ &\sim m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}\right). \end{aligned}$$

This leads to the asymptotical lower bound of $2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}$, i.e. asymptotically, for large m with $l = \lfloor \lambda m \rfloor$,

$$2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2} \leq m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}\right). \tag{3.23}$$

Combining (3.22) and (3.23), we conclude that for large m, the asymptotical upper and lower bounds coincide and equal to

$$\mu m = 2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2} \sim m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}\right). \tag{3.24}$$

Therefore the equation (3.24) gives that, fixing $l = \lfloor \lambda m \rfloor$ and asymptotically, for large m, we have

$$|{}_2\widehat{\phi}(\xi)| \leq C(1 + |\xi|)^{-\mu m} \quad \text{and} \quad {}_2\phi \in C^{\mu m};$$

and

$$|{}_1\widehat{\phi}(\xi)| \leq C(1 + |\xi|)^{-\frac{\mu}{2} m} \quad \text{and} \quad {}_1\phi \in C^{\frac{\mu}{2} m},$$

where $\mu = \frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}$. □

Remark 3.7. The proof of Theorem 3.6 also leads to the following two observations:

1. Consider pseudo-splines of type II with order $(m, m-p)$, where p is a fixed positive integer independent of m, the asymptotic rate of which is $2 - \frac{\log 3}{\log 2} \approx 0.4150$. Indeed, when $l = m - p$, $\lambda \sim \frac{l}{m} = \frac{m-p}{m} \sim 1$ for sufficiently large m. Similarly, for pseudo-splines of type I with order $(m, m-p)$, the corresponding asymptotic rate is $1 - \frac{\log 3}{2\log 2} \approx 0.2075$.

2. Assume that l is fixed for all m. The asymptotic rates of pseudo-splines of type I and II with order (m, l) are 1 and 2 respectively. This is simply because, for the fixed integer l, $\lambda \sim \frac{l}{m} \sim 0$ for sufficiently large m.

Table 3.2: Asymptotically for large m, ${}_2\phi \in C^{\mu m}$.

$m \to \infty$	$l = 0$	$l = \frac{m}{10}$	$l = \frac{m}{8}$	$l = \frac{m}{6}$	$l = \frac{m}{4}$	$l = \frac{m}{2}$	$l = m - 1$
$\mu \approx$	2.0000	1.5581	1.4857	1.3789	1.2013	0.8301	0.4150

Example 3.8. In Table 3.2, we give μ, the asymptotical rate of pseudo-spline of type II with order $(m, \lfloor \lambda m \rfloor)$, as m goes to infinity and the parameter $\lambda = \frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1$. The asymptotic rate μ_0 for pseudo-spline of type I with the same order is just $\mu_0 = \frac{\mu}{2}$.

3.3 Approximation order

We follow [17] to give a brief discussion of the approximation order of $\mathcal{P}_n$ through $\mathcal{Q}_n$ where $\mathcal{P}_n$ is given by (1.9) with the underlying refinable function ϕ and $\mathcal{Q}_n$ is given by (1.10) with the underlying tight framlets Ψ. Characterizations of approximation order of $\mathcal{Q}_n$ were given in Theorem 2.8 of [17]. Furthermore, Lemma 2.4 of [17] says that $\mathcal{P}_n = \mathcal{Q}_n$ on $L_2(\mathbb{R})$ when the tight framelets Ψ are obtained via the unitary extension principle (see Chapter 5 for the UEP) from the MRA generated by the same refinable function ϕ. The following theorem is a special case of Theorem 2.8 in [17] with the understanding $\mathcal{P}_n = \mathcal{Q}_n$.

Theorem 3.9. *Let ϕ be a pseudo-spline of order (m, l) with refinement mask a. Let $\mathcal{P}_n$ be the operator as defined in* (1.9) *with ϕ as the underlying refinable function. Then the approximation order of the operator $\mathcal{P}_n$ is $\min\{m, m_1\}$, with m_1 the order of the zero of $1 - |\widehat{a}|^2$ at the origin.*

With this, we have the followings:

Theorem 3.10. *For given nonnegative integer m and l, with $l \leq m-1$.*

1. *Let ${}_1\phi$ be the pseudo-spline of type I with order (m, l) and ${}_1\widehat{a}$ be its refinement mask. Then the corresponding operator $\mathcal{P}_n$ provides approximation order $\min\{m, 2l+2\}$.*

2. *Let ${}_2\phi$ be the pseudo-spline of type II with order (m, l) and ${}_2\widehat{a}$ be its refinement mask. Then the corresponding operator $\mathcal{P}_n$ provides approximation order $2l+2$.*

Proof. It was shown in [17] that $1 - |{}_1\widehat{a}| = O(|\cdot|^{2l+2})$. Therefore, Theorem 3.9 gives the rest of the proof of part 1. For part 2, we compute the order of zeros of $1 - |{}_2\widehat{a}|^2$ at origin. We rewrite $1 - |{}_2\widehat{a}|^2$ as

$$1 - |{}_2\widehat{a}|^2 = 1 - R_{m,l}^2(\sin^2(\xi/2)),$$

where $R_{m,l}(y)$ was defined in (2.3). It is obvious that for $\xi = 0$, $1 - R_{m,l}^2(\sin^2(\xi/2)) = 0$. Recall that the derivative of $R_{m,l}(y)$ was given by part 2 of Lemma 2.2, i.e.

$$R'_{m,l}(y) = -(m+l)\binom{m+l-1}{l} y^l (1-y)^{m-1}. \tag{3.25}$$

Applying (3.25) to take the first derivative of $1 - R_{m,l}^2(\sin^2(\xi/2))$ with respect to ξ, one obtains

$$\begin{aligned}\Big(1 - R_{m,l}^2(\sin^2(\xi/2))\Big)' &= -2R_{m,l}(\sin^2(\xi/2))R'_{m,l}(\sin^2(\xi/2))(\sin^2(\xi/2))' \\ &= 2R_{m,l}(\sin^2(\xi/2))\Bigg((m+l)\binom{m+l-1}{l} \\ &\quad \cdot \sin^{2l}(\xi/2)\cos^{2m-2}(\xi/2)\Bigg)(\sin^2(\xi/2))' \\ &= 2(m+l)\binom{m+l-1}{l}R_{m,l}(\sin^2(\xi/2)) \\ &\quad \cdot \sin^{2l+1}(\xi/2)\cos^{2m-1}(\xi/2).\end{aligned}$$

Since $R_{m,l}(\sin^2(\xi/2))$ and $\cos^{2m-1}(\xi/2)$ is 1 when $\xi = 0$, and since $\sin^{2l+1}(\xi/2)$ has zero at $\xi = 0$ of order $2l+1$, we conclude that

$$1 - |{}_2\widehat{a}(\xi)|^2 = 1 - R_{m,l}^2(\sin^2(\xi/2)) = O(|\xi|^{2l+2}).$$

Then Theorem 3.9 shows that the approximation order of $\mathcal{P}_n$ with pseudo-spline of type II as the underlying refinable function is $\min\{2m, 2l+2\} = 2l+2$, for $0 \le l \le m-1$. $\square$

Remark 3.11. The above result says that when $l \le \frac{m}{2} - 1$, the approximation order of a pseudo-spline of type I with order (m,l) and one of type II with the same order are the same, although the support of the type I is half of that of type II. When $l > \frac{m}{2} - 1$, the approximation order of type I is m and type II is $2l+2 > m$. The regularity of the type II is about two times that of the type I with the same order. Furthermore, one can obtain symmetric short Riesz Wavelets and tight framelets from pseudo-splines of type II, as we will see in Chapter 5.

Chapter 4

Linear Independence of Pseudo-splines

This chapter is to verify the linear independence of the shifts of pseudo-splines. It is easy to see from the definition of linear independence that when the function ϕ is a pseudo-spline of type I or II with order $(m, m-1)$ (which is the orthogonal refinable function for the first type or interpolatory refinable function for the second type), its shifts are linearly independent. It is also well known that a pseudo-spline of either type with order $(m, 0)$, which is a B-spline, and its shifts are linearly independent. It is very natural to ask whether an arbitrary pseudo-spline and its shifts are linearly independent. This is one of our motivations, but not the only one. The linear independence of the shifts of a compactly supported refinable function $\phi \in L_2(\mathbb{R})$ is a necessary and sufficient condition for the existence of a compactly supported dual refinable function $\phi^d \in L_2(\mathbb{R})$ of ϕ. The proof of the necessity is simple. Recall that a compactly supported refinable function $\phi^d \in L_2(\mathbb{R})$ is a dual of ϕ, if

$$\langle \phi, \phi^d(\cdot - k) \rangle = \delta(k) \tag{4.1}$$

holds for all $k \in \mathbb{Z}$ (see e.g. [6, 7]). Indeed, if there is a compactly supported function $\phi^d \in L_2(\mathbb{R})$ that is dual to ϕ, then for $b \in \ell(\mathbb{Z})$ satisfying $\phi *' b = 0$, we have

$$0 = \langle \phi *' b, \phi^d(\cdot - k) \rangle = \sum_{j \in \mathbb{Z}} b(j) \langle \phi(\cdot - j), \phi^d(\cdot - k) \rangle = b(k) \quad \text{for all } k \in \mathbb{Z}.$$

However, the proof of the sufficiency is more complicated and we refer to [41] and [42] for the details. It is well known that the construction of a compactly supported dual refinable function is a key step to construct a pair of biorthogonal wavelet systems from the given refinable function.

Next, we observe that since ${}_1\widehat{a}_{(m,l)}(\xi)$ is a trigonometric polynomial with real coefficients, we have

$${}_2\widehat{a}_{(m,l)}(\xi) = |{}_1\widehat{a}_{(m,l)}(\xi)|^2 = {}_1\widehat{a}_{(m,l)}(\xi) \cdot {}_1\widehat{a}_{(m,l)}(-\xi).$$

This leads to

$$|{}_1\widehat{\phi}_{(m,l)}(\xi)|^2 = {}_1\widehat{\phi}_{(m,l)}(\xi) \cdot {}_1\widehat{\phi}_{(m,l)}(-\xi) = {}_2\widehat{\phi}_{(m,l)}(\xi), \quad \xi \in \mathbb{R}. \tag{4.2}$$

Since both ${}_1\phi_{(m,l)}$ and ${}_2\phi_{(m,l)}$ are compactly supported and bounded, their Fourier-Laplace transforms ${}_1\widehat{\phi}_{(m,l)}(\zeta)$ and ${}_2\widehat{\phi}_{(m,l)}(\zeta)$ are analytic on $\mathbb{C}$. Hence, (4.2) holds for all $\zeta \in \mathbb{C}$, i.e.

$${}_1\widehat{\phi}_{(m,l)}(\zeta) \cdot {}_1\widehat{\phi}_{(m,l)}(-\zeta) = {}_2\widehat{\phi}_{(m,l)}(\zeta), \quad \zeta \in \mathbb{C}. \tag{4.3}$$

The identity (4.3) implies that the set of all zeros of ${}_1\widehat{\phi}_{(m,l)}(\zeta)$ is contained in that of ${}_2\widehat{\phi}_{(m,l)}(\zeta)$ for $\zeta \in \mathbb{C}$.

Applying (1.14), we conclude the following proposition:

Proposition 4.1. *Assume that the shifts of pseudo-spline ${}_2\phi_{(m,l)}$ of type II with order (m, l) are linearly independent. Then the shifts of pseudo-spline ${}_1\phi_{(m,l)}$ of type I with the same order are linearly independent.*

In the rest of this chapter, we will focus on the verification of the linear independence of the shifts of the pseudo-splines of type II.

We start with two Lemmata. The first Lemma is implied by Theorem 1 and 2 of a paper of Jia and Wang (see [38]). Instead of stating both theorems of [38] and deducing the following lemma by using them, we include here a direct proof which is essentially derived from Jia and Wang's proof of Theorem 1 and 2 in [38].

We say that a Laurent polynomial $\tilde{a}$ has symmetric zeros on $\mathbb{C} \setminus \{0\}$ if there is a $z_0 \in \mathbb{C} \setminus \{0\}$ such that

$$\tilde{a}(z_0) = \tilde{a}(-z_0) = 0.$$

Lemma 4.2. *Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function with (finitely supported) refinement mask a. The shifts of ϕ are linearly independent if and only if:*

1. *ϕ is stable;*
2. *the symbol $\tilde{a}$ does not have any symmetric zeros on $\mathbb{C} \setminus \{0\}$.*

Proof. We first show the necessity of (1) and (2). Condition (1) is necessary for the linear independence of the shifts of ϕ implied by (1.13) and (1.14). The necessity of (2) is proven by contradiction. Suppose there exists $z_0 = e^{-i\zeta_0} \in \mathbb{C} \setminus \{0\}$ such that $\tilde{a}(e^{-i\zeta_0}) = \tilde{a}(-e^{-i\zeta_0}) = 0$. Applying the Fourier-Laplace transform given in (1.2), one obtains, for any $k \in \mathbb{Z}$,

$$\widehat{\phi}(2\zeta_0 + 4k\pi) = \widehat{\phi}(\zeta_0 + 2k\pi)\tilde{a}(e^{-i\zeta_0}) = 0$$

and

$$\widehat{\phi}(2\zeta_0 + (4k+2)\pi) = \widehat{\phi}(\zeta_0 + 2k\pi + \pi)\tilde{a}(-e^{-i\zeta_0}) = 0.$$

These two identities imply that $\widehat{\phi}(2\zeta_0 + 2k\pi) = 0$ for all $k \in \mathbb{Z}$, which contradicts to the linear independence of the shifts of ϕ by (1.14).

