

PACE 2026 APPLICATION TESTS

Please read the following instructions before writing your solutions:

- Deadline: April 10, 2026.
- Submit problems with total points ≥ 10 . You will be tested on your attention to details and your mathematical writing, rather than the total points you submit.
- Most importantly, it's an opportunity for you to see if you may enjoy this program.
- Solutions must be written in English, but can be in any format, e.g. a scanned written version or a compiled latex file. It has to be submitted in pdf.
- No technical questions will be answered. If you think there is ambiguity, state it in your solution and try to resolve it by yourself.
- You are not allowed to consult any sources or collaborate with others. The necessary background is provided. You may use any statements available in this file. Any issues related to academic integrity (e.g. plagiarism) will be **handled seriously**, and will affect your future academic endeavours.

Submission link. Or scan the QR code



1. THE SYMMETRIC GROUP

Definition 1.1. A *permutation* is a bijection from $[n] := \{1, 2, \dots, n\}$ to $[n]$.

We typically write a permutation via *one-line notation*. For example, $w = 3142$ means $w(1) = 3, w(2) = 1, w(3) = 4$ and $w(4) = 2$.

Definition 1.2. The *symmetric group* S_n is the group of all permutations on $[n]$ with the group operation being composition of maps. Let $s_i = (i \ i+1)$ be the permutation that swaps i and $i + 1$, called a *simple transposition* or a *simple generator*, for $i = 1, \dots, n - 1$.

It is clear that $\{s_1, \dots, s_{n-1}\}$ generates S_n .

Definition 1.3. The *length* of a permutation w is defined to be the smallest ℓ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for some simple generators $s_{i_1}, \dots, s_{i_\ell}$. Denote the length by $\ell(w)$.

For example, the permutation 3142 has length 3 as $3142 = s_2 s_1 s_3$.

Definition 1.4. The (right) *weak Bruhat order* is a partial order defined on S_n such that $w \leq_R ws_i$ if $\ell(w) < \ell(ws_i)$ for some i . The left weak Bruhat order is defined by $w \leq_L s_i w$ if $\ell(w) < \ell(s_i w)$.

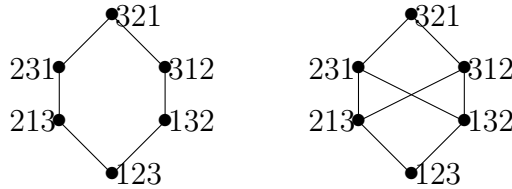


FIGURE 1. The (right) weak and strong order on S_3 .

Now define the *transpositions* T to be all the conjugates of $\{s_1, \dots, s_{n-1}\}$. In other words, $T = \{(i \ j) \mid 1 \leq i < j \leq n\}$ contains all the 2-cycles. Write $t_{ij} := (i \ j)$.

Definition 1.5. It is clear that both weak and strong Bruhat orders are graded by length. It is a standard fact (you may use freely) that $\ell(w) = |I(w)|$, where

$$I(w) = \{(i, j) \mid i < j, w(i) > w(j)\}$$

is the *inversion set* of the permutation w .

The (*strong*) *Bruhat order* is defined by $w < wt_{ij}$ if $\ell(w) < \ell(wt_{ij})$.

Write a Bruhat interval as $[u, v] := \{w \in S_n \mid u \leq w \leq v\}$. Similarly define $[u, v]_L$ and $[u, v]_R$ in the weak orders. The intervals $[\text{id}, v]$ starting from the identity element $\text{id} = 123 \cdots n$ are also called *lower intervals*.

In a partially ordered set (poset) P , we say that x is *covered* by z , or z *covers* x , denoted $x \lessdot z$, if there does not exist an element $y \in P$ such that $x < y < z$. A poset P is *ranked* if it has a decomposition $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_r$ into disjoint union such that if $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$. For example, the strong and weak Bruhat orders are ranked. A graded / ranked poset $P = P_0 \sqcup \cdots \sqcup P_r$ is called *rank-symmetric* if $|P_i| = |P_{r-i}|$ for $i = 0, 1, \dots, r$.

Definition 1.6. We say that a permutation w contains a pattern u if a subsequence of w has the same relative order as u . For example, $w = 4736\underline{215}$ contains the pattern 312 at the underlined positions. We say that w *avoids* u if w does not contain u .

