PACE 2025 APPLICATION TESTS

Please read the following instructions before writing your solutions:

- Submit problems with total points ≥ 10. You will be tested on your attention to details and your mathematical writing, rather than the total points you submit.
- Most importantly, it's an opportunity for you to see if you may enjoy this program.
- Solutions must be written in English, but can be in any format, e.g. a scanned written version or a complied latex file. It has to be sent in pdf.
- No technical questions will be answered. If you think there is ambiguity, state it in your solution and try to resolve it by yourself.
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- **Personal statement** (max 2 pages, written in English).
- Undergraduate transcript.
- Solution to this test (in pdf) sent to `pace@bicmr.pku.edu.cn`.
- One recommendation letter sent directly by the writor to `pace@bicmr.pku.edu.cn`.

1. The symmetric group

Definition 1.1. A *permutation* is a bijection from $[n] := \{1, 2, \ldots, n\}$ to $[n]$.

We typically write a permutation via *one-line notation*. For example, $w = 3142$ means $w(1) = 3$, $w(2) = 1$, $w(3) = 4$ and $w(4) = 2$.

Definition 1.2. The *symmetric group* S_n is the group of all permutations on $[n]$ with the group operation being composition of maps. Let $s_i = (i\ i{+}1)$ be the permutation that swaps i and $i + 1$, called a *simple transposition* or a *simple generator*, for $i = 1, \ldots, n - 1$.

It is clear that $\{s_1, \ldots, s_{n-1}\}$ generates S_n.

Definition 1.3. The *length* of a permutation w is defined to be the smallest ℓ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for some simple generators $s_{i_1}, \ldots, s_{i_\ell}$. Denote the length by $\ell(w)$.

For example, the permutation 3142 has length 3 as $3142 = s_2 s_1 s_3$.

Definition 1.4. The (right) *weak Bruhat order* is a partial order defined on S_n such that $w \lessdot_R ws_i$ if $\ell(w) < \ell(ws_i)$ for some i. The left weak Bruhat order is defined by $w \lessdot_L s_i w$ if $\ell(w) < \ell(s_i w)$.

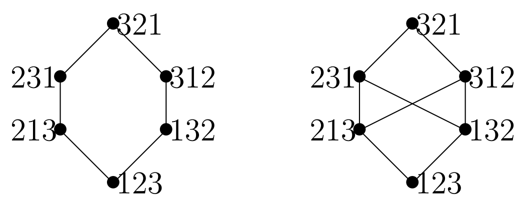


Figure 1. The (right) weak and strong order on S_3.

Now define the *transpositions* T to be all the conjugates of $\{s_1, \ldots, s_{n-1}\}$. In other words, $T = \{(i\ j) \mid 1 \le i < j \le n\}$ contains all the 2-cycles. Write $t_{ij} := (i\ j)$.

Definition 1.5. It is clear that both weak and strong Bruhat orders are graded by length. It is a standard fact (you may use freely) that $\ell(w) = |I(w)|$, where

$$I(w) = \{(i, j) \mid i < j,\ w(i) > w(j)\}$$

is the *inversion set* of the permutation w.

The *(strong) Bruhat order* is defined by $w < wt_{ij}$ if $\ell(w) < \ell(wt_{ij})$.

Write a Bruhat interval as $[u, v] := \{w \in S_n | u \le w \le v\}$. Similarly define $[u, v]_L$ and $[u, v]_R$ in the weak orders. The intervals $[\mathrm{id}, v]$ starting from the identity element $\mathrm{id} = 123 \cdots n$ are also called *lower intervals*.

In a partially ordered set (poset) P, we say that x *is covered* by z, or z *covers* x, denoted $x \lessdot z$, if there does not exist an element $y \in P$ such that $x < y < z$. A poset P is *ranked* if it has a decomposition $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_r$ into disjoint union such that if $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$. For example, the strong and weak Bruhat orders are ranked. A graded / ranked poset $P = P_0 \sqcup \cdots \sqcup P_r$ is called *rank-symmetric* if $|P_i| = |P_{r-i}|$ for $i = 0, 1, \ldots, r$.

Definition 1.6. We say that a permutation w contains a pattern u if a subsequence of w has the same relative order as u. For example, $w = 4\underline{7}36\underline{2}1\underline{5}$ contains the pattern 312 at the underlined positions. We say that w *avoids* u if w does not contain u.

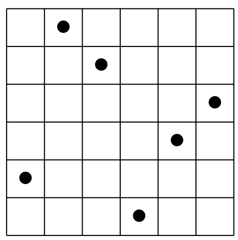
(1) **6 pt** Show that the number of permutations in S_n that avoid $\pi \in S_3$ is a fixed number that does not depend on the pattern $\pi \in S_3$. You can either provide a closed formula for this number, of provide a generating function.

(2) **6 pt** Show that $w \leq_L u$ in the left weak Bruhat order if and only if their inversion sets satisfy $I(w) \subset I(u)$.