Next, we show the sufficiency of (1) and (2), which is again shown by contradiction. Suppose that ϕ and its shifts are not linearly independent. Then, there is a $\zeta_0 \in \mathbb{C}$, such that $\widehat{\phi}(\zeta_0 + 2k\pi) = 0$ for all $k \in \mathbb{Z}$. Since $(\widehat{\phi}(2k\pi))_{k\in\mathbb{Z}} \neq \mathbf{0}$, $\zeta_0 \in \mathbb{C} \setminus \{0\}$. Applying (1.2) again, one obtains, for any $k \in \mathbb{Z}$,

$$0 = \widehat{\phi}(\zeta_0 + 4k\pi) = \widehat{\phi}(\zeta_0/2 + 2k\pi)\tilde{a}(e^{-i\zeta_0/2})$$

and

$$0 = \widehat{\phi}(\zeta_0 + (4k+2)\pi) = \widehat{\phi}(\zeta_0/2 + \pi + 2k\pi)\tilde{a}(-e^{-i\zeta_0/2}).$$

Since $\tilde{a}(z)$ does not have symmetric zeros on $\mathbb{C} \setminus \{0\}$, we conclude that at least one of the two sets of identities $\widehat{\phi}(\zeta_0/2 + 2k\pi) = 0$, $k \in \mathbb{Z}$ and $\widehat{\phi}(\zeta_0/2 + \pi + 2k\pi) = 0$, $k \in \mathbb{Z}$ holds. Let $\zeta_1 = \zeta_0/2$ or $\zeta_1 = \zeta_0/2 + \pi$. (The choice here depends on whether $\widehat{\phi}(\zeta_0/2+2k\pi) = 0$, $k \in \mathbb{Z}$ or $\widehat{\phi}(\zeta_0/2+\pi+2k\pi) = 0$, $k \in \mathbb{Z}$.) Repeating this process, one obtain $\zeta_2 = \zeta_1/2$ or $\zeta_2 = \zeta_1/2 + \pi$. Continuing the process, one obtains a set of numbers $A := \{\zeta_0, \zeta_1, \zeta_2, \cdots\}$ such that $(\widehat{\phi}(\zeta_j + 2k\pi))_{k\in\mathbb{Z}} = \mathbf{0}$ for $j = 0, 1, 2, \cdots$. However, by Proposition 2.1 of Ron in [47], the set A must be finite (also see [38]). Hence, there must exist some integers $0 \le p < q$, such that $\zeta_p = \zeta_q$. Since $\zeta_p = \frac{\zeta_0}{2^p} + r\pi$, $\zeta_q = \frac{\zeta_0}{2^q} + s\pi$ for some rational number r and s, we have

$$\frac{\zeta_0}{2^p} + r\pi = \frac{\zeta_0}{2^q} + s\pi.$$

This leads to that ζ_0 is a real number which implies that

$$\big(\widehat{\phi}(\zeta_0 + 2k\pi)\big)_{k\in\mathbb{Z}} = \mathbf{0} \quad \text{with } \zeta_0 \in \mathbb{R}.$$

This contradicts to the stability of ϕ (which is (1)) by (1.13). $\square$

Lemma 4.2 says that, in order to show the linear independence of the shifts of pseudo-splines of type II, we need to verify: (i), pseudo-splines of type II are stable; (ii), the symbol of an arbitrary pseudo-spline of type II does not have any symmetric zeros on $\mathbb{C} \setminus \{0\}$. We first prove the stability in the following lemma.

Lemma 4.3. *Pseudo-splines of type II are stable.*

Proof. Since pseudo-splines are compactly supported and belong to $L_2(\mathbb{R})$ (see Proposition 3.5), the stability of ϕ is equivalent to (1.13). Let $\phi_{(m,l)}$ be the pseudo-spline of type II with order (m,l) and $\widehat{a}_{(m,l)}$ be its refinement mask. By Definition (1.5), for each fixed $m \geq 1$ and for every $0 \leq l \leq m-1$, the following inequality

$$\cos^{2m}(\xi/2) \leq \widehat{a}_{(m,l)}(\xi)$$

holds for all $\xi \in \mathbb{R}$. Therefore, by (1.6), we have for all $\xi \in \mathbb{R}$,

$$|\widehat{B}_{2m}(\xi)| \leq |\widehat{\phi}_{(m,l)}(\xi)|. \tag{4.4}$$

Since B_{2m} is stable, the vector $(\widehat{B}_{2m}(\xi + 2k\pi))_{k\in\mathbb{Z}} \neq \mathbf{0}$ for every $\xi \in \mathbb{R}$. Hence, $(\widehat{\phi}_{(m,l)}(\xi + 2k\pi))_{k\in\mathbb{Z}} \neq \mathbf{0}$, for every $\xi \in \mathbb{R}$, which is equivalent to that $\phi_{(m,l)}$ is stable. $\square$

By Lemma 4.2 and 4.3, to show that the shifts of the corresponding pseudo-spline of type II are linearly independent, we only need to show that the symbol of it has no symmetric zeros on $\mathbb{C} \setminus \{0\}$. Now we compute the symbols of pseudo-splines of type II. Recall that the refinement mask of a pseudo-spline of type II with order (m,l) is given by (1.5), i.e.

$${}_2\widehat{a}(\xi) = \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2). \tag{4.5}$$

Using

$$\cos^2(\xi/2) = \frac{1+\cos(\xi)}{2} = \frac{2+e^{i\xi}+e^{-i\xi}}{4} = \frac{(1+e^{-i\xi})^2}{4e^{-i\xi}} \tag{4.6}$$

and

$$\sin^2(\xi/2) = \frac{1-\cos(\xi)}{2} = \frac{2-e^{i\xi}-e^{-i\xi}}{4} = \frac{-(1-e^{-i\xi})^2}{4e^{-i\xi}}, \tag{4.7}$$

one obtains

$${}_2\widehat{a}(\xi) := \frac{(1+e^{-i\xi})^{2m}}{(4e^{-i\xi})^m} \sum_{j=0}^{l} \binom{m+l}{j} \left(\frac{-(1-e^{-i\xi})^2}{4e^{-i\xi}}\right)^j \left(\frac{(1+e^{-i\xi})^2}{4e^{-i\xi}}\right)^{l-j}, \quad \text{for } \xi \in \mathbb{R}.$$

Extending the above trigonometric polynomial to the Laurent polynomial, one obtains the symbol of the pseudo-spline of type II with order (m,l):

$$ {}_2\tilde{a}(z) := \frac{(1+z)^{2m}}{(4z)^m} \sum_{j=0}^{l} \binom{m+l}{j} \left(\frac{-(1-z)^2}{4z} \right)^j \left(\frac{(1+z)^2}{4z} \right)^{l-j}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.8}$$

Before proving the main theorem of this chapter, we need to give the following proposition first. The proof of it employs *Rouché's theorem* (see e.g. [2]), which states as: Suppose two functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour C and suppose

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

then f and $f+g$ have the same number of zeros, counting multiplicities, inside C.

Proposition 4.4. *Let*

$$P(z) = \sum_{j=0}^{l} c_j z^j$$

be a polynomial with real coefficients satisfying

$$c_l > c_{l-1} > \cdots > c_0 > 0.$$

Then, all the zeros of $P(z)$ are contained within the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$.

Proof. Let

$$\rho := \max_{1 \le j \le l} \{ \frac{c_{j-1}}{c_j} \}.$$

Since c_j is strictly greater than c_{j-1}, $0 < \rho < 1$ and $\rho c_j \ge c_{j-1}$ for all $1 \le j \le l$. Consider

$$Q(z) := (\rho - z)P(z).$$

Then,

$$\begin{aligned} Q(z) &= \rho P(z) - zP(z) \\ &= \rho c_0 + (\rho c_1 - c_0)z + (\rho c_2 - c_1)z^2 + \cdots + (\rho c_l - c_{l-1})z^l - c_l z^{l+1} \\ &= g(z) + f(z), \end{aligned}$$

where

$$f(z) := -c_l z^{l+1}, \quad \text{and} \quad g(z) := \rho c_0 + (\rho c_1 - c_0)z + (\rho c_2 - c_1)z^2 + \cdots + (\rho c_l - c_{l-1})z^l.$$

Note that when $|z| = 1$, we have

$$\begin{aligned} |g(z)| &\leq \rho c_0 + (\rho c_1 - c_0) + (\rho c_2 - c_1) + \cdots + (\rho c_l - c_{l-1}) \\ &= (\rho - 1)c_0 + (\rho - 1)c_1 + \cdots + (\rho - 1)c_{l-1} + \rho c_l \\ &= \rho c_l - (1 - \rho) \sum_{j=0}^{l-1} c_j \\ &< c_l = | - c_l z^{l+1}| = |f(z)|. \end{aligned}$$

Since f and g are analytic on $\{z \in \mathbb{C} : |z| \leq 1\}$, the Rouché's theorem asserts that $Q = f + g$ has the same number of zeros as that of f in $D = \{z \in \mathbb{C} : |z| < 1\}$. Since f has $l + 1$ zeros in D, Q must have exactly $l + 1$ zeros in D. Since Q has only $l + 1$ zeros and since zeros of P is a subset of the zeros of Q, all zeros of P must be in D. $\square$

Next, we prove the main theorem of this chapter.

Theorem 4.5. *The shifts of a pseudo-spline of type II are linearly independent.*

Proof. Since Lemma 4.3 shows that pseudo-splines of type II with arbitrary orders are stable, in order to prove the linear independence of the shifts of a given pseudo-spline of type II, one needs to show that the symbol ${}_2\tilde{a}(z)$ of it has no symmetric zeros on $\mathbb{C} \setminus \{0\}$, by Lemma 4.2.

The symbol ${}_2\tilde{a}(z)$ given by (4.8) can be rewritten as

$$\begin{aligned} {}_2\tilde{a}(z) &= \frac{(1+z)^{2m}}{(4z)^m}\sum_{j=0}^{l}\binom{m+l}{j}\left(\frac{-(1-z)^2}{4z}\right)^j\left(\frac{(1+z)^2}{4z}\right)^{l-j} \\ &= \frac{(1+z)^{2m}}{(4z)^{m+l}}\sum_{j=0}^{l}\binom{m+l}{j}\big(-(1-z)^2\big)^j(1+z)^{2(l-j)} \\ &= \frac{(1+z)^{2m+2l}}{(4z)^{m+l}}\sum_{j=0}^{l}\binom{m+l}{j}\left(\frac{-(1-z)^2}{(1+z)^2}\right)^j. \end{aligned}$$

Since $z=-1$ is a zero of ${}_2\tilde{a}(z)$, while ${}_2\tilde{a}(z)=1$ when $z=1$, ${}_2\tilde{a}(z)$ having no symmetric zeros on $\mathbb{C}\setminus\{0\}$ is equivalent to

$$h(z) := \sum_{j=0}^{l}\binom{m+l}{j}\left(\frac{-(1-z)^2}{(1+z)^2}\right)^j \tag{4.9}$$

having no symmetric zeros on $\mathbb{C}\setminus\{0,1,-1\}$.

Consider

$$P(x)=\sum_{j=0}^{l} b_j x^j, \quad \text{with } b_j=\binom{m+l}{j},\ x\in\mathbb{C}.$$

We first show that Proposition 4.4 can be applied to P to conclude that the zeros of P lies inside of the unit disk of $\mathbb{C}$. For this, we need to show that for given $m>0, 0\le l\le m-1$,

$$b_{j+1} > b_j > 0, \quad 0\le j\le l-1. \tag{4.10}$$

Note that

$$\binom{m+l}{j+1} = \frac{m+l-j}{j+1}\binom{m+l}{j}.$$

Since $j\le l-1$, replacing j by $l-1$ in $\frac{m+l-j}{j+1}$, the right hand side of the above identity decreases and becomes

$$\frac{m+1}{l}\binom{m+l}{j},$$

which is larger than $\binom{m+l}{j}$ by $l\le m-1$. This shows that $b_{j+1}>b_j$, $j=0,\dots,l-1$. It is clear that $b_0=1>0$. With (4.10), applying Proposition 4.4, one concludes that all zeros of $P(x)$ must be in $\{x \in \mathbb{C} : |x| < 1\}$. Let z_0 be an arbitrary zero of h in $\mathbb{C} \setminus \{0, 1, -1\}$. Then, the above conclusion on the zeros of P implies that z_0 must satisfy

$$\left| \frac{-(1-z_0)^2}{(1+z_0)^2} \right| < 1. \tag{4.11}$$

Suppose h has symmetric zeros z_0 and $-z_0$. Then, $-z_0$ must also satisfy

$$\left| \frac{-(1+z_0)^2}{(1-z_0)^2} \right| < 1.$$

Since

$$\left| \frac{-(1-z_0)^2}{(1+z_0)^2} \right| = \frac{1}{\left| \frac{-(1+z_0)^2}{(1-z_0)^2} \right|},$$

we conclude that

$$\left| \frac{-(1-z_0)^2}{(1+z_0)^2} \right| > 1,$$

which contradicts to (4.11). This leads to that h has no symmetric zeros on $\mathbb{C} \setminus \{0, 1, -1\}$, and hence, ${}_2\tilde{a}(z)$ has no symmetric zeros on $\mathbb{C} \setminus \{0\}$. This, together with the stability of the pseudo-splines, proves the linear independence of the shifts of an arbitrary given pseudo-spline by Lemma 4.2. $\square$

Chapter 5

Riesz Wavelets and Framelets

In this chapter, we focus on the short supported Riesz wavelets and framelets derived from pseudo-splines. The study here reveals that in almost all pseudo-spline tight frame systems constructed both in [17] and here, there is one framelet whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$.

5.1 Riesz Wavelets

For a given ψ, define the *wavelet system*

$$X(\psi) := \{\psi_{n,k} = 2^{n/2}\psi(2^n \cdot -k) : n, k \in \mathbb{Z}\}.$$

We call $X(\psi)$ a *Bessel system* if for some $C_1 > 0$, and for every $f \in L_2(\mathbb{R})$,

$$\sum_{g\in X(\psi)} |\langle f, g\rangle|^2 \leq C_1 \|f\|^2_{L_2(\mathbb{R})}.$$

A Bessel system $X(\psi)$ is a *Riesz basis* if there exists $C_2 > 0$ such that,

$$C_2\|\{c_{n,k}\}\|_{\ell_2(\mathbb{Z}^2)} \leq \left\| \sum_{(n,k)\in\mathbb{Z}^2} c_{n,k}\psi_{n,k}\right\|_{L_2(\mathbb{R})}, \quad \text{for all } \{c_{n,k}\} \in \ell_2(\mathbb{Z}^2)$$

and the span of $\{\psi_{n,k} : n, k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{R})$. The function ψ is called *Riesz wavelet* if $X(\psi)$ forms a Riesz basis for $L_2(\mathbb{R})$ and $X(\psi)$ is also called the *Riesz wavelet system.*

This section is devoted to the construction of short Riesz wavelets via the MRA generated by pseudo-splines. As all pseudo-splines are compactly supported, refinable and in $L_2(\mathbb{R})$, the sequence of spaces $(V_n)_{n\in\mathbb{Z}}$ defined via (1.8) forms a MRA. Since the objective here is to construct Riesz wavelets, one needs to start with stable refinable functions, which is satisfied for all pseudo-splines as shown in Chapter 4.

We note that since constructions of exponential decay orthonormal wavelets and compactly supported pre-wavelets in the literature (see e.g. [14] and [49]) only assume the stability of refinable function ϕ, they can be applied to pseudo-splines automatically. Here we omit the details of these constructions and, instead, focus on the construction and analysis of the short Riesz wavelets. They are short because the wavelets have the same lengths of supports as the corresponding refinable functions, while other Riesz wavelets normally have longer supports than the corresponding refinable functions.

For a given stable refinable function $\phi \in L_2(\mathbb{R})$, the key step in the construction of the Riesz wavelet ψ is to select some desirable sequence b, called a *wavelet mask.* The *wavelet* ψ is then defined by b and the corresponding refinable function ϕ as

$$\psi := 2\sum_{k\in\mathbb{Z}} b(k)\phi(2\cdot -k),$$

It can be written equivalently in the Fourier domain as

$$\widehat{\psi}(\xi) = \widehat{b}(\xi/2)\widehat{\phi}(\xi/2).$$

When $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for $V_0(\phi)$, e.g. ϕ is a pseudo-spline of type I with order $(m, m-1)$, define

$$\psi := 2\sum_{k\in\mathbb{Z}} b(k)\phi(2\cdot -k) \quad \text{with} \quad b(k) = (-1)^{k-1}\overline{a(1-k)}, \quad k \in \mathbb{Z}, \tag{5.1}$$

or equivalently,

$$\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}.$$

Then the corresponding wavelet system $X(\psi)$ with the pseudo-spline of type I with order $(m, m-1)$ being the underlying refinable function forms an orthonormal basis for $L_2(\mathbb{R})$. We are interested to know whether the function ψ defined in (5.1) is a Riesz wavelet, when the refinable function ϕ is chosen to be the pseudo-splines with other orders. In fact, it was shown in [30] that it is true, when ϕ is a B-spline, i.e. a pseudo-spline with order $(m, 0)$ or when ϕ is a pseudo-spline of type II with order $(m, m-1)$. In the rest of this section we will show that for all pseudo-splines, the wavelet defined by (5.1) is a Riesz wavelet. To prove this, we use the following theorem which is the special case of Theorem 2.1 of [30]. When both refinement masks are finitely supported similar result was already obtained before in [6], [7] and [13].