- (1) Show that the number of permutations in S_n that avoid $\pi \in S_3$ is a fixed number that does not depend on the pattern $\pi \in S_3$. You can either provide a closed formula for this number, or provide a generating function. 6 pt
- (2) Show that $w \leq_L u$ in the left weak Bruhat order if and only if their inversion sets satisfy $I(w) \subset I(u)$. 6 pt
- (3) Show that if w avoids 3142 and 2413, then the weak (left or right) Bruhat intervals $[\text{id}, w]_L$ and $[\text{id}, w]_R$ are rank-symmetric. For example, if $w = 3142$, then there are two elements 1342, 3124 covered by w , but only one element 1324 covering $\text{id} = 1234$ in the right weak Bruhat interval $[\text{id}, 3142]$. 10 pt
- (4) The condition in the above question is not necessary. Find a family of permutations w that contain either 3142 or 2413 such that $[\text{id}, w]_R$ is rank-symmetric. 10 pt
- (5) For a permutation $w \in S_n$, write $w[i, j] = \#\{k \mid k \leq i, w(k) \leq j\}$ for all $i, j \in [n]$. This is called the *rank-matrix* associated with w . See Figure 2 for an example. Show 4 pt

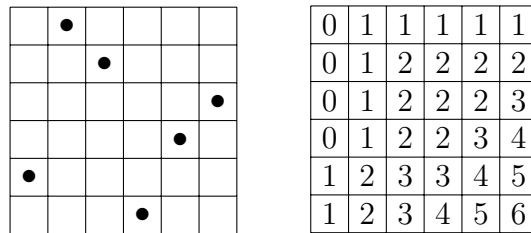


FIGURE 2. A rank matrix for permutation $w = 236514$

- that $x \leq y$ in the (strong) Bruhat order if and only if $x[i, j] \geq y[i, j]$ for all $i, j \in [n]$.
- (6) Fix a reduced word for some $w = s_{i_1} \cdots s_{i_\ell}$. Show that if $u \leq w$ in the (strong) Bruhat order, then there exists a subword of i_1, \dots, i_ℓ which is a reduced word for u . This is called the *subword property*. For example, fix $3241 = s_1 s_2 s_3 s_1$. Then $3214 = 321 < 3241$ and we can choose $s_1 s_2 s_1 = 321$ as a subword for 321. Is it true that for any reduced word $u = s_{j_1} \cdots s_{j_k}$ and $w \geq u$, there is a reduced word for w that contains j_1, \dots, j_k as a subword? 8 pt
- (7) Show that the (strong) Bruhat interval $[\text{id}, w]$ is rank-symmetric if and only if w avoids 3412 and 4231. For example, if $w = 3412$, then there are four elements 1432, 2413, 3142, 3214 covered by w , but only three elements 2134, 1324, 1243 covering id , so $[\text{id}, 3412]$ is not rank-symmetric. 10 pt
- (8) Show that the (strong) Bruhat interval $[\text{id}, w]$ is isomorphic to a boolean lattice, i.e. all subsets of some $[m] = \{1, 2, \dots, m\}$ ordered by inclusion, if and only if w avoids 321 and 3412. 10 pt
- (9) A permutation w is called *k-Grassmanian* if its descent set is contained in $\{k\}$. For example, 2356147 is 4-Grassmanian. The *Lehmer code* $L(w)$ of a permutation w is the sequence $[L(w)_1, \dots, L(w)_n]$ where $L(w)_i = \#\{j > i : w(j) < w(i)\}$. 10 pt
- (a) Use Lehmer code to give a bijection between the set of all k -Grassmanian permutation in S_n , and the set of all partitions that fit inside the $k \times (n - k)$ rectangle. Denote this bijection $\lambda(w)$.

(b) Show that for a k -Grassmanian permutation w , the set of reduced words of w are in bijection with the set of standard Young tableaux of $\lambda(w)$. (See section 2 for definition of SYT).

2. SYMMETRIC FUNCTIONS

Definition 2.1. Let $S_\infty = \bigcup_{n \geq 1} S_n$ be the *infinite symmetric group* that consists of bijections $w : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ with all but finitely many fixed points.

Definition 2.2. A permutation $w \in S_\infty$ acts on the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ and the ring of formal power series $\mathbb{Z}[[x_1, x_2, \dots]]$ by $wf(x_1, x_2, \dots) = f(x_{w(1)}, x_{w(2)}, \dots)$. A formal power series f is a *symmetric function* if $wf = f$ for all $w \in S_\infty$. Denote the ring of symmetric function as $\Lambda = \bigoplus_{d \geq 0} \Lambda^d$ where Λ^d is the degree d component.

Definition 2.3. A *composition* of n is a list $I = (i_1, \dots, i_l)$ of positive integers with sum n , denoted $I \vDash n$. The length of I is the number l , denoted $\ell(I)$. A *partition* of n is a composition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n such that $\lambda_1 \geq \dots \geq \lambda_l$, denoted $\lambda \vdash n$. The *Young diagram* of λ is the array arranged in left-justified rows of squares in which Row i from top to bottom consists of λ_i boxes. The *conjugate* of a partition λ is the the partition whose Young diagram is obtained by exchanging the rows of columns of λ , denoted λ' .