(3) **10 pt** Show that if w avoids 3142 and 2413, then the weak (left or right) Bruhat intervals $[\mathrm{id}, w]_L$ and $[\mathrm{id}, w]_R$ are rank-symmetric. For example, if $w = 3142$, then there are two elements 1342, 3124 covered by w, but only one element 1324 covering id $= 1234$ in the right weak Bruhat interval $[\mathrm{id}, 3142]$.

(4) **10 pt** The condition in the above question is not necessary. Find a family of permutations w that contain either 3142 or 2413 such that $[\mathrm{id}, w]_R$ is rank-symmetric.

(5) **8 pt** For a permutation $w \in S_n$, write $w[i, j] = \#\{k \mid k \leq i,\ w(k) \leq j\}$ for all $i, j \in [n]$. This is called the *rank-matrix* associated with w. See Figure 2 for an example. Show



0	1	1	1	1	1
0	1	2	2	2	2
0	1	2	2	2	3
0	1	2	2	3	4
1	2	3	3	4	5
1	2	3	4	5	6

FIGURE 2. A rank matrix for permutation $w = 236514$

that $x \leq y$ in the (strong) Bruhat order if and only if $x[i, j] \geq y[i, j]$ for all $i, j \in [n]$.

(6) **8 pt** Fix a reduced word for some $w = s_{i_1} \cdots s_{i_\ell}$. Show that if $u \leq w$ in the (strong) Bruhat order, then there exists a subword of $i_1, \ldots, i_\ell$ which is a reduced word for u. This is called the *subword property*. For example, fix $3241 = s_1 s_2 s_3 s_1$. Then $3214 = 321 < 3241$ and we can choose $s_1 s_2 s_1 = 321$ as a subword for 321. Is it true that for any reduced word $u = s_{j_1} \cdots s_{j_k}$ and $w \geq u$, there is a reduced word for w that contains $j_1, \ldots, j_k$ as a subword?

(7) **10 pt** Show that the (strong) Bruhat interval $[\mathrm{id}, w]$ is rank-symmetric if and only if w avoids 3412 and 4231. For example, if $w = 3412$, then there are four elements 1432, 2413, 3142, 3214 covered by w, but only three elements 2134, 1324, 1243 covering id, so $[\mathrm{id}, 3412]$ is not rank-symmetric.

(8) **10 pt** Show that the (strong) Bruhat interval $[\mathrm{id}, w]$ is isomorphic to a boolean lattice, i.e. all subsets of some $[m] = \{1, 2, \ldots, m\}$ ordered by inclusion, if and only if w avoids 321 and 3412.

(9) **10 pt** A permutation w is called *k-Grassmanian* if its descent set is contained in $\{k\}$. For example, 2356147 is 4-Grassmanian. The *Lehmer code* $L(w)$ of a permutation w is the sequence $[L(w)_1, \cdots, L(w)_n]$ where $L(w)_i = \#\{j > i : w(j) < w(i)\}$.

(a) Use Lehmer code to give a bijection between the set of all k-Grassmanian permutation in S_n, and the set of all partitions that fit inside the $k \times (n-k)$ rectangle. Denote this bijection $\lambda(w)$.

(b) Show that for a k-Grassmanian permutation w, the set of reduced words of w are in bijection with the set of standard Young tableaux of $\lambda(w)$. (See section 2 for definition of SYT).

2. Symmetric functions

Definition 2.1. Let $S_\infty = \bigcup_{n\geq 1} S_n$ be the *infinite symmetric group* that consists of bijections $w : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with all but finitely many fixed points.

Definition 2.2. A permutation $w \in S_\infty$ acts on the polynoial ring $\mathbb{Z}[x_1, x_2, \ldots]$ and the ring of formal power series $\mathbb{Z}[[x_1, x_2, \ldots]]$ by $wf(x_1, x_2, \ldots) = f(x_{w(1)}, x_{w(2)}, \ldots)$. A formal power series f is a *symmetric function* if $wf = f$ for all $w \in S_\infty$. Denote the ring of symmetric function as $\Lambda = \bigoplus_{d\geq 0} \Lambda^d$ where Λ^d is the degree d component.

Definition 2.3. A *composition* of n is a list $I = (i_1, \ldots, i_l)$ of positive integers with sum n, denoted $I \vDash n$. The length of I is the number l, denoted $\ell(I)$. A *partition* of n is a composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n such that $\lambda_1 \geq \cdots \geq \lambda_l$, denoted $\lambda \vdash n$. The *Young diagram* of λ is the array arranged in left-justified rows of squares in which Row i from top to bottom consists of λ_i boxes. The *conjugate* of a partition λ is the the partition whose Young diagram is obtained by exchanging the rows of columns of λ, denoted λ'.