Theorem 5.1. *Let a be a finitely supported refinement mask of a refinable function $\phi \in L_2(\mathbb{R})$ with $\widehat{a}(0) = 1$ and $\widehat{a}(\pi) = 0$, such that $\widehat{a}$ can be factorized into the form*

$$|\widehat{a}(\xi)| = \left|\left(\frac{1+e^{-i\xi}}{2}\right)^n \mathcal{L}(\xi)\right| = \cos^n(\xi/2)|\mathcal{L}(\xi)|, \quad \xi \in [-\pi, \pi], \tag{5.2}$$

where $\mathcal{L}$ is the Fourier series of a finitely supported sequence with $\mathcal{L}(\pi) \neq 0$. Suppose that

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \neq 0, \quad \xi \in [-\pi, \pi].$$

Define

$$\widehat{\psi}(2\xi) := e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}\widehat{\phi}(\xi)$$

and

$$\tilde{\mathcal{L}}(\xi) := \frac{\mathcal{L}(\xi)}{|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2}. \tag{5.3}$$

Assume that

$$\rho_{\mathcal{L}} := \|\mathcal{L}(\xi)\|_{L_\infty(\mathbb{R})} < 2^{n-\frac{1}{2}} \qquad \text{and} \qquad \rho_{\tilde{\mathcal{L}}} := \|\tilde{\mathcal{L}}(\xi)\|_{L_\infty(\mathbb{R})} < 2^{n-\frac{1}{2}}, \tag{5.4}$$

Then $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

As we will show, the key step in the application of the above theorem is to estimate the upper bound of $|\mathcal{L}(\xi)|$ and $|\tilde{\mathcal{L}}(\xi)|$. Recall that the refinement masks of pseudo-splines of type I and II are, for $\xi \in [-\pi, \pi]$,

$$|{}_1\widehat{a}(\xi)| := \cos^m(\xi/2)\Bigg(\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2)\Bigg)^{\frac{1}{2}} \tag{5.5}$$

and

$${}_2\widehat{a}(\xi) := \cos^{2m}(\xi/2)\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2). \tag{5.6}$$

Hence, the corresponding $\mathcal{L}$ function in (5.2) for pseudo-splines of type I is

$$|{}_1\mathcal{L}(\xi)| = \Bigg(\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2)\Bigg)^{\frac{1}{2}},$$

and for pseudo-splines of type II is

$$|{}_2\mathcal{L}(\xi)| = \sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2).$$

For $P_{m,l}(y)$ be given by (2.2) with $y = \sin^2(\xi/2)$,

$$|{}_1\widehat{a}| = \Big((1-y)^m P_{m,l}(y)\Big)^{\frac{1}{2}}, \qquad {}_2\widehat{a} = (1-y)^m P_{m,l}(y), \tag{5.7}$$

and

$$|{}_1\mathcal{L}| = (P_{m,l}(y))^{\frac{1}{2}}, \qquad |{}_2\mathcal{L}| = P_{m,l}(y). \tag{5.8}$$

For $R_{m,l}(y)$ be given in (2.3),

$$|{}_1\widehat{a}(\xi)|^2 + |{}_1\widehat{a}(\xi+\pi)|^2 = R_{m,l}(y) + R_{m,l}(1-y)$$

and

$$|{}_2\widehat{a}(\xi)|^2 + |{}_2\widehat{a}(\xi+\pi)|^2 = R^2_{m,l}(y) + R^2_{m,l}(1-y),$$

with $y = \sin^2(\xi/2)$. Hence,

$$|_1\tilde{\mathcal{L}}| = \frac{(P_{m,l}(y))^{\frac{1}{2}}}{R_{m,l}(y) + R_{m,l}(1-y)} \quad \text{and} \quad |_2\tilde{\mathcal{L}}| = \frac{P_{m,l}(y)}{R^2_{m,l}(y) + R^2_{m,l}(1-y)}. \tag{5.9}$$

The estimation of $\|_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})}$ and $\|_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})}$ are based on the following result:

Proposition 5.2. *Let m and l be given nonnegative integers with $l \leq m-1$ and $|_1\tilde{\mathcal{L}}|$ and $|_2\tilde{\mathcal{L}}|$ be defined in (5.9). Then,*

1. $\|_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} = \sup_{y\in[0,1]} \frac{(P_{m,l}(y))^{\frac{1}{2}}}{R_{m,l}(y)+R_{m,l}(1-y)} < 2^{m-\frac{1}{2}}$.
2. $\|_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} = \sup_{y\in[0,1]} \frac{P_{m,l}(y)}{R^2_{m,l}(y)+R^2_{m,l}(1-y)} < 2^{2m-\frac{1}{2}}$.

Proof. Note that from part 1 of Lemma 2.2,

$$P_{m,l}(y) = \sum_{j=0}^{l} \binom{m+l}{j} y^j (1-y)^{l-j} = \sum_{j=0}^{l} \binom{m-1+j}{j} y^j, \quad y \in [0,1], \tag{5.10}$$

hence both $(P_{m,l}(y))^{\frac{1}{2}}$ and $P_{m,l}(y)$ attain their maximum on $[0,1]$ at point 1 and the maximum values are:

$$(P_{m,l}(1))^{\frac{1}{2}} = \binom{m+l}{l}^{\frac{1}{2}} \quad \text{and} \quad P_{m,l}(1) = \binom{m+l}{l}.$$

By part 3 of Lemma 2.2, one obtains

$$\begin{aligned} \|_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} &= \sup_{y\in[0,1]} \frac{(P_{m,l}(y))^{\frac{1}{2}}}{R_{m,l}(y) + R_{m,l}(1-y)} \\ &\leq \binom{m+l}{l}^{\frac{1}{2}} \max_{y\in[0,1]} \frac{1}{R_{m,l}(y) + R_{m,l}(1-y)} \\ &\leq \frac{2^{m+l-1}\binom{m+l}{l}^{\frac{1}{2}}}{\sum_{j=0}^{l}\binom{m+l}{j}}. \end{aligned}$$

Applying part 3 of Lemma 2.1, i.e.

$$\frac{2^l \binom{m+l}{l}^{\frac{1}{2}}}{\sum_{j=0}^{l}\binom{m+l}{j}} \leq 1, \tag{5.11}$$

one obtains

$$\|{}_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} \le 2^{m-1} < 2^{m-\frac{1}{2}}.$$

The proof of part 2 is similar to that of part 1. Indeed, by part 4 of Lemma 2.2

$$\begin{aligned}
\|{}_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} &= \sup_{y\in[0,1]} \frac{P_{m,l}(y)}{R^2_{m,l}(y)+R^2_{m,l}(1-y)} \\
&\le \binom{m+l}{l} \max_{y\in[0,1]} \frac{1}{R^2_{m,l}(y)+R^2_{m,l}(1-y)} \\
&= \frac{2^{2m+2l-1}\binom{m+l}{l}}{\left(\sum_{j=0}^{l}\binom{m+l}{j}\right)^2}.
\end{aligned}$$

Applying (5.11) again, we have

$$\|{}_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} \le 2^{2m-1} < 2^{2m-\frac{1}{2}}.$$

□

Theorem 5.3. *Let ${}_k\phi$, $k=1,2$ be the pseudo-spline of type I and II with order (m,l). The refinement masks ${}_ka$, $k=1,2$, are given in (1.4) and (1.5). Define*

$$_k\widehat{\psi}(2\xi) := e^{-i\xi}\overline{{}_k\widehat{a}(\xi+\pi)}\,{}_k\widehat{\phi}(\xi), \quad k=1,2, \tag{5.12}$$

then $X({}_k\psi)$ forms a Riesz basis for $L_2(\mathbb{R})$.

Proof. To apply Theorem 5.1, we first note that

$$|{}_1\widehat{a}(\xi)|^2 + |{}_1\widehat{a}(\xi+\pi)|^2 = R_{m,l}(\sin^2(\xi/2)) + R_{m,l}(\cos^2(\xi/2)) \neq 0$$

and

$$|{}_2\widehat{a}(\xi)|^2 + |{}_2\widehat{a}(\xi+\pi)|^2 = R^2_{m,l}(\sin^2(\xi/2)) + R^2_{m,l}(\cos^2(\xi/2)) \neq 0,$$

for all $\xi\in[-\pi,\pi]$, where $R_{m,l}$ is defined in (2.3) (by part 3 and 4 of Lemma 2.2).

Next, one needs to check whether

$$\rho_{1\mathcal{L}} = \|{}_1\mathcal{L}\|_{L_\infty(\mathbb{R})} < 2^{m-\frac{1}{2}}, \quad \rho_{2\mathcal{L}} = \|{}_2\mathcal{L}\|_{L_\infty(\mathbb{R})} < 2^{2m-\frac{1}{2}}, \tag{5.13}$$

$$\rho_{{}_1\tilde{\mathcal{L}}} = \|{}_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} < 2^{m-\frac{1}{2}} \quad \text{and} \quad \rho_{{}_2\tilde{\mathcal{L}}} = \|{}_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} < 2^{2m-\frac{1}{2}}, \tag{5.14}$$

hold. Inequalities in (5.14) follows from Proposition 5.2.

For (5.13), we note that for both $k = 1$ and $k = 2$, we have

$$|{}_k\widehat{a}(\xi)|^2 + |{}_k\widehat{a}(\xi + \pi)|^2 \leq 1 \quad \text{for all } \xi \in \mathbb{R}.$$

Hence,

$$|{}_k\mathcal{L}(\xi)| \leq |{}_k\tilde{\mathcal{L}}(\xi)| \quad \text{for all } \xi \in \mathbb{R}.$$

This concludes the proof. □

Remark 5.4. The Riesz wavelet constructed in the above theorem has the same length of support and at least the same order of smoothness as that of the corresponding pseudo-spline. Its order of the vanishing moments is the same as the order of the B-spline factor of the pseudo-spline. But, in general, its dual wavelet system is not compactly supported. However, this is not a problem in some applications. In applications like image compression, the short Riesz wavelet system can be applied to obtain fast reconstruction algorithm, while the decomposition is obtained by solving a linear system of equations (see [39]). On the other hand, a compactly supported dual system can be achieved by constructing a tight frame system such that one of the generators is the Riesz wavelet defined here. This will be discussed in section 5.2.

5.2 Framelets

In this section, we connect our findings here to the tight framelets constructed from pseudo-splines in [17] via the unitary extension principle of [51] and also give a new construction. We first review the generic construction of tight frames via the unitary extension principle. The results of the previous section then reveal that every tight frame system obtained from the pseudo-spline of type I in [17] has one framelet ψ such that $X(\psi)$ itself already forms a Riesz basis for $L_2(\mathbb{R})$. Since the pseudo-splines of type I are not symmetric, the tight frame systems given in [17] are not symmetric. In this section, we make use of the symmetry of the pseudo-splines of type II to obtain symmetric tight framelets.

The construction here is based on the *unitary extension principle* (UEP) of [51]. We give a brief discussion here while the more general version and comprehensive discussions of the UEP can be found in [17] and [51].

Let $\widehat{a}$ be the refinement mask of $\phi \in L_2(\mathbb{R})$ with $\widehat{a}(0) = 1$ and let $\widehat{b}_j$, $j = 1, 2, \ldots, r$, be wavelet masks. If $\widehat{a}$ and $\widehat{b}_j$ are trigonometric polynomials that satisfy

$$\widehat{a}(\xi)\overline{\widehat{a}(\xi+\nu)} + \sum_{j=1}^{r} \widehat{b}_j(\xi)\overline{\widehat{b}_j(\xi+\nu)} = \begin{cases} 1, & \nu = 0 \\ 0, & \nu = \pi, \end{cases} \tag{5.15}$$

for all $\xi \in [-\pi, \pi]$, and $\Psi := \{\psi_1, \psi_2, \ldots, \psi_r\} \subset L_2(\mathbb{R})$ are given by

$$\widehat{\psi}_j(2\xi) := \widehat{b}_j(\xi)\widehat{\phi}(\xi), \quad j = 1, 2, \ldots, r,$$

then the UEP asserts that $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$.

In [17], three constructions of tight framelets were given for pseudo-splines of type I. The number of framelets is either two or three. Interested readers may consult [17] Section 3.1 for details. We observe that in all the three constructions, one of the framelets ψ_1 is defined by

$$\widehat{\psi}_1 := \widehat{b}_1(\xi/2)\widehat{\phi}(\xi/2),$$

where

$$\widehat{b}_1 := e^{-i\xi}\overline{\widehat{a}(\cdot + \pi)}$$

and $\widehat{a}$ is the refinement mask of a pseudo-spline. It was shown in Theorem 5.3 that $X(\psi_1)$ forms a Riesz basis for $L_2(\mathbb{R})$. This implies that all pseudo-spline tight frame systems constructed in [17] already have one of the subsystems form a Riesz basis for $L_2(\mathbb{R})$. We further remark that it was observed in [30] that the same phenomenon occurs for the tight spline frame systems constructed in [17]. This, together with our new finding here, gives insight into the redundant structure of tight frame systems given in [17].

We further illustrate this phenomenon by one of the constructions given in [17]. The construction is generic and can be applied to refinable function $\phi \in L_2(\mathbb{R})$, whose refinement mask $\widehat{a}$ satisfies:

$$|\widehat{a}|^2 + |\widehat{a}(\cdot + \pi)|^2 \leq 1. \tag{5.16}$$

Construction 5.5. Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function with its trigonometric polynomial refinement mask $\widehat{a}$ satisfying $\widehat{a}(0) = 1$ and (5.16). Let

$$T := 1 - |\widehat{a}|^2 - |\widehat{a}(\cdot + \pi)|^2,$$

and $\tau := \sqrt{T}$ where τ is obtained via the Fejér-Riesz lemma. Define

$$\widehat{b}_1(\xi) := e^{-i\xi}\overline{\widehat{a}(\xi + \pi)},$$

$$\widehat{b}_2(\xi) := \frac{\tau}{\sqrt{2}}, \; \widehat{b}_3(\xi) := e^{i\xi}\widehat{b}_2(\xi).$$

Then the masks satisfy (5.15), hence, $X(\Psi)$ forms a tight frame in $L_2(\mathbb{R})$ where $\Psi := \{\psi_1, \; \psi_2, \; \psi_3\}$ are defined by

$$\widehat{\psi}_j(\xi) := \widehat{b}_j(\xi/2)\widehat{\phi}(\xi/2), \qquad j = 1, 2, 3. \tag{5.17}$$

Since ψ_1 is defined exactly the same as (5.12) in Theorem 5.3, $X(\psi_1)$ forms a Riesz basis for $L_2(\mathbb{R})$ when ϕ is a pseudo-spline. Furthermore, since $\widehat{b}_2$ and $\widehat{b}_3$ have zeros at both 0 and π. One can check easily that neither the shifts of ψ_2 nor those of ψ_3 can form a Riesz system. Hence, $X(\psi_2)$ and $X(\psi_3)$ can not form a Riesz basis for $L_2(\mathbb{R})$.