For two partitions $\lambda \vdash n$ and $\mu \vdash n$, we say that λ *dominates* μ , denoted $\lambda \supseteq \mu$, if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all i . We say that μ *is contained in* λ , denoted $\mu \subseteq \lambda$, if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i \leq \ell(\mu)$. For $\mu \subseteq \lambda$, the *skew shape* λ/μ is the shape obtained from the Young diagram of λ by removing the Young diagram of μ . When μ is empty, the skew shape λ/μ is the partition λ .

We do not distinguish a partition with its Young diagram. Sometimes we may also allow some parts to be 0 or treat λ with infinite length by appending 0's. Try to understand it in context. We are interested in basis of Λ^n . Let $\lambda \vdash n$.

Definition 2.4. Define the following *monomial symmetric functions* $\{m_\lambda\}$'s, *elementary symmetric functions* $\{e_\lambda\}$'s, *complete homogeneous symmetric functions* $\{h_\lambda\}$'s, *power sum symmetric functions* $\{p_\lambda\}$'s as follows:

$$m_\lambda = \sum_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} \cdots \text{ where } \alpha = (\alpha_1, \alpha_2, \dots) = w\lambda \text{ for some } w \in S_\infty,$$

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k} \text{ where } e_m = \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k} \text{ where } h_m = \sum_{i_1 \leq i_2 \leq \dots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k} \text{ where } p_m = x_1^m + x_2^m + \dots.$$

It should be clear that $\{m_\lambda \mid \lambda \vdash n\}$ form a basis of Λ^n , and the fact that $\{e_\lambda \mid \lambda \vdash n\}$ form a basis of Λ^n is called the *fundamental theorem of symmetric functions*.

Definition 2.5. A *semistandard Young tableau* (SSYT) of *shape* λ/μ and *type* α is a filling of the boxes of the shape λ/μ using α_i copies of i for $i = 1, 2, \dots$ such that each row is weakly increasing and each column is strictly increasing from top to bottom. Also write $\text{wt}(T) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$ as the *weight* of an SSYT T .

Further, define a *standard Young tableaux* of shape $\lambda \vdash n$ to be a SSYT that uses each of the numbers 1 to n exactly once.

Denote the set of SSYTs with shape λ by $\text{SSYT}(\lambda/\mu)$, and the set of SSYTs with shape λ and content α by $\text{SSYT}(\lambda/\mu, \alpha)$. The *Kostka number* $K_{\lambda/\mu, \alpha}$ is the cardinality of $\text{SSYT}(\lambda/\mu, \alpha)$.

Definition 2.6. The *Schur* function is $s_{\lambda/\mu} = \sum_{T \in \text{SSYT}(\lambda/\mu)} \text{wt}(T)$.

Example 2.7. The following is an SSYT of shape $(4, 4, 3, 1)$ and weight $x_1^2 x_2^2 x_3^3 x_5 x_6 x_7 x_{10} x_{12}$.

1	1	2	5
2	3	3	10
3	6	7	
12			

A *horizontal strip* (resp. *vertical strip*) is a skew shape in which there is only one cell in each column (resp. row).

Theorem 2.8 (Pieri's rule). *We have*

$$s_{\lambda} e_k = \sum_{\mu} s_{\mu},$$

where the sum runs over all partitions μ such that μ/λ is a vertical strip of length k .

(1) What is the number of compositions of n ? What is the number of partitions of n ?

2 pt You can either give a closed formula or provide a generating function.

(2) Let $\lambda, \mu \vdash n$. Show that if $\ell(\lambda) + \ell(\mu) \geq n + 1$, then $\lambda' \supseteq \mu$ and $\mu' \supseteq \lambda$.

4 pt

(3) Write the formal power series $\prod_{i=1}^{\infty} (1 + x_i + x_i^2)$ in elementary symmetric functions.

5 pt

(4) Show that s_{λ} is always a symmetric function. Hint: the infinite symmetric group is generated by simple transpositions $s_i = (i \ i+1)$ so it suffices to find an involution on the set of semistandard Young tableaux.

5 pt

(5) Show that $\{s_{\lambda} \mid \lambda \vdash n\}$ form a basis of Λ^n by showing that there exists a semistandard Young tableau of shape λ and type $\mu \vdash n$ if and only if $\lambda \supseteq \mu$, and that there exists a unique semistandard Young tableau of shape λ and type λ .

5 pt

(6) Consider the basis $b_{\lambda} = \sum_{\mu \leq \lambda, |\mu|=|\lambda|} m_{\mu}$ of Λ . Is each Schur function s_{λ} expanded nonnegatively into this basis?

8 pt

(7) Show Girard-Newton's formula:

6 pt

$$\sum_{i \neq n} (-1)^{n-i-1} e_i p_{n-i} = n e_n.$$

Hint: A combinatorial proof is based on some combinatorial interpretation of e_i and p_{n-i} in terms of special Young tableaux.