For two partitions $\lambda \vdash n$ and $\mu \vdash n$, we say that λ *dominates* μ, denoted $\lambda \trianglerighteq \mu$, if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i. We say that *μ is contained in λ*, denoted $\mu \subseteq \lambda$, if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i \leq \ell(\mu)$. For $\mu \subseteq \lambda$, the *skew shape* λ/μ is the shape obtained from the Young diagram of λ by removing the Young diagram of μ. When μ is empty, the skew shape λ/μ is the partition λ.

We do not distinguish a partition with its Young diagram. Sometimes we may also allow some parts to be 0 or treat λ with infinite length by appending 0's. Try to understand it in context. We are interested in basis of Λ^n. Let $\lambda \vdash n$.

Definition 2.4. Define the following *monomial symmetric functions* $\{m_\lambda\}$'s, *elementary symmetric functions* $\{e_\lambda\}$'s, *complete homogeneous symmetric functions* $\{h_\lambda\}$'s, *power sum symmetric functions* $\{p_\lambda\}$'s as follows:

$$\begin{aligned}
m_\lambda =& \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots \text{ where } \alpha = (\alpha_1, \alpha_2, \ldots) = w\lambda \text{ for some } w \in S_\infty,\\
e_\lambda =& e_{\lambda_1} \cdots e_{\lambda_k} \text{ where } e_m = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m},\\
h_\lambda =& h_{\lambda_1} \cdots h_{\lambda_k} \text{ where } h_m = \sum_{i_1 \leq i_2 \leq \cdots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m},\\
p_\lambda =& p_{\lambda_1} \cdots p_{\lambda_k} \text{ where } p_m = x_1^m + x_2^m + \cdots .
\end{aligned}$$

It should be clear that $\{m_\lambda \,|\, \lambda \vdash n\}$ form a basis of Λ^n, and the fact that $\{e_\lambda \,|\, \lambda \vdash n\}$ form a basis of Λ^n is called the *fundamental theorem of symmetric functions*.

Definition 2.5. A *semistandard Young tableau* (SSYT) of *shape* λ/μ and *type* α is a filling of the boxes of the shape λ/μ using α_i copies of i for $i = 1, 2, \ldots$ such that each row is weakly increasing and each column is strictly increasing from top to bottom. Also write $\mathrm{wt}(T) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$ as the *weight* of an SSYT T.

Further, define a *standard Young tableaux* of shape $\lambda \vdash n$ to be a SSYT that uses each of the numbers 1 to n exactly once.

Denote the set of SSYTs with shape λ by $\mathrm{SSYT}(\lambda/\mu)$, and the set of SSYTs with shape λ and content α by $\mathrm{SSYT}(\lambda/\mu, \alpha)$. The *Kostka number* $K_{\lambda/\mu,\alpha}$ is the cardinality of $\mathrm{SSYT}(\lambda/\mu, \alpha)$.

Definition 2.6. The *Schur* function is $s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/mu)} \mathrm{wt}(T)$.

Example 2.7. The following is an SSYT of shape $(4, 4, 3, 1)$ and weight $x_1^2 x_2^2 x_3^3 x_5 x_6 x_7 x_{10} x_{12}$.

1	1	2	5
2	3	3	10
3	6	7	
12			

A *horizontal strip* (resp. *vertical strip*) is a skew shape in which there is only one cell in each column (resp. row).

Theorem 2.8 (Pieri's rule)**.** *We have*

$$s_\lambda e_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions μ such that μ/λ is a vertical strip of length k.

(1) What is the number of compositions of n? What is the number of partitions of n? You can either give a closed formula or provide a generating function.
2 pt

(2) Let $\lambda, \mu \vdash n$. Show that if $\ell(\lambda) + \ell(\mu) \geq n + 1$, then $\lambda' \trianglerighteq \mu$ and $\mu' \trianglerighteq \lambda$.
4 pt

(3) Write the formal power series $\prod_{i=1}^{\infty}(1 + x_i + x_i^2)$ in elementary symmetric functions.
5 pt

(4) Show that s_λ is always a symmetric function. Hint: the infinite symmetric group is generated by simple transpositions $s_i = (i\ i{+}1)$ so it suffices to find an involution on the set of semistandard Young tableaux.
5 pt

(5) Show that $\{s_\lambda \,|\, \lambda \vdash n\}$ form a basis of Λ^n by showing that there exists a semistandard Young tableau of shape λ and type $\mu \vdash n$ if and only if $\lambda \trianglerighteq \mu$, and that there exists a unique semistandard Young tableau of shape λ and type λ.
5 pt

(6) Consider the basis $b_\lambda = \sum_{\mu \trianglelefteq \lambda, |\mu| = |\lambda|} m_\mu$ of Λ. Is each Schur function s_λ expanded nonnegatively into this basis?
8 pt

(7) Show Girard-Newton's formula:
6 pt

$$\sum_{i \neq n} (-1)^{n-i-1} e_i p_{n-i} = n e_n.$$

Hint: A combinatorial proof is based on some combinatorial interpretation of e_i and p_{n-i} in terms of special Young tableaux.

