Computer Search for Finite Saturated Pure Partial Planes of Certain Orders

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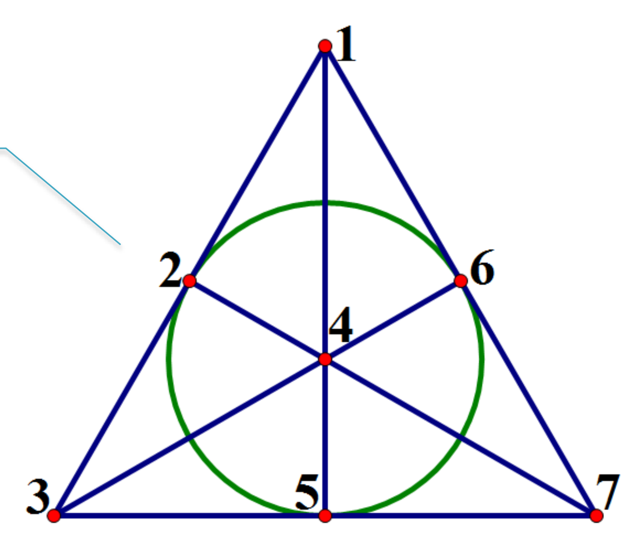
Project Background

The open problem of the existence of Finite Projective Planes of certain orders has interested mathematicians for decades.

Definition: A ***Finite Projective Plane (FPP) of order n*** is a collection of n^2+n+1 lines and n^2+n+1 points, such that:

- Every line contains n+1 points
- Every point is on n+1 lines
- Any two distinct lines intersect at exactly 1 point
- Any two distinct points lie together on exactly 1 line

"Lines": 6 line segments and 1 circle.
"Points": 7 intersections



L1={1,2,3}
L2={1,4,5}
L3={1,6,7}
L4={2,4,7}
L5={2,5,6}
L6={3,4,6}
L7={3,5,7}

Finite Projective Plane of order 2

Research Problem

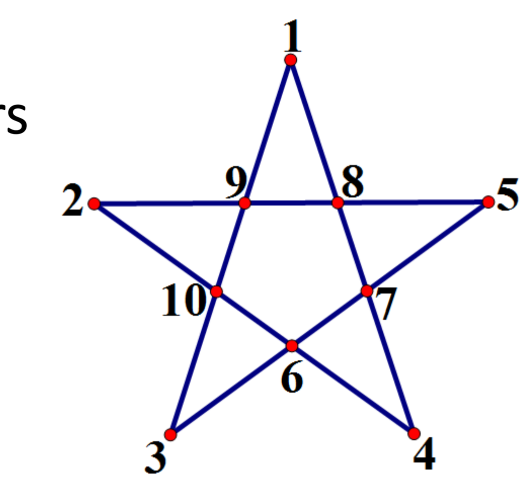
My Project focuses on a highly related mathematical object, which has not been studied before.

Definition: A ***Saturated Pure Partial Plane (SPPP) of order n and size m*** is a collection of m lines and n^2+n+1 points, such that:

- Every line contains n+1 points
- Any two distinct lines intersect at exactly 1 point
- No more lines can be added to this collection of lines such that the above two conditions hold.

The **GOAL** is to

- Find all SPPP for small orders (up to isomorphism)
- Study their properties
- Generalize to higher orders



L1={1,9,10,3}
L2={3,6,7,5}
L3={5,8,9,2}
L4={2,10,6,4}
L5={4,7,8,1}

Example of a SPPP of order 3, size 5

Technical Approach

The search space is **too large**!!

The number of possible lines is exponential in n, and the number of possible combinations of those lines is **exponential in exponential in n**. Ridiculously large!

Need to do **Depth First Search (DFS)** combined with clever ways to use **symmetry**.

Possible Starting Configuration (By Hands) → DFS (C/C++) with checks to reduce search space → One Valid SPPP → All Valid SPPPs → Isomorphism Testing → Geometric Representation & Generalization

Current Results & Observations

order	Size of FPP (if exists)	Largest size SPPP (not FPP)	Smallest size SPPP	Number of SPPPs
2	7	X	7	1
3	13	5	5	2
4	21	13	9	4
5	31	19	7	≥99

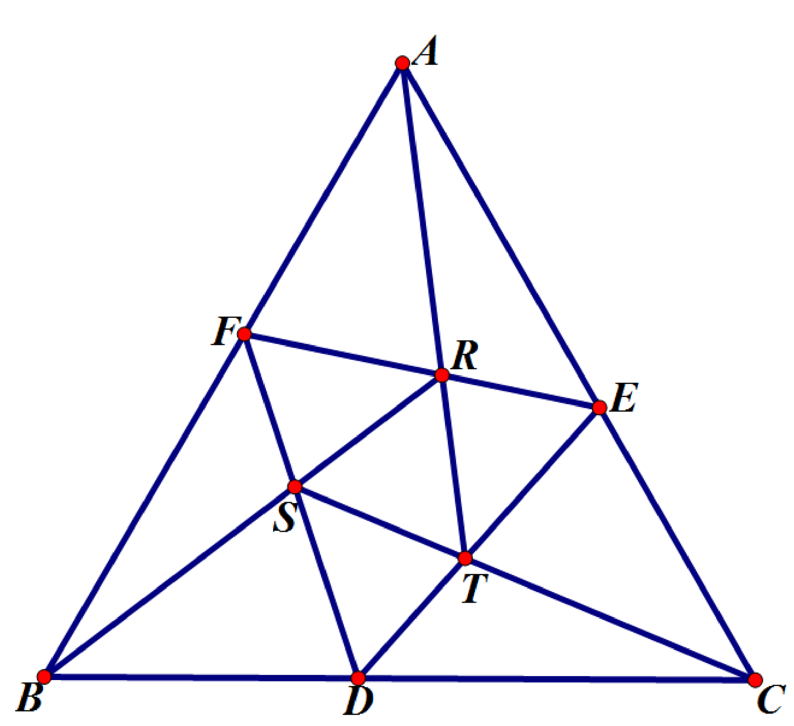
A FPP is always a SPPP, and is the largest size SPPP.

We have finished the search for cases of order **n≤4** and got partial results for n=5. Some observations:

- If n is odd, the smallest size SPPP has size n+2. (Proved)
- There is a huge gap between the size of the FPP and the size of the largest SPPP, which is not a FPP.

Properties about some SPPP

Here is an example of a SPPP of order n=4, size m=9.
Its **core structure**:



ABC, DEF, RST are all regular triangles. Then what is the **ratio AE/AF** if we'd like to keep the same ratio in each layer and keep on going inside?

The answer is $\sqrt[3]{2}$.
A nice number coming out of nowhere!!

Future Work

Technically, to **search for all SPPPs of some higher orders (n=5,6)**, we need to further exploit symmetry since the search space grows really fast.

Theoretically, there are many meaningful asymptotic questions to ask, as n tends to infinity, what is:

- the approximate number of SPPPs?
- the ratio of the size of the largest SPPPs over n^2+n+1?
- The ratio of the size of the largest SPPP that is not a FPP over n^2+n+1?

$n \to \infty$?