Definition 2.9. Let G be a simple graph. A coloring of the vertices $\kappa : V(G) \rightarrow \mathbb{Z}_{\geq 1}$ is *proper* if $\kappa(a) \neq \kappa(b)$ for adjacent vertices a and b . The *chromatic symmetric function* is $X_G(x_1, x_2, \dots) := \sum_{\kappa \text{ proper}} \prod_{v \in V(G)} x_{\kappa(v)}$. This is clearly symmetric.

For a composition $I \vDash n$, there is a unique partition $\rho(I) \vdash n$ which consists of the parts of I by reordering. Write $e_I := e_{\rho(I)}$.

(1) Let P_n be the path graph on n vertices, that is, its vertices can be labeled v_1, \dots, v_n where v_i is connected with v_{i+1} for $i \in [n-1]$. Show that $X_{P_n} = \sum_{I \vDash n} w_I e_I$ where

$$w_I = i_1(i_2 - 1)(i_3 - 1) \cdots (i_k - 1) \text{ if } I = (i_1, \dots, i_k).$$

(2) Let C_n be the cycle graph on n vertices, that is, its vertices can be labeled v_1, \dots, v_n where v_i is connected with v_{i+1} for $i = 1, \dots, n$ and $v_{n+1} := v_1$. Show that $X_{C_n} = \sum_{I \vDash n} (i_1 - 1) w_I e_I$.

3. COXETER GROUPS

Definition 3.1. A *Coxeter system* (W, S) is a group W (possibly infinite!) and a finite set $S \subset W$ of generators of W , for which W admits a presentation (“Coxeter presentation”) of a very particular form. Namely, there must be a matrix $(m_{st})_{s,t \in S}$ satisfying $m_{ss} = 1$ for each $s \in S$, and $m_{st} = m_{ts} \in \{2, 3, \dots\} \cup \{\infty\}$ for $s \neq t \in S$, such that

$$W = \langle s \in S \mid (st)^{m_{st}} = \text{id for any } s, t \in S \text{ with } m_{st} < \infty \rangle.$$

When $m_{st} = \infty$, there is no corresponding relation between s and t . The elements of S are often called *simple reflections* and the *rank* of the Coxeter system (W, S) is defined as $|S|$.

Definition 3.2. For each $w \in W$, one can write $w = s_1 \cdots s_k$ for some $s_1, \dots, s_k \in S$. The sequence (s_1, \dots, s_k) is called an *expression* or a *word* for w . We use a notational shorthand \underline{w} to denote the sequence (s_1, \dots, s_k) , when the product $s_1 \cdots s_k$ is equal to w . That is, the notation \underline{w} indicates both an element $w \in W$ and a particular choice of expression for w . The *length* of w , denoted by $\ell(w)$, is the minimal k for which w admits an expression (s_1, \dots, s_k) . Any expression for w with this minimal length $\ell(w)$ is called a *reduced expression*.

The following theorem is crucial in the theory of Coxeter groups.

Theorem 3.3 (Exchange Condition). *Let $\underline{w} = (s_1, s_2, \dots, s_k)$ be a reduced expression for w , and $t \in S$. If $\ell(wt) < \ell(w)$ then there exists i such that $1 \leq i \leq k$ and $wt = s_1 s_2 \cdots \widehat{s_i} \cdots s_k$.*

Let $(s_{i_1}, \dots, s_{i_k})$ and $(s_{j_1}, \dots, s_{j_k})$ be two arbitrary expressions of the same length. If we can apply a sequence of braid relations to obtain $(s_{j_1}, \dots, s_{j_k})$ from $(s_{i_1}, \dots, s_{i_k})$ we say that they are related by braid relations. The following is a beautiful theorem.

Theorem 3.4 (Matsumoto’s Theorem). *Any two reduced expressions for $w \in W$ are related by braid relations.*

(1) Give a realization of the dihedral group D_{2n} (the symmetric group of the regular n -gon) as a Coxeter system.

(2) Find (and prove!) a Coxeter presentations of 3 generators for the dihedral group D_{12} (the symmetric group of the regular hexagon), which has 12 elements.

(3) Show that any Coxeter group W admits a sign representation: an action of W on \mathbb{R} where each simple reflection acts by -1 . Deduce that $\ell(ws) \neq \ell(w)$, for any $w \in W$ and $s \in S$. 5 pt

(4) Using the exchange condition to show that for $w \in W$, the set $\{s \in S \mid \ell(ws) < \ell(w)\}$ is equal to the set $\{s \in S \mid w \text{ admits a reduced expression ending in } s\}$, which is called the *right descent set*. 5 pt

(5) Using the exchange condition and Matsumoto's theorem to show that *the word problem in a Coxeter group is solvable*, that is, there is an algorithm to determine whether two words in the generators represent the same element. 10 pt

4. COMBINATORIAL HODGE THEORY

We assume some basic knowledge of \mathfrak{sl}_2 -representations in this section.