When the above construction is applied to pseudo-splines of type I, it leads to one of the constructions of [17]. However, the framelets are neither symmetric nor antisymmetric. One can obtain symmetric or antisymmetric framelets by applying Construction 3.4 in [25] which converts any tight frame system into another one with symmetric and antisymmetric framelets. However, the new tight frame system is generated by a different MRA since pseudo-splines of type I are not symmetric, unless the order is $(m, 0)$. A detailed discussion can be found in Section 4 of [25].

It is clear that pseudo-splines of type II also satisfy (5.16). One can apply Construction 5.5 to obtain a set of tight framelets. We note that ψ_1 obtained in Construction 5.5 is already symmetric, however, ψ_2 and ψ_3 are not, which is due to the fact that τ is not symmetric. Applying Construction 3.4 in [25], one can convert ψ_1, ψ_2 and ψ_3 to a set of five symmetric or antisymmetric tight framelets. It was further shown in [25] that the Construction 3.4 leads to new tight framelets from the same MRA as the old tight framelets whenever the old ones are derived from the MRA generated by a symmetric refinable function. We forgo the idea of giving the details of this construction and leave it to readers by consulting [25], because next we will give a different approach that leads to a symmetric tight frame system with only three generators. The ideas of this construction are based on those of [10] and one of the constructions of [17]. Again, the construction is generic and can be applied to any symmetric refinable function whose mask is a trigonometric polynomial and satisfies (5.16).

Construction 5.6. Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function with its trigonometric polynomial refinement mask $\widehat{a}$ satisfying $\widehat{a}(0) = 1$ and (5.16). Moreover we assume that ϕ, hence its refinement mask $\widehat{a}$, is symmetric about the origin. Let

$$T = 1 - |\widehat{a}|^2 - |\widehat{a}(\cdot + \pi)|^2 \quad \text{and} \quad \mathcal{A} := \frac{\sqrt{T}}{2},$$

where $\sqrt{T}$ is obtained via the Fejér-Riesz lemma. Define

$$\widehat{b}_1(\xi) := e^{-i\xi}\overline{\widehat{a}(\xi + \pi)}, \quad \widehat{b}_2(\xi) := \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) \quad \text{and} \quad \widehat{b}_3(\xi) := e^{-i\xi}\overline{\widehat{b}_2(\xi + \pi)}.$$

Let $\Psi := \{\psi_1,\ \psi_2,\ \psi_3\}$, where

$$\widehat{\psi}_j(\xi) := \widehat{b}_j(\xi/2)\widehat{\phi}(\xi/2), \qquad j = 1, 2, 3. \tag{5.18}$$

Then $X(\Psi)$ is tight frame for $L_2(\mathbb{R})$. Moreover, ψ_1 is symmetric about $\frac{1}{2}$, ψ_2 is symmetric about $\frac{1}{4}$ and ψ_3 is antisymmetric about $\frac{1}{4}$. We also note that since ψ_1 is defined exactly the same as (5.12) in Theorem 5.3, $X(\psi_1)$ forms a Riesz basis for $L_2(\mathbb{R})$ when ϕ is a pseudo-spline. Furthermore, since $\widehat{b}_2$ and $\widehat{b}_3$ have zeros at both 0 and π. One can check easily that neither the shifts of ψ_2 nor those of ψ_3 can form a Riesz system. Hence, $X(\psi_2)$ and $X(\psi_3)$ cannot form a Riesz basis for $L_2(\mathbb{R})$.

Proof. In order to verify that $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$, one needs to show that the masks $\{\widehat{a}, \widehat{b}_1, \widehat{b}_2, \widehat{b}_3\}$ satisfy (5.15). Note that

$$\widehat{b}_1 = e^{-i\xi}\overline{\widehat{a}(\cdot + \pi)} \quad \text{and} \quad \widehat{b}_3 = e^{-i\xi}\overline{\widehat{b}_2(\cdot + \pi)},$$

Hence,

$$\widehat{a}\overline{\widehat{a}(\cdot + \pi)} + \sum_{j=1}^{3}\widehat{b}_j\overline{\widehat{b}_j(\cdot + \pi)} = \widehat{a}\overline{\widehat{a}(\cdot + \pi)} - \widehat{a}\overline{\widehat{a}(\cdot + \pi)} + \widehat{b}_2\overline{\widehat{b}_2(\cdot + \pi)} - \widehat{b}_2\overline{\widehat{b}_2(\cdot + \pi)} = 0.$$

Next, we show that

$$|\widehat{a}|^2 + \sum_{j=1}^{3}|\widehat{b}_j|^2 = 1. \tag{5.19}$$

Since

$$|\widehat{a}|^2 + |\widehat{b}_1|^2 = |\widehat{a}|^2 + |\widehat{a}(\cdot + \pi)|^2,$$

it remains to show that

$$|\widehat{b}_2|^2 + |\widehat{b}_3|^2 = 1 - |\widehat{a}|^2 - |\widehat{a}(\cdot + \pi)|^2 = T.$$

Since

$$|\mathcal{A}(\xi)|^2 = \frac{1}{4}T(\xi) = \frac{1}{4}\Big(1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi + \pi)|^2\Big)$$

and since T is π-periodic, the spectral factorization (which is based on the Fejér-Riesz lemma) leads to the function $\mathcal{A}(\xi)$ also to be π-periodic. Furthermore, the Fourier coefficients of $\mathcal{A}(\xi)$ are real. Hence, we have

$$\mathcal{A}(\xi) = \mathcal{A}(\xi + \pi), \quad \text{and} \quad |\mathcal{A}(\xi)|^2 = |\mathcal{A}(-\xi)|^2, \quad \text{for all } \xi \in \mathbb{R}. \tag{5.20}$$

Since

$$b_2(\xi) = \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) \quad \text{and} \quad \widehat{b}_3(\xi) = e^{-i\xi}\overline{\widehat{b}_2(\cdot + \pi)} = e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi),$$

applying (5.20), one obtains

$$\begin{aligned}
|\widehat{b}_2(\xi)|^2 &= \Big(\mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi)\Big)\Big(\overline{\mathcal{A}(\xi)} + e^{i\xi}\overline{\mathcal{A}(-\xi)}\Big) \\
&= |\mathcal{A}(\xi)|^2 + |\mathcal{A}(-\xi)|^2 + e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} + e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)} \\
&= 2|\mathcal{A}(\xi)|^2 + e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} + e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)}
\end{aligned}$$

and

$$\begin{aligned}
|\widehat{b}_3(\xi)|^2 &= \Big(e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi)\Big)\Big(e^{i\xi}\overline{\mathcal{A}(-\xi)} - \overline{\mathcal{A}(\xi)}\Big) \\
&= |\mathcal{A}(\xi)|^2 + |\mathcal{A}(-\xi)|^2 - e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} - e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)} \\
&= 2|\mathcal{A}(\xi)|^2 - e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} - e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)}.
\end{aligned}$$

Hence,

$$|\widehat{b}_2(\xi)|^2 + |\widehat{b}_3(\xi)|^2 = 4|\mathcal{A}(\xi)|^2 = T(\xi),$$

which gives (5.19) and thus concludes that the masks $\{\widehat{a}, \widehat{b}_1, \widehat{b}_2, \widehat{b}_3\}$ satisfy (5.15). Therefore, $X(\Psi)$ is indeed a tight frame for $L_2(\mathbb{R})$ by the unitary extension principle.

Now we show that ψ_1 is symmetric about $\frac{1}{2}$ while ψ_2 is symmetric about $\frac{1}{4}$ and ψ_3 is antisymmetric about $\frac{1}{4}$. It is well known that a function $f \in L_2(\mathbb{R})$, is symmetric about the point $\gamma_1 \in \mathbb{R}$ if and only if

$$f(x) = f(2\gamma_1 - x) \quad \text{a.e.},$$

which is equivalent to

$$\widehat{f}(\xi) = e^{-i2\gamma_1\xi}\widehat{f}(-\xi) \quad \text{a.e.}. \tag{5.21}$$

Similarly, a function $f \in L_2(\mathbb{R})$ is antisymmetric about the point $\gamma_2 \in \mathbb{R}$ if and only if

$$f(x) = -f(2\gamma_2 - x) \quad \text{a.e.},$$

which is equivalent to

$$\widehat{f}(\xi) = -e^{-i2\gamma_2\xi}\widehat{f}(-\xi) \quad \text{a.e.}. \tag{5.22}$$

By the definition of $\widehat{b}_1$ and the fact that $\widehat{a}$ is symmetric about the origin and 2π-periodic, one obtains

$$\widehat{b}_1(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)} = e^{-2i\xi}\big(e^{i\xi}\overline{\widehat{a}(-\xi+\pi)}\big) = e^{-2i\xi}\widehat{b}_1(-\xi).$$

Since ϕ is symmetric about the origin, then by (5.21) one obtains

$$\widehat{\phi}(\xi) = \widehat{\phi}(-\xi), \quad \text{for all } \xi \in \mathbb{R}. \tag{5.23}$$

Therefore,

$$\widehat{\psi}_1(\xi) = \widehat{b}_1(\xi/2)\widehat{\phi}(\xi/2) = e^{-i\xi}\widehat{b}_1(-\xi/2)\widehat{\phi}(-\xi/2) = e^{-i\xi}\widehat{\psi}_1(-\xi),$$

which, by (5.21), means that ψ_1 is symmetric about $\frac{1}{2}$. Similarly by the definition of $\widehat{b}_2$, one obtains

$$\widehat{b}_2(\xi) = \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) = e^{-i\xi}\big(\mathcal{A}(-\xi) + e^{i\xi}\mathcal{A}(\xi)\big) = e^{-i\xi}\widehat{b}_2(-\xi).$$

Applying (5.23) and the definition of $\widehat{\psi}_2$, one obtains,

$$\widehat{\psi}_2(\xi) = \widehat{b}_2(\xi/2)\widehat{\phi}(\xi/2) = e^{-i\frac{\xi}{2}}\widehat{b}_2(-\xi/2)\widehat{\phi}(-\xi/2) = e^{-i\frac{\xi}{2}}\widehat{\psi}_2(-\xi),$$

which, by (5.21), means that ψ_2 is symmetric about $\frac{1}{4}$. Similarly, we can show that ψ_3 is antisymmetric about $\frac{1}{4}$. $\square$

The approximation order provided by a tight frame $X(\Psi)$ can be characterized by the approximation order of the corresponding operator $\mathcal{Q}_n$ (see [17]), which is defined in (1.10). We have shown in Section 3.3 that, for the operator $\mathcal{P}_n$ defined in (1.9), we have $\mathcal{Q}_n f = \mathcal{P}_n f$, for $f \in L_2(\mathbb{R})$, provided that Ψ is derived from the UEP and the underlying MRA is generated by the same ϕ as that defines $\mathcal{P}_n$. Therefore, by Theorem 3.10, if we start from the pseudo-spline of type II with order (m, l) in Construction 5.6, the tight frame system $X(\Psi)$ provides approximation order $2l + 2$.

In the end, we give one example of (anti)symmetric tight framelets constructed from Construction 5.6 using pseudo-splines of type II with order $(3, 1)$.

Example 5.7. Let $\widehat{a}$ to be the mask of pseudo-spline of type II with order $(3, 1)$ i.e.

$$\widehat{a}(\xi) = \cos^6(\xi/2)\big(1 + 3\sin^2(\xi/2)\big).$$

We define

$$\widehat{b_1}(\xi) := e^{-i\xi}\overline{\widehat{a}(\xi+\pi)} = e^{-i\xi}\sin^6(\xi/2)\big(1 + 3\cos^2(\xi/2)\big),$$

$$\widehat{b_2}(\xi) := \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) \quad \text{and} \quad \widehat{b_3}(\xi) := e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi),$$

where

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\Big(&0.00123930398199e^{-4i\xi} + 0.00139868605052e^{-2i\xi} - 0.22813823298962 \\ &+ 0.44712319189971e^{2i\xi} - 0.22162294894260e^{4i\xi}\Big).\end{aligned}$$

The graphs of Ψ are given by (b)-(d) in Figure 5.1. The tight frame system has approximation order 4.

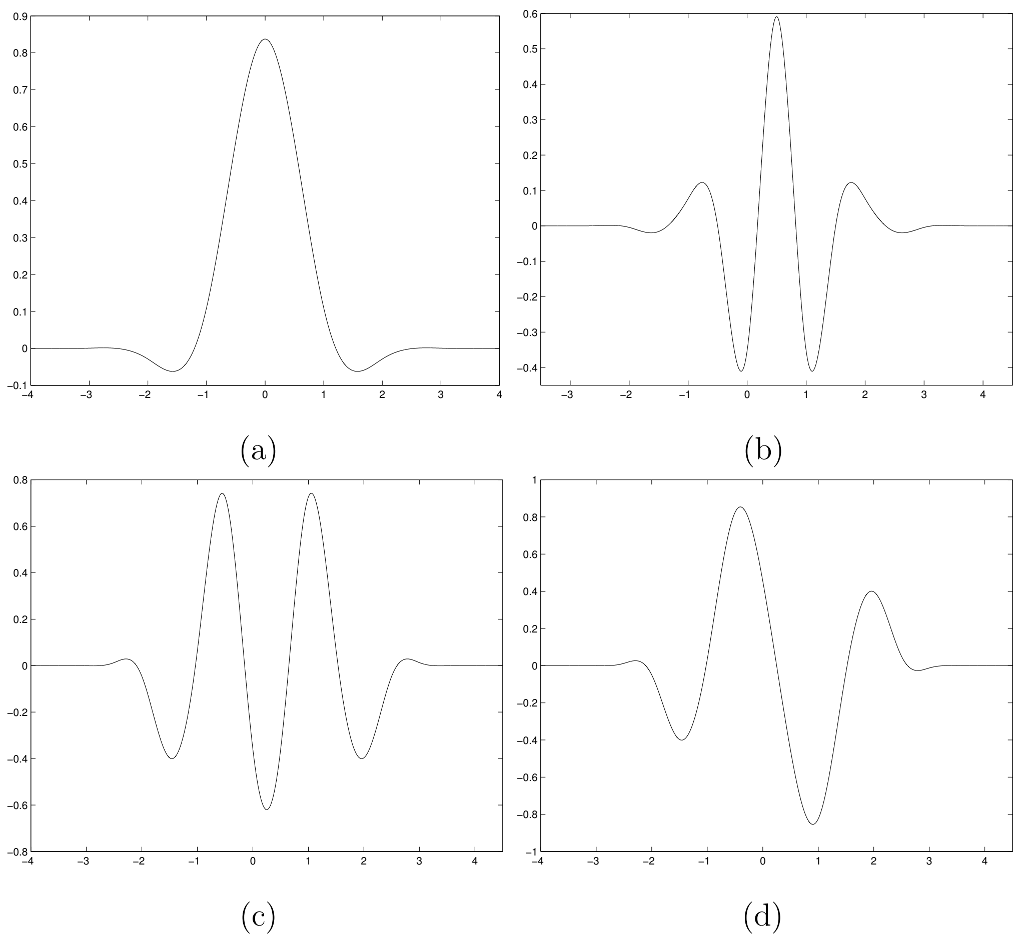

Figure 5.1: (a) is the pseudo-spline of type II with order $(3,1)$ and (b)-(d) are the corresponding (anti)symmetric tight framelets.