Definition 2.9. Let G be a simple graph. A coloring of the vertices $\kappa : V(G) \to \mathbb{Z}_{\geq 1}$ is *proper* if $\kappa(a) \neq \kappa(b)$ for adjacent vertices a and b. The *chromatic symmetric function* is $X_G(x_1, x_2, \ldots) := \sum_{\kappa \text{ proper}} \prod_{v \in V(G)} x_{\kappa(v)}$. This is clearly symmetric.

For a composition $I \vDash n$, there is a unique partition $\rho(I) \vdash n$ which consists of the parts of I by reordering. Write $e_I := e_{\rho(I)}$.

(1) [10 pt] Let P_n be the path graph on n vertices, that is, its vertices can be labeled $v_1, \ldots, v_n$ where v_i is connected with v_{i+1} for $i \in [n-1]$. Show that $X_{P_n} = \sum_{I \vDash n} w_I e_I$ where

$$w_I = i_1(i_2 - 1)(i_3 - 1) \cdots (i_k - 1) \text{ if } I = (i_1, \ldots, i_k).$$

(2) [10 pt] Let C_n be the cycle graph on n vertices, that is, its vertices can be labeled $v_1, \ldots, v_n$ where v_i is connected with v_{i+1} for $i = 1, \ldots, n$ and $v_{n+1} := v_1$. Show that $X_{C_n} = \sum_{I \vDash n} (i_1 - 1) w_I e_I$.

3. Coxeter groups

Definition 3.1. A *Coxeter system* (W, S) is a group W (possibly infinite!) and a finite set $S \subset W$ of generators of W, for which W admits a presentation ("Coxeter presentation") of a very particular form. Namely, there must be a matrix $(m_{st})_{s,t \in S}$ satisfying $m_{ss} = 1$ for each $s \in S$, and $m_{st} = m_{ts} \in \{2, 3, \ldots\} \cup \{\infty\}$ for $s \neq t \in S$, such that

$$W = \langle s \in S | (st)^{m_{st}} = \text{id for any } s, t \in S \text{ with } m_{st} < \infty \rangle .$$

When $m_{st} = \infty$, there is no corresponding relation between s and t. The elements of S are often called *simple reflections* and the *rank* of the Coxeter system (W, S) is defined as $|S|$.

Definition 3.2. For each $w \in W$, one can write $w = s_1 \cdots s_k$ for some $s_1, \ldots, s_k \in S$. The sequence $(s_1, \ldots, s_k)$ is called an *expression* or a *word* for w. We use a notational shorthand $\underline{w}$ to denote the sequence $(s_1, \ldots, s_k)$, when the product $s_1 \cdots s_k$ is equal to w. That is, the notation $\underline{w}$ indicates both an element $w \in W$ and a particular choice of expression for w. The *length* of w, denoted by $\ell(w)$, is the minimal k for which w admits an expression $(s_1, \ldots, s_k)$. Any expression for w with this minimal length $\ell(w)$ is called a *reduced expression*.

The following theorem is crucial in the theory of Coxeter groups.

Theorem 3.3 (Exchange Condition)**.** *Let $\underline{w} = (s_1, s_2, \ldots, s_k)$ be a reduced expression for w, and $t \in S$. If $\ell(wt) < \ell(w)$ then there exists i such that $1 \leq i \leq k$ and $wt = s_1 s_2 \cdots \widehat{s_i} \cdots s_k$.*

Let $(s_{i_1}, \ldots, s_{i_k})$ and $(s_{j_1}, \ldots, s_{j_k})$ be two arbitrary expressions of the same length. If we can apply a sequence of braid relations to obtain $(s_{j_1}, \ldots, s_{j_k})$ from $(s_{i_1}, \ldots, s_{i_k})$ we say that they are related by braid relations. The following is a beautiful theorem.

Theorem 3.4 (Matsumoto's Theorem)**.** *Any two reduced expressions for $w \in W$ are related by braid relations.*

(1) [5 pt] Give a realization of the dihedral group D_{2n} (the symmetric group of the regular n-gon) as a Coxeter system.

(2) [5 pt] Find (and prove!) a Coxeter presentations of 3 generators for the dihedral group D_{12} (the symmetric group of the regular hexagon), which has 12 elements.

(3) **5 pt** Show that any Coxeter group W admits a sign representation: an action of W on $\mathbb{R}$ where each simple reflection acts by -1. Deduce that $\ell(ws) \neq \ell(w)$, for any $w \in W$ and $s \in S$.

(4) **5 pt** Using the exchange condition to show that for $w \in W$, the set $\{s \in S \mid \ell(ws) < \ell(w)\}$ is equal to the set $\{s \in S \mid w$ admits a reduced expression ending in $s\}$, which is called the *right descent set*.