Definition 4.1. Fix a finite-dimensional graded real vector space

$$H = \bigoplus_{i \in \mathbb{Z}} H^i$$

and a symmetric non-degenerate graded bilinear form

$$\langle -, - \rangle : H \times H \longrightarrow \mathbb{R}.$$

By “graded” we mean that $\langle H^i, H^j \rangle = 0$ if $i \neq -j$. It is immediate that $\langle -, - \rangle$ induces an isomorphism between H^{-i} and $(H^i)^*$. We say that the graded vector space H satisfies *Poincaré duality*. Thus, if $b_i = \dim(H^i)$ (the i -th “Betti number” of H), then $b_i = b_{-i}$ for all $i \in \mathbb{Z}$. Our convention is such that the mirror of Poincaré duality is in degree zero.

Definition 4.2. We say a degree two linear map

$$L : H^i \longrightarrow H^{i+2}$$

is a *Lefschetz operator* if $\langle La, b \rangle = \langle a, Lb \rangle$ for all $a, b \in H$. If L is a Lefschetz operator, then it is said to satisfy *hard Lefschetz* if for all $i \geq 0$,

$$L^i : H^{-i} \longrightarrow H^i$$

is an isomorphism.

Definition 4.3. Let L be a Lefschetz operator. For each $i \geq 0$, define the *Lefschetz form* on H^{-i} with respect to L as

$$(a, b)_L^{-i} = \langle a, L^i b \rangle$$

for $a, b \in H^{-i}$.

We will use the notation H^{\min} to denote the nonzero graded component of H of minimal degree, which is well defined as long as $H \neq 0$.

Definition 4.4. For all $i \geq 0$ set

$$P_L^{-i} = \ker(L^{i+1}) \cap H^{-i},$$

which is called a *primitive subspace*. Assume $H^{\text{odd}} = 0$ or $H^{\text{even}} = 0$ and that L is a Lefschetz operator satisfying hard Lefschetz. We say that $(H, \langle -, - \rangle, L)$ satisfies the *Hodge–Riemann bilinear relations* if the restriction of the Lefschetz form to the primitive subspace

$$(-, -)_L^{\min+2i} \Big|_{P_L^{\min+2i}}$$

is $(-1)^i$ -definite.

(1) Show that a degree two operator L on H satisfies hard Lefschetz if and only if there is an action of $\mathfrak{sl}_2(\mathbb{R}) = \langle e, f, h \rangle$ on H with e acting as L and $h \cdot v = iv$ for all $v \in H^i$. Moreover, show that this $\mathfrak{sl}_2(\mathbb{R})$ -action is unique.

(2) If L satisfies hard Lefschetz, show that the betti numbers satisfy

$$\dots \leq b_{-4} \leq b_{-2} \leq b_0 \geq b_2 \geq b_4 \geq \dots, \quad \dots \leq b_{-3} \leq b_{-1} = b_1 \geq b_3 \geq \dots.$$

(3) Assume $H^{\text{odd}} = 0$. Find a formula for the signature of the Lefschetz form in degree zero in terms of the betti numbers of H .

(4) Suppose that $(H, \langle -, - \rangle, L)$ satisfies Hodge–Riemann and K is an L -invariant graded subspace of H satisfying Poincaré duality. Show that

$$(K, \langle -, - \rangle_K, L|_K)$$

satisfies Hodge–Riemann (up to a sign). In particular, the restriction of the Lefschetz form from H^{-i} to K^{-i} is non-degenerate, for each $i \geq 0$.

(5) This exercise will explore $H^*(\text{Gr}(2, 4), \mathbb{R})$, the cohomology of the Grassmannian of 2-planes in \mathbb{C}^4 , using a combinatorial model. Let $P(2, 4)$ denote the set of Young diagrams which fit inside a 2×2 rectangle. The degree of a partition will be -4 plus twice the number of boxes; for example the partition $(2, 1)$ has degree $+2$. Two Young diagrams are complementary if one can be rotated 180 degrees in order to fill the full 2×2 rectangle with the other: for example $(2, 1)$ and $(1, 0)$ are complementary. Let H be the graded vector space with basis $\{v_\lambda\}_{\lambda \in P(2,4)}$. Place a symmetric bilinear pairing on H , where $\langle v_\lambda, v_\mu \rangle = 1$ if λ and μ are complementary, and is equal to zero otherwise. Place an operator $L : H^i \rightarrow H^{i+2}$ on this space, where $Lv_\lambda = \sum_\mu v_\mu$ is the sum over the Young diagrams $\mu \in P(2, 4)$ obtained from λ by adding one box.

Prove that L is a Lefschetz operator satisfying hard Lefschetz and the Hodge–Riemann bilinear relations. (Hint: Compute a basis for each primitive subspace.)