Chapter 6

Compactly Supported Biorthogonal Wavelets

In this chapter, we will construct pairs of smooth compactly supported biorthogonal Riesz wavelets using pseudo-splines. As we will see in a minute that this can be done by constructing dual refinable functions from pseudo-splines, which satisfy any prescribed regularity. Recall that two wavelet systems $X(\psi)$ and $X(\psi^d)$ are said to be *biorthogonal (Riesz) wavelet bases*, if they are Riesz wavelet systems and for all $f \in L_2(\mathbb{R})$,

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi^d_{j,k}.$$

Moreover, we call ψ and ψ^d *biorthogonal (Riesz) wavelets* and call ψ^d the *dual wavelet* of ψ. The difference between the Riesz wavelets constructed in this chapter and those constructed in the last chapter is that both the Riesz wavelets and its dual wavelets constructed here are compactly supported, while the dual Riesz wavelets of the short Riesz wavelets constructed in Section 5.1 are not compactly supported in general. However, by starting from the same pseudo-spline, both of the wavelets in each pair of compactly supported biorthogonal Riesz wavelets constructed in this section have longer supports than that of the underlying short Riesz wavelet.

We first give a general framework of the MRA-based construction of biorthogonal wavelets starting from a given refinable function. Constructions of biorthogonal wavelets have been extensively studied in the literature. The interested reader can find general discussions in [6, 7, 11, 15, 27, 30] and the references there.

Let $\phi \in L_2(\mathbb{R})$ be a compactly supported stable refinable function with finitely supported refinement mask a. The first step of the construction of a pair of compactly supported biorthogonal wavelets is to find a compactly supported stable refinable function $\phi^d \in L_2(\mathbb{R})$ with finitely supported refinement mask a^d satisfying

$$\langle \phi, \phi^d(\cdot - k)\rangle = \delta(k), \quad k \in \mathbb{Z}. \tag{6.1}$$

If a stable refinable function $\phi^d \in L_2(\mathbb{R})$ satisfies (6.1), we call it the *(biorthogonal) dual refinable function* of ϕ, or just *dual* of ϕ for simplicity. A necessary condition for ϕ and ϕ^d to satisfy (6.1) is

$$\widehat{a}\overline{\widehat{a}^d} + \widehat{a}(\cdot + \pi)\overline{\widehat{a}^d(\cdot + \pi)} = 1. \tag{6.2}$$

We call a^d a *dual refinement mask*, or just *dual mask* for convenience. Most of constructions starts with finding a^d to satisfy (6.2). Suppose we have the dual mask a^d in hand. We then need to check whether the corresponding refinable function ϕ^d is in $L_2(\mathbb{R})$ and stable, which can be done through the transition operator (see e.g. [7, 43, 53]). With the stable dual pair ϕ and ϕ^d and their refinement masks a and a^d satisfying (6.2), the dual pair of wavelets can be constructed (see e.g. [7] and [15]) as

$$\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi), \quad \text{and} \quad \widehat{\psi}^d(2\xi) = \widehat{b}^d(\xi)\widehat{\phi}^d(\xi), \tag{6.3}$$

where

$$\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}^d(\xi + \pi)} \quad \text{and} \quad \widehat{b}^d(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi + \pi)}. \tag{6.4}$$

Then the corresponding wavelet systems $X(\psi)$ and $X(\psi^d)$ form biorthogonal Riesz wavelet bases for $L_2(\mathbb{R})$ (see e.g. [6, 7, 30]). Since the mask a is assumed through out this chapter to be finitely supported, the wavelet mask b^d is also finitely supported. Therefore, ψ^d can be written as a finite linear combination of shifts of ϕ^d, which means that ψ^d has the same regularity as ϕ^d.

As we see from this framework, the key step in construction is to design a desirable pair of stable refinable functions satisfying (6.1). The existence of smooth dual refinable function for pseudo-splines is guaranteed by the linear independence of the shifts of them (see e.g. [41]). We shall talk about this in details in the following section. In the rest of this chapter, we shall focus on the constructions of dual refinable functions ϕ^d from pseudo-splines with prescribed regularity.

6.1 Duals of Pseudo-splines

In this section, we construct biorthogonal dual refinable functions from pseudo-splines, which can satisfy arbitrarily high order of regularity. The regularity here is the same as defined in Chapter 3.

We first give the existence of dual refinable functions with the prescribed regularity which immediately follows from the result of [41].

Theorem 6.1. *[41]. Let $\phi \in L_2(\mathbb{R})$ be compactly supported refinable function whose shifts are linearly independent. Then, for an arbitrary $\alpha > 0$, there exists a compactly supported refinable function $\phi^d \in L_2(\mathbb{R})$ with regularity α, such that ϕ^d is the biorthogonal dual refinable function of ϕ.*

Applying this theorem together with the fact that the shifts of pseudo-splines are linearly independent, we have:

Corollary 6.2. *Let ϕ be a pseudo-spline. Then, for an arbitrary $\alpha > 0$, there exists a compactly supported refinable function $\phi^d \in L_2(\mathbb{R})$ with regularity α, such that ϕ^d is the biorthogonal dual refinable function of ϕ.*

Remark 6.3.

1. The original theorem of [41] is stated in a different way. The compactly supported refinable function ϕ is assumed in [41] to be stable and have a minimal support. (A stable refinable function ϕ having a minimal support means, according to [41], that its symbol does not have symmetric zeros on $\mathbb{C} \setminus \{0\}$.) This is equivalent to that ϕ has linearly independent shifts by Lemma 4.2 (see also [38]).

2. In the approach taken by [41], for a given compactly supported refinable function $\phi \in L_2(\mathbb{R})$ with linearly independent shifts, the existence of a compactly supported dual refinable function satisfying any desired regularity is reduced to the existence of a compactly supported dual refinable function in $L_2(\mathbb{R})$. The proof of existence of a compactly supported dual refinable function in $L_2(\mathbb{R})$ for a given ϕ starts with a finitely supported dual mask of some refinable distribution, which is derived by solving (6.6) numerically and may not even be pre-stable. Then use this mask and another sequence obtained by truncating the standard infinite dual mask of a to derive a finitely supported dual mask of a whose corresponding refinable function is in $L_2(\mathbb{R})$ and stable. To obtain a dual refinable function with higher regularity, it repeats the above processing by constructing an L_2 dual of function $B_m * \phi$ instead of ϕ. To see this (see also [41]), let us consider $B_m * \phi$, with any given $m \geq 1$, where B_m is B-spline of order m whose Fourier transform is

$$\widehat{B_m} := \left(\frac{1 - e^{-i\xi}}{i\xi} \right)^m.$$

It can be easily verified that $B_m * \phi$ has linearly independent shifts. If there is a compactly supported dual refinable function $g \in L_2(\mathbb{R})$ of $B_m * \phi$, then $\phi^d := B_m(-\cdot) * g$ is a compactly supported dual of ϕ with regularity at least

$m-1-\varepsilon$. Indeed, since for compactly supported functions $\phi, \phi^d \in L_2(\mathbb{R})$,

$$\langle \phi, \phi^d(\cdot - k)\rangle = \delta(k), \quad k \in \mathbb{Z},$$

is equivalent to

$$[\widehat{\phi}, \widehat{\phi}^d] = 1,$$

(see e.g. [7] and [15]), we have

$$[\widehat{\phi}, \ \overline{\widehat{B_m}\widehat{g}}] = [\widehat{B_m}\widehat{\phi}, \widehat{g}] = 1.$$

Next, we explore a constructive way to get duals of pseudo-splines with prescribed regularities. For this, we first note that if the pseudo-spline of type II with order (m,l) has a compactly supported dual refinable function with regularity α, then we can obtain a compactly supported refinable function with regularity at least α that is dual to the pseudo-spline of type I with the same order. Indeed, for the pseudo-spline ${}_2\phi_{(m,l)}$ of type II with order (m,l), let ${}_2\phi^d \in L_2(\mathbb{R})$ be its compactly supported dual refinable function with regularity α. Since ${}_2\widehat{\phi}_{(m,l)} = |{}_1\widehat{\phi}_{(m,l)}|^2 = {}_1\widehat{\phi}_{(m,l)} \cdot {}_1\overline{\widehat{\phi}}_{(m,l)}$, we have

$$1 = [{}_2\widehat{\phi}_{(m,l)}, {}_2\widehat{\phi}^d] = [{}_1\widehat{\phi}_{(m,l)} \cdot {}_1\overline{\widehat{\phi}}_{(m,l)} \ , {}_2\widehat{\phi}^d] = [{}_1\widehat{\phi}_{(m,l)} \ , {}_1\widehat{\phi}_{(m,l)} \cdot {}_2\widehat{\phi}^d].$$

Therefore,

$$ {}_1\widehat{\phi}^d := {}_1\widehat{\phi}_{(m,l)} \cdot {}_2\widehat{\phi}^d \tag{6.5}$$

is a compactly supported dual refinable function with the regularity at least α by the fact that ${}_1\widehat{\phi}_{(m,l)} \in L_\infty(\mathbb{R})$. Hence, we only need to construct dual refinable functions of pseudo-splines of type II. In the rest of this section, we focus on discussions of dual refinable functions of pseudo-splines of type II with any prescribed regularity.

Construction of compactly supported dual refinable function ϕ^d always starts from constructing a dual mask a^d from a such that (6.2) is satisfied. This can be done whenever the symbol $\tilde{a}(z)$ does not have symmetric zeros on $\mathbb{C}\setminus\{0\}$. In fact, it is well known that in this case (see e.g. [7], [15] and [31]) one can always find $\tilde{a}^d(z)$ such that

$$\tilde{a}(z)\tilde{a}^d(z^{-1}) + \tilde{a}(-z)\tilde{a}^d(-z^{-1}) = 1, \quad z \in \mathbb{C}\setminus\{0\}. \tag{6.6}$$

Indeed, let

$$\tilde{a}_e(z^2) := \sum_{j\in\mathbb{Z}} a(2j)z^{2j} \quad \text{and} \quad \tilde{a}_o(z^2) := \sum_{j\in\mathbb{Z}} a(2j+1)z^{2j}.$$

Then,

$$\tilde{a}(z) = \tilde{a}_e(z^2) + z\tilde{a}_o(z^2) \quad \text{and} \quad \tilde{a}(-z) = \tilde{a}_e(z^2) - z\tilde{a}_o(z^2). \tag{6.7}$$

Since $\tilde{a}(z)$ does not have symmetric zeros on $\mathbb{C}\setminus\{0\}$, $\tilde{a}_e(z^2)$ and $\tilde{a}_o(z^2)$ do not have common zeros on $\mathbb{C}\setminus\{0\}$ by (6.7). Then the Hilbert's Nullstellensatz assures the existence of Laurent polynomials $\tilde{q}_e$ and $\tilde{q}_o$ such that

$$\tilde{a}_e(z^2)\tilde{q}_e(z^2) + \tilde{a}_o(z^2)\tilde{q}_o(z^2) = \frac{1}{2}z^{2k}, \quad \text{for all } z \in \mathbb{C}\setminus\{0\} \text{ and } k \in \mathbb{N}. \tag{6.8}$$

Let

$$\tilde{q}(z) := \tilde{q}_e(z^2) + z^{-1}\tilde{q}_o(z^2),$$

and define

$$\tilde{a}^d(z) := z^{2k}\tilde{q}(z^{-1}).$$

Then, $\tilde{a}$ and $\tilde{a}^d$ satisfy (6.6) by applying (6.7) and (6.8). Let a^d be the coefficients of $\tilde{a}^d(z)$. We conclude that $\widehat{a}$ and $\widehat{a}^d$ satisfy (6.2).

The solutions to (6.6) can be obtained by solving a polynomial equation utilizing **Maple** and **Singular** [26], which is an Ad-hoc construction, although sometimes it can be very efficient in both univariate and multivariate constructions (see e.g. [49]). The more efficient and systematic way of solving equation (6.6) is the method called *construction by cosets* (CBC), which was suggested in [11] and [27]. The method starts with a dual mask of a given refinement mask, then lifts the dual mask to a new dual mask whose underlying refinable function satisfies a desired order of the Stang-Fix condition. It should also be pointed out that the CBC algorithm gives the minimal support of the dual refinable functions for a given order of the Strang-Fix condition. All approaches of solving equation (6.6) normally derive dual refinable functions that satisfy given order of Strang-Fix condition. The regularity has to be checked one by one numerically using methods given in [9, 15, 28, 34, 52], although the regularity of a refinable function seems to increase as the order of Strang-Fix condition increases by numerical tests. Furthermore, since (6.6) is only a necessary condition for the underlying refinable functions ϕ and ϕ^d to be a dual pair for any given solution of equation (6.6), one needs to further check the stability of ϕ^d, which can also be done numerically by methods given in [15] and [43].

Our method for pseudo-splines is similar to the both methods above in the aspect that we also start with a dual refinement mask satisfying very mild conditions, then create new dual masks from it. The difference is that we obtain new dual masks from this initial mask, whose underlying refinable functions are stable and have prescribed regularities. Since the regularity of a compactly supported refinable function implies its order of the Strang-Fix condition (see e.g. [8, 44, 48]), and since once the prescribed regularity is given, the method gives a dual with the given regularity by choosing a proper parameter, our approach gains more than what the above methods may offer to pseudo-splines.

We start from an arbitrary pseudo-spline ϕ of type II with order (m, l), $m \geq 2$, $0 \leq l \leq m-1$, whose refinement mask is a. The first step is to find an initial finitely supported dual mask b. As we will see that for the case $m = 1$, the construction and the regularity analysis have already been considered in [7] (also see [15]).

Condition 6.4. Let b be a finitely supported mask satisfying:

1. b is a (real-valued) dual mask of a, i.e.

$$\widehat{a}(\xi)\overline{\widehat{b}(\xi)} + \widehat{a}(\xi+\pi)\overline{\widehat{b}(\xi+\pi)} = 1;$$

2. $\widehat{b}$ is real-valued and nonnegative;

3. The refinable distribution ϑ, corresponding to the refinement mask b, is pre-stable.

Remark 6.5. Note that we did not require ϑ to be a function, and just require that it is pre-stable. Actually, by Corollary 6.2, there always exists a mask b such that ϑ is a compactly supported stable refinable function in $L_2(\mathbb{R})$, which is a much more strong condition than part (3) above. For a given refinement mask a, it is not difficult to find such an initial dual mask b by CBC method of [11] and [27]. Once we have this b, the prescribed regularity dual refinable function can be built up.