(5) **10 pt** Using the exchange condition and Matsumoto's theorem to show that *the word problem in a Coxeter group is solvable*, that is, there is an algorithm to determine whether two words in the generators represent the same element.

4. Combinatorial Hodge theory

We assume some basic knowledge of $\mathfrak{sl}_2$-representations in this section.

Definition 4.1. Fix a finite-dimensional graded real vector space

$$H = \bigoplus_{i \in \mathbb{Z}} H^i$$

and a symmetric non-degenerate graded bilinear form

$$\langle -, - \rangle : H \times H \longrightarrow \mathbb{R}.$$

By "graded" we mean that $\langle H^i, H^j \rangle = 0$ if $i \neq -j$. It is immediate that $\langle -, - \rangle$ induces an isomorphism between H^{-i} and $(H^i)^*$. We say that the graded vector space H satisfies *Poincaré duality*. Thus, if $b_i = \dim(H^i)$ (the i-th "Betti number" of H), then $b_i = b_{-i}$ for all $i \in \mathbb{Z}$. Our convention is such that the mirror of Poincare duality is in degree zero.

Definition 4.2. We say a degree two linear map

$$\mathrm{L} : H^i \longrightarrow H^{i+2}$$

is a *Lefschetz operator* if $\langle La, b \rangle = \langle a, Lb \rangle$ for all $a, b \in H$. If L is a Lefschetz operator, then it is said to satisfies *hard Lefschetz* if for all $i \geq 0$,

$$\mathrm{L}^i : H^{-i} \longrightarrow H^i$$

is an isomorphism.

Definition 4.3. Let L be a Lefschetz operator. For each $i \geq 0$, define the *Lefschetz form* on H^{-i} with respect to L as

$$(a, b)_L^{-i} = \langle a, L^i b \rangle$$

for $a, b \in H^{-i}$.

We will use the notation $H^{\min}$ to denote the nonzero graded component of H of minimal degree, which is well defined as long as $H \neq 0$.

Definition 4.4. For all $i \geq 0$ set

$$P_L^{-i} = \ker\left(L^{i+1}\right) \cap H^{-i},$$

which is called a *primitive subspace*. Assume $H^{\text{odd}} = 0$ or $H^{\text{even}} = 0$ and that L is a Lefschetz operator satisfying hard Lefschetz. We say that $(H, \langle -, - \rangle, L)$ satisfies the *Hodge–Riemann bilinear relations* if the restriction of the Lefschetz form to the primitive subspace

$$(-,-)_L^{\min+2i}\Big|_{P_L^{\min+2i}}$$

is $(-1)^i$-definite.

(1) [5 pt] Show that a degree two operator L on H satisfies hard Lefschetz if and only if there is an action of $\mathfrak{sl}_2(\mathbb{R}) = \langle e, f, h \rangle$ on H with e acting as L and h $\cdot v = iv$ for all $v \in H^i$. Moreover, show that this $\mathfrak{sl}_2(\mathbb{R})$-action is unique.

(2) [5 pt] If L satisfies hard Lefschetz, show that the betti numbers satisfy

$$\cdots \le b_{-4} \le b_{-2} \le b_0 \ge b_2 \ge b_4 \ge \cdots, \quad \cdots \le b_{-3} \le b_{-1} = b_1 \ge b_3 \ge \cdots.$$

(3) [5 pt] Assume $H^{\text{odd}} = 0$. Find a formula for the signature of the Lefschetz form in degree zero in terms of the betti numbers of H.

(4) [5 pt] Suppose that $(H, \langle -, - \rangle, L)$ satisfies Hodge-Riemann and K is an L-invariant graded subspace of H satisfying Poincaré duality. Show that

$$(K, \langle -, - \rangle_K, L|_K)$$

satisfies Hodge–Riemann (up to a sign). In particular, the restriction of the Lefschetz form from H^{-i} to K^{-i} is non-degenerate, for each $i \ge 0$.

(5) [10 pt] This exercise will explore $H^*(\text{Gr}(2,4), \mathbb{R})$, the cohomology of the Grassmannian of 2-planes in $\mathbb{C}^4$, using a combinatorial model. Let $P(2,4)$ denote the set of Young diagrams which fit inside a 2×2 rectangle. The degree of a partition will be -4 plus twice the number of boxes; for example the partition $(2,1)$ has degree $+2$. Two Young diagrams are complementary if one can be rotated 180 degrees in order to fill the full 2×2 rectangle with the other: for example $(2,1)$ and $(1,0)$ are complementary. Let H be the graded vector space with basis $\{v_\lambda\}_{\lambda \in P(2,4)}$. Place a symmetric bilinear pairing on H, where $\langle v_\lambda, v_\mu \rangle = 1$ if λ and μ are complementary, and is equal to zero otherwise. Place an operator $L : H^i \to H^{i+2}$ on this space, where $Lv_\lambda = \sum_\mu v_\mu$ is the sum over the Young diagrams $\mu \in P(2,4)$ obtained from λ by adding one box.