5. PLÜCKER RELATIONS AND GRÖBNER BASES

Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring of n variables over \mathbb{C} .

Definition 5.1. A term order $<$ is a total order on the monomials of R such that it is

- multiplicative: $x^{\mathbf{b}} := \prod_{i=1}^n x_i^{b_i} < x^{\mathbf{c}}$ if and only if $x^{\mathbf{a}+\mathbf{b}} < x^{\mathbf{a}+\mathbf{c}}$;
- artinian: $1 < x^{\mathbf{a}}$.

Given a polynomial $f = \sum_{\mathbf{a}} c_{\mathbf{a}} x^{\mathbf{a}}$, the leading term $\text{in}_{<}(f) = c_{\mathbf{a}} x^{\mathbf{a}}$ if $x^{\mathbf{a}}$ is the largest under $<$ among monomials appearing in f with non-zero coefficient.

Definition 5.2. Fix a term order $<$, the *initial ideal* of an ideal $I \subset R$ is

$$\text{in}_{<}(I) := \langle \text{in}_{<}(f) : f \in I \rangle.$$

Suppose $I = \langle f_1, \dots, f_m \rangle$. Then $\{f_1, \dots, f_m\}$ is said to be a *Gröbner basis* if

$$\text{in}_<(I) = \langle \text{in}_<(f_i) : i \in [m] \rangle.$$

A Gröbner basis $\{f_1, \dots, f_m\}$ is *reduced* if each $\text{in}_<(f_i)$ has coefficient 1 and the only monomial appearing anywhere in $\{f_1, \dots, f_m\}$ that is divisible by $\text{in}_<(f_i)$ is $\text{in}_<(f_i)$ itself.

- (1) If $\text{in}_<(I) = \langle \text{in}_<(f_i) : i \in [m] \rangle$, then $I = \langle f_1, \dots, f_m \rangle$.

5 pt

- (2) Every ideal $I \subset R$ has a finite Gröbner basis for every term order.

5 pt

- (3) For every term order $<$, there is a unique reduced Gröbner basis.

5 pt

Definition 5.3. For a term order $<$ and an ideal $I \subset R$, define the *standard monomials* of R/I to be the monomials that are not in the initial ideal $\text{in}_<(I)$.

- (1) Show that the set of standard monomials is a \mathbb{C} -vector space basis of R/I .

5 pt

- (2) Find a characterization of all monomial orders on $\mathbb{C}[x_1, x_2]$. Generalize your result to $R = \mathbb{C}[x_1, \dots, x_n]$.

5 pt

The two most frequently used monomial orders are the purely lexicographical order “ $<_{\text{lex}}$ ” and the reverse lexicographical order “ $<_{\text{rlex}}$ ”. These are defined as follows. We assume that an order is given on the variables, say, $x_1 > x_2 > \dots > x_n$. We then put $x^\alpha <_{\text{lex}} x^\beta$ if there exists $i \in [n]$, such that $\alpha_j = \beta_j$ for all $j < i$, and $\alpha_i < \beta_i$. In contrast to “ $<_{\text{lex}}$ ”, the reverse lexicographic order “ $<_{\text{revlex}}$ ” is a linear extension of the natural grading on R . We define $x^\alpha <_{\text{revlex}} x^\beta$ if $\sum_i \alpha_i < \sum_i \beta_i$ or if $\sum_i \alpha_i = \sum_i \beta_i$ and there exists $i \in [n]$ such that $\alpha_j = \beta_j$ for all $j > i$ and $\alpha_i > \beta_i$.

Let $R = \mathbb{C}[x_{1,1}, \dots, x_{k,n}]$ be the ring of polynomial functions on $\text{Mat}_{k \times n}$ the space of $k \times n$ matrices (we assume $k \leq n$ throughout this section).

Definition 5.4. For $\mathbf{a} = \{a_1 < \dots < a_k\} \in \binom{[n]}{k}$ a k -element subset of $[n]$, denote $[\mathbf{a}]$ the determinant of the $k \times k$ submatrix of $\text{Mat}_{k \times n}$ with column index set \mathbf{a} . We extend the notation of $[\mathbf{a}]$ to all sequences

$$\mathbf{a} = (a_1, \dots, a_k) \in [n]^k$$

where we use the convention that for any permutation $\sigma \in S_n$,

- (1) $[a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k)}] = (-1)^{\ell(\sigma)} \cdot [a_1, \dots, a_k]$.

In particular, this implies that $[\mathbf{a}] = 0$ if $a_i = a_j$ for some $i \neq j \in [n]$.