The idea here is to use the mask $\widehat{c} := \widehat{a}\widehat{b}$. Let η be the corresponding refinable distribution of c. We will show that c and η satisfy the following properties:

Proposition 6.6. *Let ϕ be a pseudo-spline of type II with mask a and ϑ be the refinable distribution corresponding to the mask b, which satisfies all the conditions in Condition 6.4. Let $\widehat{c} = \widehat{a}\widehat{b}$ and η be the corresponding refinable distribution. Then:*

1. *$\widehat{c}$ is real-valued and nonnegative;*

2. *η belongs to $L_2(\mathbb{R})$;*

3. *η is stable.*

Proof. Part 1 is immediate by the fact that both $\widehat{a}$ and $\widehat{b}$ are real-valued and nonnegative.

Part 2 can be established by using Lemma 6.2.1 of [15]. Indeed, since the trigonometric polynomial $\widehat{c}$ is nonnegative, symmetric, i.e. $\widehat{c}(\xi) = \widehat{c}(-\xi)$, and $\widehat{c}(\xi)+\widehat{c}(\xi+\pi) = 1$, there exists (by Fejér-Riesz Lemma) a trigonometric polynomial

$\widehat{h}$ such that $|\widehat{h}|^2 = \widehat{c}$ and $|\widehat{h}(\xi)|^2 + |\widehat{h}(\xi+\pi)|^2 = 1$. Let f be the corresponding refinable distribution to mask h. Lemma 6.2.1 of [15] gives that $\widehat{f} \in L_2(\mathbb{R})$. Since $|\widehat{f}|^2 = \widehat{\eta}$, we conclude that $\widehat{\eta} \in L_1(\mathbb{R})$. Hence, η is compactly supported and continuous, which gives that $\eta \in L_2(\mathbb{R})$.

For part 3, since η is compactly supported and belongs to $L_2(\mathbb{R})$, we only need to show that η is pre-stable by checking whether $\widehat{\eta}$ has 2π-periodic zeros or not (see e.g. [35]). We first prove that the set of all zeros of $\widehat{\phi}$ is $\{2\pi p\}_{p\in\mathbb{Z}\setminus\{0\}}$. Note that $\widehat{\phi}$ can be written as $\widehat{\phi} = \widehat{B}_{2m}\widehat{g}$ where g is a refinable distribution with refinement mask d defined by

$$\widehat{d}(\xi) := \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2).$$

Applying the following identity of Lemma 2.2 and letting $y = \sin^2(\xi/2)$,

$$\sum_{j=0}^{l} \binom{m+l}{j} y^j(1-y)^{l-j} = \sum_{j=0}^{l} \binom{m-1+j}{j} y^j, \quad y \in \mathbb{R}, \tag{6.9}$$

the mask $\widehat{d}$ can be rewritten as

$$\widehat{d}(\xi) = \sum_{j=0}^{l} \binom{m-1+j}{j} \sin^{2j}(\xi/2).$$

Then it is obvious that $\widehat{d} \geq 1$ on $\mathbb{R}$, which implies that $\widehat{g} > 0$ on $\mathbb{R}$. Therefore, the set of all zeros of $\widehat{\phi}$ is the same as that of $\widehat{B}_{2m}$ which is exactly $\{2\pi p\}_{p\in\mathbb{Z}\setminus\{0\}}$.

Now we shall prove the pre-stability of η by contradiction. Suppose that ξ_0 is a 2π-periodic zero of $\widehat{\eta}$, i.e.

$$\widehat{\eta}(\xi_0 + 2\pi k) = \widehat{\phi}(\xi_0 + 2\pi k)\widehat{\vartheta}(\xi_0 + 2\pi k) = 0,$$

for all $k \in \mathbb{Z}$. Since by assumption, $\widehat{\vartheta}$ does not have 2π-periodic zeros, there must be some $k_0 \in \mathbb{Z}$, such that $\widehat{\phi}(\xi_0 + 2\pi k_0) = 0$. Since the zero set of $\widehat{\phi}$ is $\{2\pi p\}_{p\in\mathbb{Z}\setminus\{0\}}$, there exists $p_0 \in \mathbb{Z}\setminus\{0\}$ such that $\xi_0 + 2\pi k_0 = 2\pi p_0$, i.e. $\xi_0 = 2\pi(p_0 - k_0) =: 2\pi m_0$.

This gives that $\widehat{\eta}(\xi_0+2\pi k) = \widehat{\eta}(2\pi m_0+2\pi k) = 0$ for all $k \in \mathbb{Z}$. In particular, when $k = -m_0$, we have $\widehat{\eta}(0) = 0$. Since $\widehat{\phi}(0) = 1$, we must have $\widehat{\vartheta}(0) = 0$. However, since $\widehat{b}(0) = 1$, we should have that $\widehat{\vartheta}(0) = 1$. This is a contradiction. □

Having the mask c in hand, we first note by the construction of c and the fact that b is a dual mask of a, we have

$$\widehat{c}(\xi) + \widehat{c}(\xi + \pi) = 1.$$

Thus

$$\Big(\widehat{c} + \widehat{c}(\cdot + \pi)\Big)^{2n-1} = 1, \quad \text{for } n \geq 2. \tag{6.10}$$

The first n terms of the binomial expansion in (6.10) is

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{2n-1-j} \widehat{c}^{j}(\cdot + \pi) = \widehat{c}^{n} \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j} \widehat{c}^{j}(\cdot + \pi). \tag{6.11}$$

Since $\widehat{c} = \widehat{a}\widehat{b}$, we can factorize one $\widehat{a}$ out from the right hand side of (6.11) and the rest is denoted as $\widehat{a}^d$. As we shall see in a moment that the mask a^d is indeed a dual mask of a and the corresponding refinable function ϕ^d is indeed a dual of ϕ. The detailed construction is given as the followings.

Construction 6.7. Let ϕ be pseudo-spline of type II with order (m, l) and a be its refinement mask. Let b be the initial dual mask of a satisfying all the conditions in Condition 6.4, and $\widehat{c} = \widehat{a}\widehat{b}$. Then define mask a^d as

$$\widehat{a}^d := \widehat{b} \cdot \widehat{c}^{n-1} \cdot \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j} \Big(1 - \widehat{c}\Big)^{j}. \tag{6.12}$$

The corresponding refinable function is defined as

$$\widehat{\phi}^d(\xi) := \prod_{j=1}^{\infty} \widehat{a}^d(2^{-j}\xi).$$

Remark 6.8. The idea here is not new. Similar idea as given in the above construction can also be found in [29, 54, 56, 57]. Furthermore, this idea was used in [31] to construct multivariate biorthogonal wavelets via the multivariate interpolatory refinable functions, which also leads to the dual refinable functions of box splines with arbitrarily high regularity. The interested reader may consult these papers for details. Here, we not only give a construction, but also give a more precise regularity analysis for the construction. It is also worth to point out that one can choose the power $2n$ instead of $2n-1$ in (6.10). The argument presented here still works after a proper adjustment of the last term in the summation of the definition of $\widehat{a}^d$(see e.g. [31]). Finally, we note that all the dual refinable functions obtained by Construction 6.7 are symmetric, which is desirable in many applications.

To ensure that the corresponding refinable functions ϕ^d is indeed a dual of ϕ, we need to verify that (see e.g. [7] or [53]): (1), a^d is a dual mask of a, i.e. a and a^d satisfy (6.2); (2), ϕ^d is stable. For the first condition, we note that the first n terms of the expansion of (6.10) is exactly $\widehat{a}\overline{\widehat{a}^d}$ and the last n terms of the expansion of (6.10) is exactly $\widehat{a}(\cdot+\pi)\overline{\widehat{a}^d(\cdot+\pi)}$ by applying the identity $\widehat{c}(\cdot+\pi)=1-\widehat{c}$. Thus, the first condition follows from identity (6.10). For the second condition, since ϕ^d is compactly supported, the stability of ϕ^d will follow from that: (1), ϕ^d is pre-stable; (2), $\phi^d\in L_2(\mathbb{R})$. We will prove the pre-stability of ϕ^d in Proposition 6.10 and $\phi^d\in L_2(\mathbb{R})$ in Theorem 6.11. In fact, Theorem 6.11 says more than $\phi^d\in L_2(\mathbb{R})$. It shows that the regularity exponent of ϕ^d increases as we choose larger n in Construction 6.7.

The proof of the following proposition employs the following lemma of [31].

Lemma 6.9. *[31]. Let ϕ_1 and ϕ_2 be two compactly supported refinable functions in $L_2(\mathbb{R})$ with refinement masks a_1 and a_2. Suppose the set of all zeros of $\widehat{a}_1$ contains that of the mask $\widehat{a}_2$. If ϕ_1 is pre-stable, then ϕ_2 is pre-stable.*

Proposition 6.10. *Let ϕ^d be the compactly supported refinable distribution with refinement mask a^d given in* (6.12)*. Then ϕ^d is pre-stable.*

Proof. To show the the pre-stability of ϕ^d, we prove that the set of all zeros of $\widehat{a}^d$ coincides with that of $\widehat{c}$. With this, the pre-stability of ϕ^d follows from the pre-stability of η by applying Lemma 6.9. In fact, since for $\xi \in \mathbb{R}$

$$\widehat{c}(\xi) \geq 0, \quad \text{and} \quad \widehat{c}(\xi) + \widehat{c}(\xi + \pi) = 1,$$

one obtains that $0 \leq \widehat{c} \leq 1$. Applying (6.9) with $m = n$, $l = n-1$, $y = 1 - \widehat{c}$ and by the fact that $\widehat{c} \leq 1$, one obtains

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j}(1-\widehat{c})^j = \sum_{j=0}^{n-1} \binom{n-1+j}{j} (1-\widehat{c})^j \geq 1.$$

Since

$$\widehat{a}^d = \widehat{b} \cdot \widehat{c}^{n-1} \cdot \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j}(1-\widehat{c})^j,$$

we have that the set of all zeros of $\widehat{a}^d$ coincides with that of $\widehat{b}\widehat{c}^{n-1}$. Furthermore, since $\widehat{c} = \widehat{a}\widehat{b}$ and since

$$\widehat{b}\widehat{c}^{n-1} = \widehat{b}(\widehat{a}\widehat{b})^{n-1} = \widehat{a}^{n-1}\widehat{b}^n,$$

the set of all zeros of $\widehat{c}$ coincides with that of $\widehat{b}\widehat{c}^{n-1}$ and, hence, coincides with that of $\widehat{a}^d$. $\square$

Now we shall analyze the regularity of ϕ^d by estimating the decay of $|\widehat{\phi}^d|$, and show that the regularity of ϕ^d increases as the parameter n in Construction 6.7 increases.

Let

$$\mathcal{L} := \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{n-1-j}(1-\widehat{c})^j. \tag{6.13}$$

Then,

$$\widehat{a}^d = \widehat{b}\widehat{c}^{n-1}\mathcal{L}.$$

This gives that

$$\widehat{\phi}^d(\xi) = \widehat{\vartheta}(\xi)\widehat{\eta}^{n-1}(\xi)\prod_{j=1}^{\infty}\mathcal{L}(2^{-j}\xi). \tag{6.14}$$

Since $|\widehat{\vartheta}|$ is uniformly bounded and since $\widehat{\eta} = \widehat{\vartheta}\widehat{\phi}$, we have

$$|\widehat{\vartheta}\widehat{\eta}^{n-1}| = |\widehat{\vartheta}^n\widehat{\phi}^{n-1}| \le C|\widehat{\phi}^{n-1}|.$$

Recall that the optimal decay of $|\widehat{\phi}|$ was given in Theorem 3.4, i.e.

$$|\widehat{\phi}(\xi)| \le C(1+|\xi|)^{-s},$$

where

$$s := 2m - \frac{\log P_{m,l}(\frac{3}{4})}{\log 2} \tag{6.15}$$

and

$$P_{m,l}(y) = \sum_{j=0}^{l}\binom{m+l}{j}y^j(1-y)^{l-j}. \tag{6.16}$$

Consequently we have

$$|\widehat{\vartheta}(\xi)\widehat{\eta}^{n-1}(\xi)| \le C(1+|\xi|)^{-s(n-1)}. \tag{6.17}$$

Since, by (6.9),

$$\mathcal{L} = \sum_{j=0}^{n-1}\binom{n-1+j}{j}(1-\widehat{c})^j,$$

and since $0 \le \widehat{c} \le 1$, one can see that $\mathcal{L}$ reaches its maximum value at $\widehat{c} = 0$ (note that $\widehat{c}(\pi) = 0$). Therefore

$$\max_{\xi\in[0,2\pi]}|\mathcal{L}(\xi)| = \binom{2n-1}{n}.$$

Then Lemma 7.1.1 of [15] gives that

$$\prod_{j=1}^{\infty}\mathcal{L}(2^{-j}\xi) \le C(1+|\xi|)^{\frac{\log\binom{2n-1}{n}}{\log 2}},$$

and hence, by (6.14), (6.17) and the above inequality, one obtains,

$$|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\gamma}, \tag{6.18}$$

where

$$\gamma := s(n-1) - \frac{\log \binom{2n-1}{n}}{\log 2}. \tag{6.19}$$

Hence $\phi^d \in C^{\gamma-1-\varepsilon}$.

We note that the estimate given here is not optimal. It leads to a lower bound of the regularity of ϕ^d. We remark that the optimal Sobolev regularity of a given refinable function can be obtained via its mask by applying transfer operator (see [15], [52] and references in there). Although the transfer operator approach is very efficient to compute the exact Sobolev regularity for each given refinable function, it cannot be used to analyze the regularity for a set of refinable functions obtained through a systematic construction.

In the following theorem we will show that for pseudo-splines of type II with order $m \geq 2$, the decay rate γ of $|\widehat{\phi}^d|$ increases as n increases. Moreover, an asymptotical analysis of the regularity of ϕ^d is provided.

Theorem 6.11. *Let ϕ^d be the compactly supported refinable functions with refinement mask a^d given in* (6.12)*. The decays of $\widehat{\phi}^d$ is given by* (6.18)*. Then:*

1. *The decay rate γ of $\widehat{\phi}^d$ given in* (6.19) *increases as n increases. Consequently, ϕ^d is continuous for all $n \geq 2$ and its regularity exponent increases as n increases, where $\phi^d \in C^{\gamma-1-\varepsilon}$ for all $\varepsilon > 0$. In particular, $\phi^d \in L_2(\mathbb{R})$ for all $n \geq 2$.*

2. *Asymptotically for large n with fixed m, the decay rate γ is μn, where $\mu = s-2$ with s defined in* (6.15)*. Consequently we have,*

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\mu n}, \quad \phi^d \in C^{\mu n},$$

   asymptotically for large n

Proof. For part 1, we first show that γ increases as n increases, which is equivalent to show that

$$M := sn - \frac{\log \binom{2n+1}{n+1}}{\log 2} - s(n-1) + \frac{\log \binom{2n-1}{n}}{\log 2} > 0.$$

Simplifying M, one obtains

$$\begin{aligned} M &= s - \frac{\log \frac{\binom{2n+1}{n+1}}{\binom{2n-1}{n}}}{\log 2} \\ &= s - \frac{\log \frac{4n+2}{n+1}}{\log 2} \\ &= s - 2 - \frac{\log \frac{n+\frac{1}{2}}{n+1}}{\log 2} \\ &> s - 2. \end{aligned}$$

Since the decay rate s decreases as l increases and increases as m increases (see Proposition 3.5), and $s > 2.678$ for $m = 2$, $l = 1$, we have that $s > 2.678$ for all $m \geq 2$ and $0 \leq l \leq m-1$. Hence, we have $M > s - 2 > 0$. Consequently, the regularity exponent $\gamma - 1 - \varepsilon$ of ϕ^d increases as n increases. Since for $n = 2$ we have that

$$\gamma = s - \frac{\log 3}{\log 2} > 2.678 - \frac{\log 3}{\log 2} > 1.09,$$

this proves that ϕ^d is continuous for all $n \geq 2$ and, hence, $\phi^d \in L_2(\mathbb{R})$ for all $n \geq 2$.