Prove that L is a Lefschetz operator satisfying hard Lefschetz and the Hodge–Riemann bilinear relations. (Hint: Compute a basis for each primitive subspace.)

5. Plücker relations and Gröbner bases

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring of n variables over $\mathbb{C}$.

Definition 5.1. A term order $<$ is a total order on the monomials of R such that it is

- multiplicative: $x^{\mathbf{b}} := \prod_{i=1}^n x_i^{b_i} < x^{\mathbf{c}}$ if and only if $x^{\mathbf{a}+\mathbf{b}} < x^{\mathbf{a}+\mathbf{c}}$;
- artinian: $1 < x^{\mathbf{a}}$.

Given a polynomial $f = \sum_{\mathbf{a}} c_{\mathbf{a}} x^{\mathbf{a}}$, the leading term $\text{in}_<(f) = c_{\mathbf{a}} x^{\mathbf{a}}$ if $x^{\mathbf{a}}$ is the largest under $<$ among monomials appearing in f with non-zero coefficient.

Definition 5.2. Fix a term order $<$, the *initial ideal* of an ideal $I \subset R$ is

$$\text{in}_<(I) := \langle \text{in}_<(f) : f \in I \rangle.$$

Suppose $I = \langle f_1, \dots f_m \rangle$. Then $\{f_1, \dots, f_m\}$ is said to be a *Gröbner basis* if

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(f_i) : i \in [m] \rangle.$$

A Gröbner basis $\{f_1, \dots, f_m\}$ is *reduced* if each $\mathrm{in}_<(f_i)$ has coefficient 1 and the only monomial appearing anywhere in $\{f_1, \dots, f_m\}$ that is divisible by $\mathrm{in}_<(f_i)$ is $\mathrm{in}_<(f_i)$ itself.

(1) If $\mathrm{in}_<(I) = \langle \mathrm{in}_<(f_i) : i \in [m] \rangle$, then $I = \langle f_1, \dots f_m \rangle$.

5 pt

(2) Every ideal $I \subset R$ has a finite Gröbner basis for every term order.

5 pt

(3) For every term order $<$, there is a unique reduced Gröbner basis.

5 pt

Definition 5.3. For a term order $<$ and an ideal $I \subset R$, define the *standard monomials* of R/I to be the monomials that are not in the initial ideal $\mathrm{in}_<(I)$.

(1) Show that the set of standard monomials is a $\mathbb{C}$-vector space basis of R/I.

5 pt

(2) Find a characterization of all monomial orders on $\mathbb{C}[x_1, x_2]$. Generalize your result to $R = \mathbb{C}[x_1, \dots, x_n]$.

5 pt

The two most frequently used monomial orders are the purely lexicographical order "$<_{\mathrm{lex}}$" and the reverse lexicographical order "$<_{\mathrm{rlex}}$". These are defined as follows. We assume that an order is given on the variables, say, $x_1 > x_2 > \dots > x_n$. We then put $x^\alpha <_{\mathrm{lex}} x^\beta$ if there exists $i \in [n]$, such that $\alpha_j = \beta_j$ for all $j < i$, and $\alpha_i < \beta_i$. In contrast to "$<_{\mathrm{lex}}$", the reverse lexicographic order "$<_{\mathrm{revlex}}$" is a linear extension of the natural grading on R. We define $x^\alpha <_{\mathrm{revlex}} x^\beta$ if $\sum_i \alpha_i < \sum_i \beta_i$ or if $\sum_i \alpha_i = \sum_i \beta_i$ and there exists $i \in [n]$ such that $\alpha_j = \beta_j$ for all $j > i$ and $\alpha_i > \beta_i$.

Let $R = \mathbb{C}[x_{1,1}, \dots, x_{k,n}]$ be the ring of polynomial functions on $\mathrm{Mat}_{k \times n}$ the space of $k \times n$ matrices (we assume $k \leq n$ throughout this section).

Definition 5.4. For $\mathbf{a} = \{a_1 < \dots < a_k\} \in \binom{[n]}{k}$ a k-element subset of $[n]$, denote $[\mathbf{a}]$ the determinant of the $k \times k$ submatrix of $\mathrm{Mat}_{k \times n}$ with column index set $\mathbf{a}$. We extend the notation of $[\mathbf{a}]$ to all sequences

$$\mathbf{a} = (a_1, \dots, a_k) \in [n]^k$$

where we use the convention that for any permutation $\sigma \in S_n$,

$$[a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k)}] = (-1)^{\ell(\sigma)} \cdot [a_1, \dots, a_k]. \tag{1}$$

In particular, this implies that $[\mathbf{a}] = 0$ if $a_i = a_j$ for some $i \neq j \in [n]$.