Let $S \subseteq R$ be the subalgebra generated by the set $\{[\mathbf{a}] : \mathbf{a} \in \binom{[n]}{k}\}$. Then S can be presented as a quotient of polynomial ring $\mathbb{C}[[\mathbf{a}] : \mathbf{a} \in \binom{[n]}{k}]/\mathcal{J}$ where \mathcal{J} is generated by the following *straightening relations*:

Definition 5.5. Let $s \in [k], \alpha \in \binom{[n]}{s-1}, \beta \in \binom{[n]}{k+1}$ and $\gamma \in \binom{[n]}{k-s}$ where elements in α, β, γ are in increasing order. The straightening relation attached to α, β, γ is

$$\sum_{I \in \binom{[k+1]}{s}} (-1)^{\text{sgn}(I)} [\alpha_1, \dots, \alpha_{s-1}, \beta_{i'_1}, \dots, \beta_{i'_{k-s+1}}] \cdot [\beta_{i_1}, \dots, \beta_{i_s}, \gamma_1, \dots, \gamma_{k-s}]$$

where $I = \{i_1 < \dots < i_s\} \subset [k+1], \{i'_1 < \dots < i'_{k-s+1}\} := [k+1] \setminus I$, and $\text{sgn}(I) := \sum_{j=1}^s i_j - \binom{s+1}{2}$.

Consider the ordering on the variables $[\mathbf{a}]$ with $\mathbf{a} = \{a_1 < \dots < a_k\}$ given by $[\mathbf{a}] < [\mathbf{b}]$ if there exists $i \in [k]$ such that $a_j = b_j$ for all $j < i$ and $a_i < b_i$.

- (1) For a SSYT T of shape (d^k) with entries in $[n]$, define $[T] := \prod_{i=1}^d [T_i] \in \mathbb{C}[[\mathbf{a}]]$,
10 pt where $T_i \in \binom{[n]}{k}$ is the set of entries in the i -th column of T . Show that every degree d homogeneous polynomial in $\mathbb{C}[[\mathbf{a}]]$ can be written as \mathbb{C} -linear combinations of $\{[T] : T \in \text{SSYT}(d^k)\}$.
- (2) Show that the straightening relations form a Gröbner basis of \mathcal{J} with respect to the
10 pt term order “ $<_{\text{revlex}}$ ” and describe the initial ideal.

6. EHRHART THEORY

Definition 6.1. A *convex d -polytope* is the convex hull of finitely many points in \mathbb{R}^d . These points are called the *vertices* of P . When all of the vertices of P have integer coordinates, we call it a *lattice polytope*.

Definition 6.2. Let P be the convex hull of $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^d$. For any linear function $a(x) = a_1x_1 + \dots + a_dx_d$ on \mathbb{R}^d , let $h = \max(a(v_1), \dots, a(v_m))$. The *supporting hyperplane* of P at a is $H_{a,P} = \{x \in \mathbb{R}^d \mid a(x) = h\}$. The *supporting face* of P at a is $F_{a,P} := P \cap H_{a,P}$. The set of faces of P is the set of nonempty supporting faces of P for all possible a . The *facets* of P are exactly the faces of dimension $d-1$.

Let $t \in \mathbb{R}$. The t -dilate of P is $tP := \{t \cdot p \mid p \in P\}$, where the point $t \cdot p$ is obtained from the point p by multiplying all the coordinates of p by the real number t .

Definition 6.3. The *Ehrhart function* (in the variable t) of a d -polytope P is $L_P(t) = |(tP) \cap \mathbb{Z}^d|$, that is, the number of lattice points (integer vectors) in the t -dilate of P .

Theorem 6.4. *The Ehrhart function $L_P(t)$ of a lattice polytope P is a polynomial in t .*

- (1) Compute the Ehrhart polynomial of the *hypercube*, which is the convex hull of all
2 pt binary vectors in \mathbb{R}^d . A binary vector has coordinate either 0 or 1.
- (2) Compute the Ehrhart polynomial of the *standard d -simplex*, which is the convex hull
2 pt of $d+1$ standard unit vectors.
- (3) Let P° be the *relative interior* of P , which is P with all boundaries/facets removed.
3 pt Show by inclusion-exclusion that $|tP^\circ \cap \mathbb{Z}^d|$ is also a polynomial in t .

- (4) Compute the Ehrhart polynomial of the relative interior of the standard d -simplex, and show that it satisfies the *Ehrhart–Macdonald reciprocity*: $L_P(-t) = (-1)^d L_{P^\circ}(t)$. 3 pt

A *d-simplex* is the convex hull of $d + 1$ affinely independent points. A d -simplex is *unimodular* if it is lattice isomorphic to the standard d -simplex. (Two polytopes P, Q are *lattice isomorphic*, denoted $P \approx Q$, if there exists an affine isomorphism A such that $A|_{\mathbb{Z}^n}$ is a bijection onto \mathbb{Z}^n and $AP = Q$.)