For part 2, we consider the asymptotic behavior of γ when n is large. Note that

$$\gamma = (n-1)s - \frac{\log \binom{2n-1}{n}}{\log 2} = n\left((1 - \frac{1}{n})s - \frac{\frac{1}{n}\log \binom{2n-1}{n}}{\log 2}\right).$$

By Stirling approximation, i.e. $n! \sim \sqrt{2\pi} e^{(n+\frac{1}{2})\log n - n}$ (see e.g. [24]), we have

$$\begin{aligned} \log n! &\sim \log(\sqrt{2\pi} e^{(n+\frac{1}{2})\log n - n}) \\ &\sim \log \sqrt{2\pi} + (n + \frac{1}{2}) \log n - n \\ &\sim \big(n \log n - n\big) \frac{\log \sqrt{2\pi} + (n + \frac{1}{2}) \log n - n}{n \log n - n} \\ &\sim \big(n \log n - n\big) \frac{(1 + \frac{1}{2n}) \log n - 1}{\log n - 1} \\ &\sim n \log n - n. \end{aligned} \tag{6.20}$$

Applying (6.20) one obtains,

$$\begin{aligned}\log\binom{2n-1}{n} &= \log(2n-1)! - \log n! - \log(n-1)! \\ &\sim (2n-1)\log(2n-1) - n\log n - (n-1)\log(n-1).\end{aligned}$$

Applying the above approximation to the estimate of β one obtains

$$\begin{aligned}\gamma &\sim n\left(s - \frac{\frac{1}{n}\big((2n-1)\log(2n-1) - n\log n - (n-1)\log(n-1)\big)}{\log 2}\right) \\ &\sim n\left(s - \frac{2\log(2n-1) - \log n - \log(n-1)}{\log 2}\right) \\ &\sim n\left(s - \frac{2\log 2}{\log 2}\right) = n(s-2).\end{aligned}$$

Thus we have shown that $\gamma \sim (s-2)n$, asymptotically for large n. Consequently, one obtains that for large n,

$$|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\mu n}, \quad \phi^d \in C^{\mu n},$$

with $\mu = s - 2$. $\square$

So far we have shown in Proposition 6.10 that ϕ^d is pre-stable and proved in part 1 of Theorem 6.11 that $\phi^d \in L_2(\mathbb{R})$. Furthermore, ϕ^d is compactly supported as one can easily see from the Construction 6.7. Therefore, we conclude that ϕ^d is stable. Having the stability of ϕ^d, together with a and a^d satisfying (6.2), Theorem 3.14 of [53] (also see [7]) leads to the conclusion that ϕ and ϕ^d is a pair of dual refinable functions, i.e.

$$\langle \phi, \phi^d(\cdot - k)\rangle = \delta(k).$$

Therefore the corresponding pair of biorthogonal Riesz wavelets ψ and ψ^d can be constructed by (6.3) and (6.4), and the systems $X(\psi)$ and $X(\psi^d)$ form a pair of biorthogonal Riesz wavelet bases for $L_2(\mathbb{R})$.

Remark 6.12. The pair of masks $\widehat{a}$, $\widehat{a}^d$ in Construction 6.7 can be viewed as one of many possible factorizations of the trigonometric polynomial

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{2n-1-j} \widehat{c}^j(\cdot + \pi)$$

given by (6.11). In fact, we can choose factorization $\widehat{h}$ and $\widehat{h}^d$ arbitrarily such that

$$\widehat{h}\widehat{h}^d = \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{2n-1-j} \widehat{c}^j(\cdot + \pi).$$

When the compactly supported refinable functions corresponding to the masks h and h^d are in $L_2(\mathbb{R})$ and pre-stable, a dual pair of compactly supported biorthogonal wavelet systems can be derived from them. For example, let $n' > 0$ and define

$$\widehat{h} := \widehat{c}^{n'} \quad \text{and} \quad \widehat{h}^d := \sum_{j=0}^{n-1} \binom{2n-1}{j} \widehat{c}^{2n-2-n'-j}(1-\widehat{c})^j, \ n' \geq 1.$$

As long as n and n' are chosen properly, one can get a desired dual pair of refinement masks for a dual pair of compactly supported refinable functions. In particular, let $\widehat{c} = \cos^2(\xi/2)$ be the mask of piecewise linear B-spline which is interpolatory. Then, the construction here coincides with the biorthogonal wavelet construction given in [7].

For the dual mask a^d given in Construction 6.7, we cannot have an explicit form of it in general, because we need to find mask b numerically first. For some special pseudo-splines, however, we do have an explicit form for all the dual masks constructed from Construction 6.7. In the next section we will give a detailed construction of dual refinable functions from pseudo-splines of type II with order $(m, m-1)$.

6.2 Duals of a Special Case

Let ϕ be pseudo-spline of type II with order $(m, m-1)$ with $m \geq 1$, i.e. an interpolatory refinable function, and let a be its refinement mask. Since ϕ is interpolatory, the mask $\widehat{a}$ satisfies $\widehat{a}+\widehat{a}(\cdot+\pi)=1$. Hence, $\widehat{b}$ in Condition 6.4 can be simply chosen to be 1, and the corresponding refinable distribution is $\widehat{\vartheta}=1$. Then all the conditions in Condition 6.4 are satisfied. Following the construction given by (6.12), one can obtain the dual mask $\widehat{a}^d$ as

$$\widehat{a}^d := \widehat{a}^{n-1}\sum_{j=0}^{n-1}\binom{2n-1}{j}\widehat{a}^{n-1-j}\Big(1-\widehat{a}\Big)^j. \tag{6.21}$$

The corresponding refinable function $\widehat{\phi}^d$ can be defined as

$$\widehat{\phi}^d(\xi) := \prod_{j=1}^{\infty}\widehat{a}^d(2^{-j}\xi).$$

Since $\widehat{b}=1$, we have $\widehat{c}=\widehat{b}\widehat{a}=\widehat{a}$, where c given in Proposition 6.6. Therefore, the trigonometric polynomial $\mathcal{L}$ defined in (6.13) can now be written as,

$$\mathcal{L}=\sum_{j=0}^{n-1}\binom{2n-1}{j}\widehat{a}^{n-1-j}\Big(1-\widehat{a}\Big)^j.$$

This gives that

$$\widehat{\phi}^d(\xi)=\widehat{\phi}^{n-1}(\xi)\prod_{j=1}^{\infty}\mathcal{L}(2^{-j}\xi).$$

Since $\widehat{a}$ satisfies $0\le\widehat{a}\le 1$ and since $\widehat{\vartheta}=1$, following a similar argument in Section 6.1 we have that

$$|\widehat{\phi}^d(\xi)|\le C(1+|\xi|)^{-\beta}, \tag{6.22}$$

where the decay rate β satisfies

$$\beta = s(n-1)-\frac{\log\binom{2n-1}{n}}{\log 2} \tag{6.23}$$

with $s'=2m-\frac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}$, and $P_{m,l}(y)$ defined in (6.16). Hence $\phi^d\in C^{\beta-1-\varepsilon}$.

The decay estimates for $|\widehat{\phi}^d|$ here are not accurate. However, for the simplest case when $m=1$, i.e. $\widehat{a}=\cos^2(\xi/2)$, we do have optimal decay estimate for $|\widehat{\phi}^d|$.

Indeed, in this case

$$\begin{aligned}\widehat{a}^d &= \cos^{2n-2}(\xi/2)\sum_{j=0}^{n-1}\binom{2n-1}{j}\cos^{2(n-1-j)}(\xi/2)\sin^{2j}(\xi/2)\\ &= \cos^{2n-2}(\xi/2)\sum_{j=0}^{n-1}\binom{2n-1}{j}\sin^{2j}(\xi/2)(1-\sin^2(\xi/2))^{(n-1-j)}.\end{aligned}$$

The optimal decay of $\widehat{\phi}^d$ is

$$|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\rho}, \tag{6.24}$$

where

$$\rho := 2(n-1) - \frac{\log P_{n,n-1}(\frac{3}{4})}{\log 2}. \tag{6.25}$$

The complete construction and analysis for this special case have already been given by [7] (see also [15]). In fact, by applying the approach in Remark 6.12 this leads to their construction of a pair of biorthogonal compactly supported symmetric wavelets with any prescribed regularity.

The following table gives the decay rates of $|\widehat{\phi}^d|$ in (6.23) with some choices of m and n.

β	n=2	n=3	n=4	n=5	n=6
$m=2$	1.0931	2.0342	2.9049	3.7350	4.5386
$m=3$	1.6871	3.2222	4.6870	6.1110	7.5086
$m=4$	2.2411	4.3282	6.3459	8.3230	10.2736

Table 6.1: In the above estimates of β, $|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\beta}$.

Next, we will give an asymptotical analysis of the decay of $\widehat{\phi}^d$ given in (6.22) in terms of its refinement mask $\widehat{a}^d$ given in (6.21). For $m=1$, the asymptotical analysis of decay of $\widehat{\phi}^d$ given in (6.24) can be done by following the analysis in [7] or [15], which leads to the optimal decay rate $0.4150\cdots$.

Proposition 6.13. *Let ϕ^d be the refinable function with the refinement mask a^dgiven in* (6.21)*. The decay of $\widehat{\phi}^d$ is given by* (6.22)*. Then:*

1. *For fixed $m \geq 2$ and asymptotically for large n, we have*

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\nu n} \quad \textit{and} \quad \phi^d \in C^{\nu n},$$

where $\nu = 2(m-1) - \frac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}$.

2. *For fixed $n \geq 2$ and asymptotically for large m, we have*

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\sigma m} \quad \textit{and} \quad \phi^d \in C^{\sigma m},$$

where $\sigma = \big(2 - \frac{\log 3}{\log 2}\big)(n-1)$.

Proof. Part 1 is immediate from part 2 of Theorem 6.11 by letting $s = 2m - \frac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}$.

For part 2, let m be asymptotically large and n be fixed. Then,

$$\begin{aligned} \beta &= (n-1)\left(2m - \frac{\log P_{m,m-1}(\frac{3}{4})}{\log 2}\right) - \frac{\log \binom{2n-1}{n}}{\log 2} \\ &\sim m(n-1)\left(2 - \frac{\frac{1}{m}\log P_{m,m-1}(\frac{3}{4})}{\log 2}\right). \end{aligned}$$

Recall that we have already shown in Theorem 3.6 (see also [15, 58, 40]) that

$$\frac{1}{m}P_{m,m-1}\big(\frac{3}{4}\big) \sim \log 3. \tag{6.26}$$

Applying (6.26) one obtains

$$\beta \sim m(n-1)\left(2 - \frac{\log 3}{\log 2}\right) =: \sigma m.$$

Thus we have shown that with fixed n,

$$|\widehat{\phi}^d(\xi)| \leq C(1+|\xi|)^{-\sigma m} \quad \text{and} \quad \phi^d \in C^{\sigma m},$$

with $\sigma = (n-1)\big(2 - \frac{\log 3}{\log 2}\big)$. □

	m=1	m=2	m=3	m=4	m=5
ν	0.4150	0.6781	1.2721	1.8251	2.3532

Table 6.2: In the above estimates of ν, $|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\nu n}$, asymptotically for large n.

	n=2	n=3	n=4	n=5	n=6
σ	0.4150	0.8301	1.2451	1.6601	2.0752

Table 6.3: In the above estimates of σ, $|\widehat{\phi}^d(\xi)| \le C(1+|\xi|)^{-\sigma m}$, asymptotically for large m.

Table 6.2 and 6.3 provide some numerical results for the asymptotic rates μ and σ given by Proposition 6.13.

We shall now give two examples of biorthogonal Riesz wavelets constructed in this section. In the first example, we start with pseudo-spline of type II with order $(2,1)$ and $n = 2$; in the second one, we start with pseudo-spline of type II with order $(3,2)$ and $n = 2$.

Example 6.14. We first choose $\widehat{a}$ to be the refinement mask of a pseudo-spline of type II with order $(2,1)$, i.e.

$$\widehat{a} = \cos^4(\xi/2)(1 + 2\sin^2(\xi/2)).$$

By Construction 6.7 with $n = 2$ we have that

$$\widehat{a}^d := \widehat{a}\big(3 - 2 \cdot \widehat{a}\big).$$

Define wavelet masks and wavelets as

$$\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}^d(\xi+\pi)} \quad \text{and} \quad \widehat{b}^d(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)};$$

$$\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi) \quad \text{and} \quad \widehat{\psi}^d(2\xi) = \widehat{b}^d(\xi)\widehat{\phi}^d(\xi),$$

where $\widehat{\phi}$ and $\widehat{\phi}^d$ are the refinable functions corresponding to the refinement masks $\widehat{a}$ and $\widehat{a}^d$. The systems $X(\psi)$ and $X(\psi^d)$ form a pair of biorthogonal wavelet bases for $L_2(\mathbb{R})$. The figures of ϕ, ϕ^d, ψ and ψ^d are given in Figure 6.1.

Example 6.15. We first choose $\widehat{a}$ to be the refinement mask of a pseudo-spline of type II with order $(3, 2)$, i.e.

$$\widehat{a} = \cos^6(\xi/2)(1 + 3\sin^2(\xi/2) + 6\sin^4(\xi/2)).$$

By Construction 6.7 with $n = 2$ we have that

$$\widehat{a}^d := \widehat{a}\big(3 - 2 \cdot \widehat{a}\big).$$

Define wavelet masks and wavelets as

$$\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}^d(\xi+\pi)} \quad \text{and} \quad \widehat{b}^d(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)};$$

$$\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi) \quad \text{and} \quad \widehat{\psi}^d(2\xi) = \widehat{b}^d(\xi)\widehat{\phi}^d(\xi),$$

where $\widehat{\phi}$ and $\widehat{\phi}^d$ are the refinable functions corresponding to the refinement masks $\widehat{a}$ and $\widehat{a}^d$. The systems $X(\psi)$ and $X(\psi^d)$ form a pair of biorthogonal wavelet bases for $L_2(\mathbb{R})$. The figures of ϕ, ϕ^d, ψ and ψ^d are given in Figure 6.2.