Let $S \subseteq R$ be the subalgebra generated by the set $\{[\mathbf{a}] : \mathbf{a} \in \binom{[n]}{k}\}$. Then S can be presented as a quotient of polynomial ring $\mathbb{C}[[\mathbf{a}] : \mathbf{a} \in \binom{[n]}{k}]/\mathcal{J}$ where $\mathcal{J}$ is generated by the following *straightening relations*:

Definition 5.5. Let $s \in [k], \alpha \in \binom{[n]}{s-1}, \beta \in \binom{[n]}{k+1}$ and $\gamma \in \binom{[n]}{k-s}$ where elements in α, β, γ are in increasing order. The straightening relation attached to α, β, γ is

$$\sum_{I \in \binom{[k+1]}{s}} (-1)^{\mathrm{sgn}(I)} [\alpha_1, \ldots, \alpha_{s-1}, \beta_{i'_1}, \ldots, \beta_{i'_{k-s+1}}] \cdot [\beta_{i_1}, \ldots, \beta_{i_s}, \gamma_1, \ldots, \gamma_{k-s}]$$

where $I = \{i_1 < \cdots < i_s\} \subset [k+1]$, $\{i'_1 < \cdots < i'_{k-s+1}\} := [k+1] \setminus I$, and $\mathrm{sgn}(I) := \sum_{j=1}^{s} i_j - \binom{s+1}{2}$.

Consider the ordering on the variables $[\mathbf{a}]$ with $\mathbf{a} = \{a_1 < \cdots < a_k\}$ given by $[\mathbf{a}] < [\mathbf{b}]$ if there exists $i \in [k]$ such that $a_j = b_j$ for all $j < i$ and $a_i < b_i$.

(1) 10 pt For a SSYT T of shape (d^k) with entries in $[n]$, define $[T] := \prod_{i=1}^{d} [T_i] \in \mathbb{C}[[\mathbf{a}]]$, where $T_i \in \binom{[n]}{k}$ is the set of entries in the i-th column of T. Show that every degree d homogeneous polynomial in $\mathbb{C}[[\mathbf{a}]]$ can be written as $\mathbb{C}$-linear combinations of $\{[T] : T \in \mathrm{SSYT}(d^k)\}$.

(2) 10 pt Show that the straightening relations form a Gröbner basis of $\mathcal{J}$ with respect to the term order "$<_{\mathrm{revlex}}$" and describe the initial ideal.

6. Ehrhart Theory

Definition 6.1. A *convex d-polytope* is the convex hull of finitely many points in $\mathbb{R}^d$. These points are called the *vertices* of P. When all of the vertices of P have integer coordinates, we call it a *lattice polytope*.

Definition 6.2. Let P be the convex hull of $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^d$. For any linear function $a(x) = a_1 x_1 + \cdots + a_d x_d$ on $\mathbb{R}^d$, let $h = \max(a(v_1), \ldots, a(v_m))$. The *supporting hyperplane* of P at a is $H_{a,P} = \{x \in \mathbb{R}^d \mid a(x) = h\}$. The *supporting face* of P at a is $F_{a,P} := P \cap H_{a,P}$. The set of faces of P is the set of nonempty supporting faces of P for all possible a. The *facets* of P are exactly the faces of dimension $d - 1$.

Let $t \in \mathbb{R}$. The t-dilate of P is $tP := \{t \cdot p \mid p \in P\}$, where the point $t \cdot p$ is obtained from the point p by multiplying all the coordinates of p by the real number t.

Definition 6.3. The *Ehrhart function* (in the variable t) of a d-polytope P is $L_P(t) = |(tP) \cap \mathbb{Z}^d|$, that is, the number of lattice points (integer vectors) in the t-dilate of P.

Theorem 6.4. *The Ehrhart function $L_P(t)$ of a lattice polytope P is a polynomial in t.*

(1) 2 pt Compute the Ehrhart polynomial of the *hypercube*, which is the convex hull of all binary vectors in $\mathbb{R}^d$. A binary vector has coordinate either 0 or 1.

(2) 2 pt Compute the Ehrhart polynomial of the *standard d-simplex*, which is the convex hull of $d + 1$ standard unit vectors.

(3) 3 pt Let P° be the *relative interior* of P, which is P with all boundaries/facets removed. Show by inclusion-exclusion that $|tP^\circ \cap \mathbb{Z}^d|$ is also a polynomial in t.

(4) Compute the Ehrhart polynomial of the relative interior of the standard d-simplex,
3 pt and show that it satisfies the *Ehrhart–Macdonald reciprocity:* $L_P(-t) = (-1)^d L_{P^\circ}(t)$.

A *d-simplex* is the convex hull of $d+1$ affinely independent points. A d-simplex is *unimodular* if it is lattice isomorphic to the standard d-simplex. (Two polytopes P, Q are *lattice isomorphic*, denoted $P \approx Q$, if there exists an affine isomorphism A such that $A|_{\mathbb{Z}^n}$ is a bijection onto $\mathbb{Z}^n$ and $AP = Q$.)