Definition 6.5. A *polytopal complex* is a finite collection C of polytopes in some \mathbb{R}^d with the following two properties: If $P \in C$ and F is a face of P , then $F \in C$; and if $P, Q \in C$ then $F = P \cap Q \in C$ and F is common face of both P and Q . The polytopes in C are also called faces. A (geometric) *simplicial complex* is a polytopal complex in which all faces are simplices. An *abstract simplicial complex* is a set Δ of subsets of a finite set V , such that Δ is closed under taking subsets. A *subdivision* of a polytopal complex C is a polytopal complex C' such that $\cup C = \cup C'$ and every face of C' is contained in a face of C . A *triangulation* is a subdivision in which all faces are simplices. A *unimodular triangulation* is a triangulation in which all simplices are unimodular.

Definition 6.6. Let Δ be an abstract simplicial complex on the ground set $[n]$. Let k be some field. The *Stanley–Reisner ring* of Δ is the quotient $k[\Delta] := k[x_1, \dots, x_n]/I_\Delta$ where I_Δ is the *Stanley–Reisner ideal*, generated by the monomials corresponding to non-faces of Δ

$$I_\Delta = \langle x_{i_1} \cdots x_{i_r} \mid \{i_1, \dots, i_r\} \notin \Delta \rangle.$$

When P is a polytope, we use $k[P]$ to denote the Stanley–Reisner ring of the polytopal complex associated with P .

Definition 6.7. Let $R = k[x_1, \dots, x_n]/I = \bigoplus_{t \geq 0} R_t$ be a polynomial ring, where R_t is the k -vector space spanned by all the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_1 + \cdots + \alpha_n = t$. The *Hilbert function/polynomial* of R is $H_R(t) := \dim_k R_t$.

- (1) Let P be a d -polytope, and let f_i be the number of i -dimensional faces in P , for $0 \leq i \leq d$. Show that the Hilbert polynomial of $k[P]$ is equal to $\sum_{i=0}^d f_i \binom{t-1}{i}$. 3 pt

- (2) Let Δ be a unimodular triangulation of a d -polytope P . Prove that the Ehrhart polynomial of P equals the Hilbert function of the Stanley–Reisner ring of Δ , i.e., 5 pt

$$L_P(t) = H_{k[\Delta]}(t).$$

- (3) If P, Q are both d -polytopes, show that $L_{P \cup Q} = L_P + L_Q - L_{P \cap Q}$. If $P \approx Q$, show that $L_P = L_Q$. 2 pt

- (4) Prove that the Ehrhart function is a polynomial, and prove the Ehrhart–Macdonald reciprocity for any lattice polytope. 5 pt

7. q -ANALOGUES

We say $f(q)$ is a *q -analogue* (or a *q -deformation*) of A if $\lim_{q \rightarrow 1} f(q) = A$. Some classical examples of q -analogues include:

- (1) The q -integer $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ is a q -analogue of $n \in \mathbb{N}$.
- (2) The q -factorial $[n]_q! := [1]_q [2]_q \cdots [n]_q$ is a q -analogue of $n!$.

- (3) The q -binomial coefficient $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ is a q -analogue of $\binom{n}{k}$.
- (4) The Hecke algebra of a Weyl group W is a q -analogue of the group algebra of W .
- (5) The flag variety over a finite field $GL_n(\mathbb{F}_q)/B$ is a q -analogue of S_n .

Most q -analogues often come with nice enumerative and geometric properties. If a number a count the cardinality of the set A and $f(q)$ is a q -analogue of a , then $f(q)$ is often the generating function of some natural statistic on A . Moreover, there often exist an algebraic variety X such that the q analogue $f(q)$ of a count the size of $X(\mathbb{F}_q)$. They sometime also appear as Hilbert series of some representations.

- (1) Let $R_{k,n}$ denote the set of Young diagrams that fit inside the $k \times (n - k)$ rectangle.

3 pt Note that $\#R_{k,n} = \binom{n}{k}$. Find a statistic $st : R_{k,n} \rightarrow \mathbb{N}$ on $R_{k,n}$ such that

$$\binom{n}{k}_q = \sum_{a \in R_{k,n}} q^{st(a)}.$$

- (2) Recall that $n! = \#S_n$. Find a statistic $st : S_n \rightarrow \mathbb{N}$ on S_n such that

3 pt
$$[n]_q! = \sum_{w \in S_n} q^{st(w)}.$$

- (3) The (complete) flag variety $FL(F)$ over a field F is defined to be

5 pt
$$FL_n(F) := \{V_1 \subset V_2 \subset \cdots \subset V_n \subset F^n \mid \dim V_i = i\}.$$

Show that $|FL_n(\mathbb{F}_q)| = [n]_q!$.

- (4) A special case of the flag variety is the Grassmanian, defined as follows.

5 pt
$$\text{Gr}_k(F^n) = \{V \subset F^n \mid \dim(V) = k\}.$$

Show that $|\text{Gr}_k(\mathbb{F}_q^n)| = \binom{n}{k}_q$.