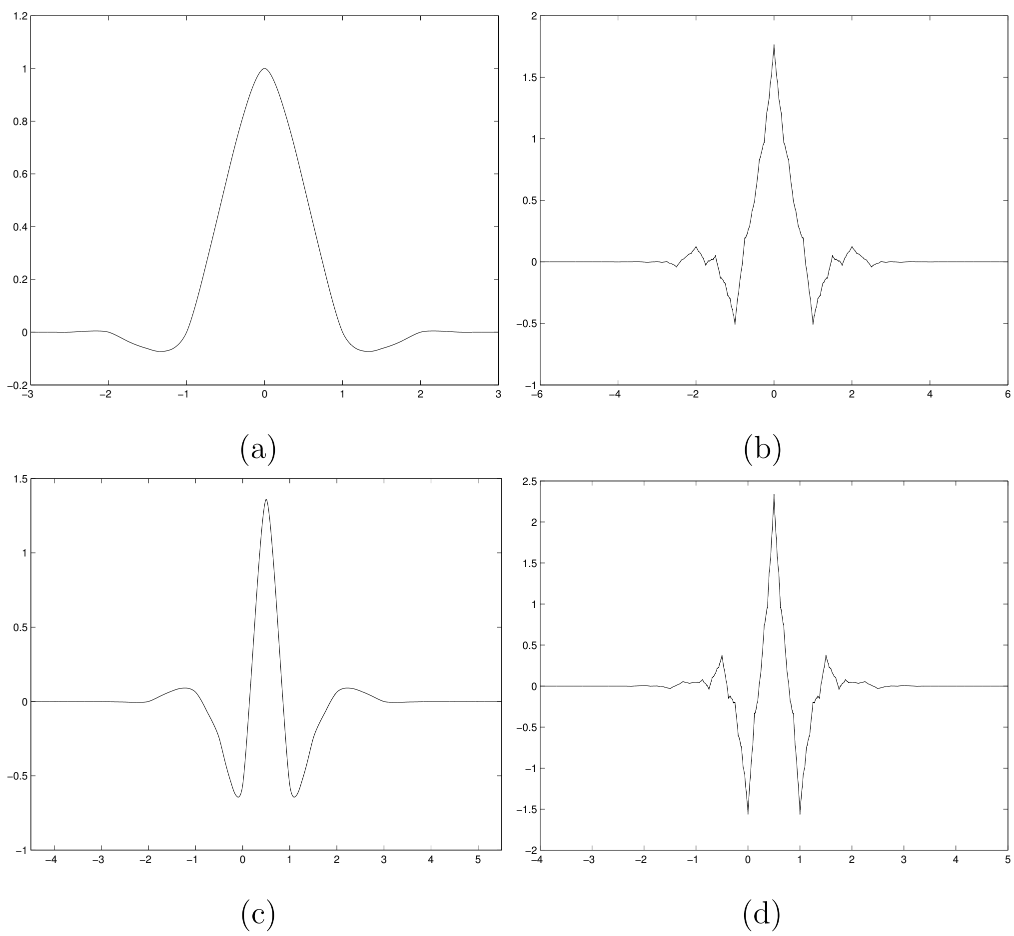

Figure 6.1: The figures of ϕ and ϕ^d in Example 6.14 are given in graphs (a) and (b). Figures of the corresponding Riesz wavelets ψ and ψ^d are given in (c) and (d).

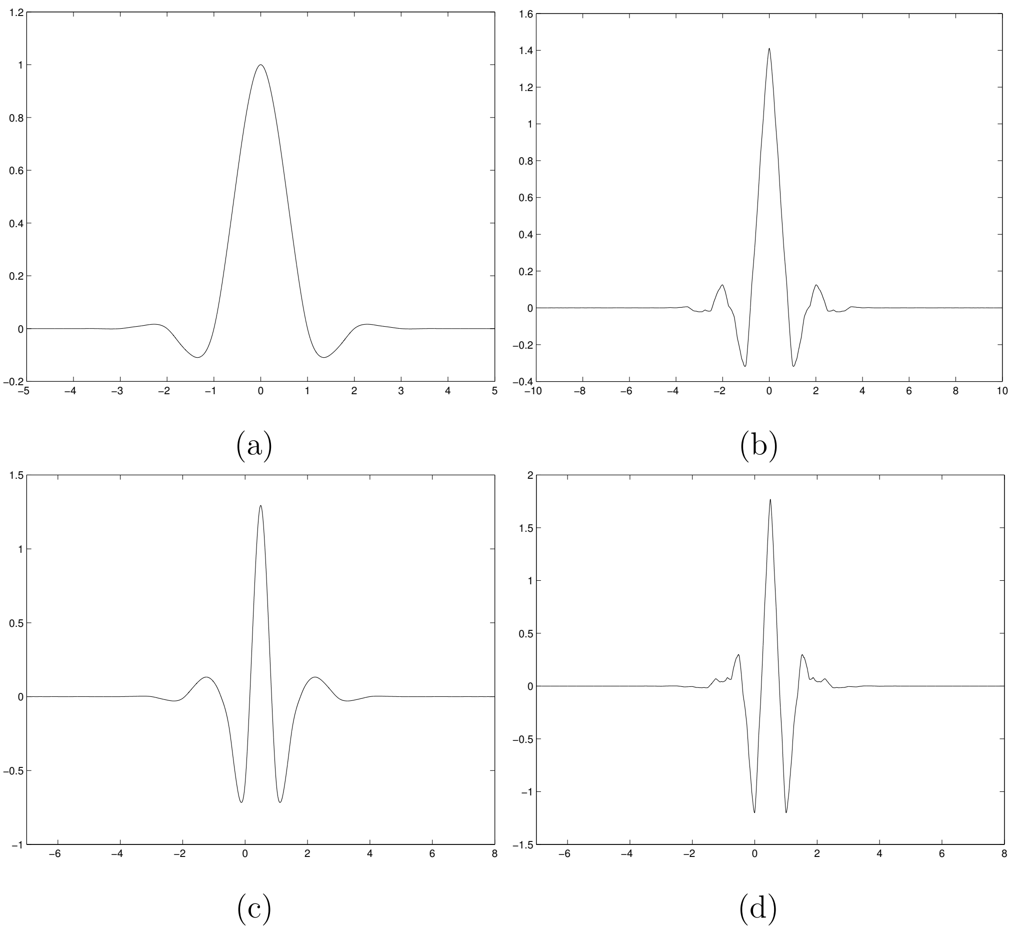

Figure 6.2: The figures of ϕ and ϕ^d in Example 6.15 are given in graphs (a) and (b). Figures of the corresponding Riesz wavelets ψ and ψ^d are given in (c) and (d).

Bibliography

[1] C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.

[2] J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 7th ed., Boston : McGraw-Hill Higher Education, 2004.

[3] C. de Boor, R. DeVore and A. Ron, *On the construction of multivariate (pre)wavelets*, Constr. Approx. **9** (1993), 123–166.

[4] A. Ben-Artzi and A. Ron, *On the integer translates of a compactly supported function: dual bases and linear projectors*, SIAM J. Math. Anal. **21** (1990), 1550–1562.

[5] A. Cohen and J. P. Conze, *Régularité des bases d'ondelettes et mesures ergodiques*, Rev. Math. Iberoamer **8** (1992), 351–365.

[6] A. Cohen and I. Daubechies, *A stability criterion for biorthogonal wavelet bases and their related subband coding scheme*, Duke Math. J. **68** (2) (1992), 313–335.

[7] A. Cohen, I. Daubechies and J. C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **45** (1992), 485–560.

[8] A. S. Cavaretta, W. Dahmen and C. A. Micchelli, *Stationary subdivision*, Memoir Amer. Math. Soc. 453, Providence 1991.

[9] A. Cohen, K. Grochenig and L. Villemoes, *Regularity of multivariate refinable functions*, Constr. Approx. **15 (2)** (1999), 241–255.

[10] C. K. Chui and Wenjie He, *Compactly supported tight frames associated with refinable functions*, Appl. Comput. Harmon. Anal. **8** (3) (2000), 293–319.

[11] Di-Rong Chen, B. Han and S. D. Riemenschneider, *Construction of multivariate biorthogonal wavelets with arbitrary vanishing moments*, Adv. Comput. Math. **13 (2)** (2000), 131–165.

[12] C. C. Chen and K. M. Koh, Principles and Techniques in Combinatorics, World Scientific, Singapore, 1992.

[13] A. Cohen, *"Biorthogonal Wavelets"* in Wavelets: A Tutorial in Theory and Applications, ed. by C. K. Chui, Academic Press, San Diego, 1992.

[14] C. K. Chui and J. Z. Wang, *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Math. Soc. **330** (1992), 903–915.

[15] I. Daubechies, *Ten Lectures on Wavelets*, in: CBMS Conf. Series in Appl. Math., vol. 61, SIAM, Philadelphia, 1992.

[16] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.

[17] I. Daubechies, B. Han, A. Ron, and Z. Shen, *Framelets: MRA-based constructions of wavelet frames*, Appl. Comput. Harmon. Anal. **14** (1) (2003), 1–46.

[18] W. Dahmen and C. A. Micchelli, *Translates of multivariate splines*, Linear Algebra Appl. **52** (1983), 217–234.

[19] W. Dahmen and C. A. Micchelli, *On the local linear independence of translates of a box spline*, Studia Math. **82** (1985), 243–262.

[20] B. Dong and Z. Shen, *Pseudo-splines, wavelets and framelets*, preprint (2004).

[21] B. Dong and Z. Shen, *Linear independence of pseudo-splines*, Proc. Amer. Math. Soc., to appear.

[22] B. Dong and Z. Shen, *Construction of biorthogonal wavelets from pseudo-splines*, preprint (2005).

[23] S. Dubuc, *Interpolation through an iterative scheme*, J. Math. Anal. Appl. **114** (1986), 185–204.

[24] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. New York: Wiley, 1968.

[25] S. Goh, Z. Lim and Z. Shen, *Symmetric and antisymmetric tight wavelet frames*, preprint (2004).

[26] G.-M. Greuel, G. Pfister and H. Schönemann, Singular version 1.2 User Manual. In Reports On Computer Algebra, number **21**. Centre for Computer Algebra, University of Kaiserslautern, June 1998.

[27] B. Han, *Analysis and construction of optimal multivariate biorthogonal wavelets with compact support*, SIAM J. Math. Anal. **31 (2)** (1999), 274–304.

[28] B. Han, *Computing the smoothness exponent of a symmetric multivariate refinable function*, SIAM J. Matrix Anal. Appl. **24 (3)** (2003), 693–714.

[29] B. Han, *On dual tight wavelet frames*, Appl. Comput. Harmon. Anal. **4** (1997) 380–413.

[30] B. Han and Z. Shen, *Wavelets with short support*, preprint (2003).

[31] H. Ji, S. D. Riemenschneider and Z. Shen, *Multivariate compactly supported fundamental refinable functions, duals and biorthogonal wavelets*, Stud. Appl. Math. **102** (1999), 173–204.

[32] R. Q. Jia, *Linear independence of translates of a box spline*, J. Approx. Theory **40** (1984), 158–160.

[33] R. Q. Jia, *Local linear independence of the translates of a box spline*, Constr. Approx. **1** (1985), 175–182.

[34] R. Q. Jia, *Characterization of smoothness of multivariate refinable functions in Sobolev spaces*, Trans. Amer. Math. Soc. **351 (10)** (1999), 4089–4112.

[35] R. Q. Jia and C. A. Micchelli, *Using the refinement equations for the construction of pre-wavelets. II. Powers of two*, Curves and Surfaces, 209–246, Academic Press, Boston, MA, 1991.

[36] R. Q. Jia and Z. Shen, *Multiresolution and wavelets*, Proc. Edinburgh Math. Soc. **37** (1994), 271–300.

[37] Q. T. Jiang and Z. Shen, *On existence and weak stability of matrix refinable functions*, Constr. Approx. **15** (1999), 337–353.

[38] R. Q. Jia and J. Z. Wang, *Stability and linear independence associated with wavelet decompositions*, Proc. Amer. Math. Soc., **117** (4) (1993), 1115–1124.

[39] A. Khodakovsky, P. Schröder and W. Sweldens, *Progressive geometry compression*, Proceedings of SIG-GRAPH, 2000.

[40] K. S. Lau and Q. Y. Sun, *Asymptotic regularity of Daubechies' scaling functions*, Proc. Amer. Math. Soc. **128** (2000) 1087–1095.

[41] P. G. Lemarié-Rieusset, *On the existence of compactly supported dual wavelets*, Appl. Comput. Harmon. Anal. **3** (1997), 117–118.

[42] P. G. Lemarié-Rieusset, *Fonctions d'échelle interpolantes, polynômes de Bernstein et ondelettes non stationnaires*, Rev. Mat. Iberoamericana **13** (1) (1997), 91–188.

[43] W. Lawton, S. L. Lee and Z. Shen, *Stability and orthonormality of multivariate refinable functions*, SIAM J. Math. Anal. **28** (1997), 999–1014.

[44] Y. Meyer, *Ondelettes et Opérateurs I: Ondelettes*, Hermann Éditeurs, 1990.

[45] G. Polya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis* **vol. II**, Springer-Verlag, Berlin, 1971.

[46] A. Ron, *Factorization theorems of univariate splines on regular grids*, Israel J. Math. **70** (1) (1990), 48–68.

[47] A. Ron, *A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution*, Constr. Approx. **5** (1989), 297–308.

[48] A. Ron, *Smooth refinable functions provide good approximation orders*, SIAM J. Math. Anal. **28** (1997), 731–748.

[49] S. D. Riemenschneider and Z. Shen, *Wavelets and pre-wavelets in low dimensions*, J. Approx. Theory **71** (1) (1992), 18–38.

[50] A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Canad. Math. J. **47** (1995), 1051–1094.

[51] A. Ron and Z. Shen, *Affine systems in* $L_2(\mathbb{R}^d)$*: the analysis of the analysis operator*, J. Funct. Anal. **148** (2) (1997), 408–447.

[52] A. Ron and Z. Shen, *The Sobolev regularity of refinable functions*, J. Approx. Theory **106** (2000), 185–225.

[53] Z. Shen, *Refinable function vectors*, SIAM J. Math. Anal. **29** (1998), 235–250.

[54] M. Salvatori and P. M. Soardi, *Multivariate compactly supported biorthogonal spline wavelets*, Ann. Mat. Pura Appl. (4) **181** (2) (2002), 161–179.

[55] I. Selesnick, *Smooth wavelet tight frames with zero moments*, Appl. Comput. Harmon. Anal. **10** (2) (2001) 163–181.

[56] X. L. Shi and Q. Y. Sun, *A class of M-dilation scaling functions with regularity growing proportionally to filter support width*, Proc. Amer. Math. Soc. **126** (1998) 3501–3506.

[57] R. Strichartz, *Construction of orthonormal wavelets, Wavelets: Mathematics and Applications*, Stud. Adv. Math., CRC, Boca Raton, FL, (1994) 23–50.

[58] H. Volker, *On the regularity of wavelets*, IEEE Trans. Information Theory. **38** (1992) 872–876.