Definition 6.5. A *polytopal complex* is a finite collection C of polytopes in some $\mathbb{R}^d$ with the following two properties: If $P \in C$ and F is a face of P, then $F \in C$; and if $P, Q \in C$ then $F = P \cap Q \in C$ and F is common face of both P and Q. The polytopes in C are also called faces. A (geometric) *simplicial complex* is a polytopal complex in which all faces are simplices. An *abstract simplicial complex* is a set Δ of subsets of a finite set V, such that Δ is closed under taking subsets. A *subdivision* of a polytopal complex C is a polytopal complex C' such that $\cup C = \cup C'$ and every face of C' is contained in a face of C. A *triangulation* is a subdivision in which all faces are simplicies. A *unimodular triangulation* is a triangulation in which all simplices are unimodular.

Definition 6.6. Let Δ be an abstract simplicial complex on the ground set $[n]$. Let k be some field. The *Stanley–Reisner ring* of Δ is the quotient $k[\Delta] := k[x_1, \dots, x_n]/I_\Delta$ where I_Δ is the *Stanley–Reisner ideal*, generated by the monomials corresponding to non-faces of Δ

$$I_\Delta = \langle x_{i_1} \cdots x_{i_r} \mid \{i_1, \dots, i_r\} \notin \Delta \rangle.$$

When P is a polytope, we use $k[P]$ to denote the Stanley–Reisner ring of the polytopal complex associated with P.

Definition 6.7. Let $R = k[x_1, \dots, x_n]/I = \bigoplus_{t \geq 0} R_t$ be a polynomial ring, where R_t is the k-vector space spanned by all the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_1 + \cdots + \alpha_n = t$. The *Hilbert function/polynomial* of R is $H_R(t) := \dim_k R_t$.

(1) Let P be a d-polytope, and let f_i be the number of i-dimensional faces in P, for
3 pt $0 \leq i \leq d$. Show that the Hilbert polynomial of $k[P]$ is equal to $\sum_{i=0}^d f_i \binom{t-1}{i}$.

(2) Let Δ be a unimodular triangulation of a d-polytope P. Prove that the Ehrhart
5 pt polynomial of P equals the Hilbert function of the Stanley–Reisner ring of Δ, i.e.,

$$L_P(t) = H_{k[\Delta]}(t).$$

(3) If P, Q are both d-polytopes, show that $L_{P \cup Q} = L_P + L_Q - L_{P \cap Q}$. If $P \approx Q$, show
2 pt that $L_P = L_Q$.

(4) Prove that the Ehrhart function is a polynomial, and prove the Ehrhart–Macdonald
5 pt reciprocity for any lattice polytope.

7. q-analogues

We say $f(q)$ is a *q-analogue* (or a *q-deformation)* of A if $\lim_{q \to 1} f(q) = A$. Some classical examples of q-analogues include:

(1) The q-integer $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ is a q-analogue of $n \in \mathbb{N}$.
(2) The q-factorial $[n]_q! := [1]_q [2]_q \cdots [n]_q$ is a q-analogue of $n!$.

(3) The q-binomial coefficient $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ is a q-analogue of $\binom{n}{k}$.
(4) The Hecke algebra of a Weyl group W is a q-analogue of the group algebra of W.
(5) The flag variety over a finite field $GL_n(\mathbb{F}_q)/B$ is a q-analogue of S_n.

Most q-analogues often come with nice enumerative and geometric properties. If a number a count the cardinality of the set A and $f(q)$ is a q-analogue of a, then $f(q)$ is often the generating function of some natural statistic on A. Moreover, there often exist an algebraic variety X such that the q analogue $f(q)$ of a count the size of $X(\mathbb{F}_q)$. They sometime also appear as Hilbert series of some representations.

(1) [3 pt] Let $R_{k,n}$ denote the set of Young diagrams that fit inside the $k \times (n-k)$ rectangle. Note that $\#R_{k,n} = \binom{n}{k}$. Find a statistic $st : R_{k,n} \to \mathbb{N}$ on $R_{k,n}$ such that

$$\binom{n}{k}_q = \sum_{a \in R_{k,n}} q^{st(a)}.$$

(2) [3 pt] Recall that $n! = \#S_n$. Find a statistic $st : S_n \to \mathbb{N}$ on S_n such that

$$[n]_q! = \sum_{w \in S_n} q^{st(w)}.$$

(3) [5 pt] The (complete) flag variety $FL(F)$ over a field F is defined to be

$$FL_n(F) := \{V_1 \subset V_2 \subset \cdots \subset V_n \subset F^n | \dim V_i = i\}.$$

Show that $|FL_n(\mathbb{F}_q)| = [n]_q!$.

(4) [5 pt] A special case of the flag variety is the Grassmanian, defined as follows.

$$\mathrm{Gr}_k(F^n) = \{V \subset F^n \mid \dim(V) = k\}.$$

Show that $|\mathrm{Gr}_k(\mathbb{F}_q^n)| = \binom{n}{k}_q$.