Symmetric structures in the strong Bruhat order

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Overview

1. Background on the (strong) Bruhat order
2. Self-dual intervals
3. Billey-Postnikov decompositions
4. The automorphism group of the Bruhat graph

Background on the (strong) Bruhat order

- The symmetric group $\mathfrak{S}_n$ is generated by the **simple transpositions** $S = \{s_i = (i\ i{+}1) \mid i = 1, 2, \ldots, n{-}1\}$ with relations

$$\begin{cases} s_i^2 = \mathrm{id}, \\ s_i s_j = s_j s_i \text{ for } |i-j| > 1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \end{cases}$$

- The **Coxeter length** $\ell(w)$ of a permutation $w \in \mathfrak{S}_n$ is the smallest ℓ such that $w = s_{i_1} \cdots s_{i_\ell}$.
- Such an expression is called a **reduced word** of w.
- The **reflections** $T = \{t_{ij} = (i\ j) \mid 1 \leq i < j \leq n\}$ are the conjugates of simple transpositions.
- The (strong) **Bruhat order** is generated by

$$w \leq wt_{ij} \text{ if } \ell(w) < \ell(wt_{ij}).$$

- The **cover relations** of the Bruhat order are given by

$$w \lessdot wt_{ij} \text{ if } \ell(w) + 1 = \ell(wt_{ij}).$$

Background on the (strong) Bruhat order

- The Bruhat order encodes topological information of the full **flag variety** $\mathrm{Fl}_n = \{\emptyset = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n \mid \dim V_i = i\}$.
- The flag variety admits a **Bruhat decomposition** $Fl(\mathbb{C}^n) = \coprod_{w \in \mathfrak{S}_n} \Omega_w$ into **open Schubert cells**.
- The **Schubert varieties** are $X(w) := \overline{\Omega_w}$.
- The Bruhat order is given by $u \leq w$ if $X(u) \subset X(w)$.

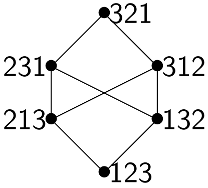


Figure: The (strong) Bruhat order on $\mathfrak{S}_3$

Smooth permutations

Definition

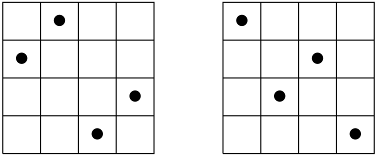
A permutation w is **smooth** if its corresponding Schubert variety X_w is smooth.

The following result is very classical.

Theorem (Lakshmibai-Sandhya 1990, Carrell 1994)

The followings are equivalent for $w \in \mathfrak{S}_n$:

1. *the interval $[\mathrm{id}, w]$ in the Bruhat order is rank-symmetric;*
2. *w avoids 3412 and 4231;*
3. *w is smooth.*



Self-dual permutations

A poset P is **self-dual** if it admits an order-reversing involution.

Theorem (Gaetz and G. 2020)

The followings are equivalent for $w \in \mathfrak{S}_n$:

1. *the bipartite graphs Γ_w and Γ^w are isomorphic;*
2. *w avoids the smooth patterns 3412 and 4231 as well as 34521, 45321, 54123, and 54312;*
3. *w is polished;*
4. *$[\mathrm{id}, w]$ in the Bruhat order is self-dual.*

We call these permutations **self-dual**.
This is a strictly stronger condition than smoothness.

Self-dual permutations

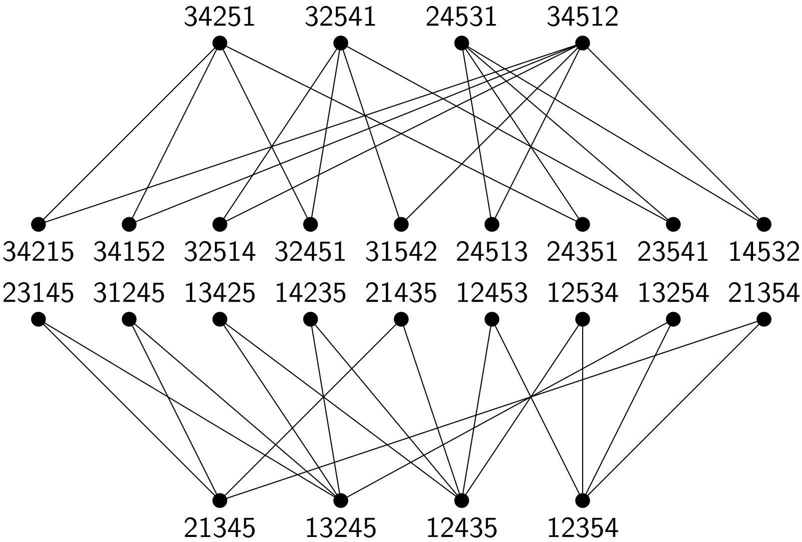
Recall condition (1): the bipartite graphs Γ_w and Γ^w are isomorphic.

For $w \in \mathfrak{S}_n$ and $k = 0, 1, \ldots, \ell(w)$, let

$$P_k^w := \{u \leq w \mid \ell(u) = k\}.$$

Let Γ_w and Γ^w be the bipartite graphs on $P_1^w \sqcup P_2^w$ and $P_{\ell(w)-1}^w \sqcup P_{\ell(w)-2}^w$ respectively with edges given by cover relations of the Bruhat order.

Example of Γ_w and Γ^w for $w = 34521$



Comparison with an observation of Billey and Postnikov

Condition (1) says that to check self-duality, one needs to only look at **two** ranks from top and from bottom.

What about rank-symmetry?

Theorem (Gaetz and G. 2022+, conjectured by Billey-Postnikov 2005)

For $w \in \mathfrak{S}_n$, if

$$|P_k^w| = |P_{\ell(w)-k}^w| \text{ for } k \leq n-3,$$

then w is smooth. Moreover, the bound $n-3$ is tight.

In particular, this bound is linear, not constant.

Self-dual permutations

Recall condition (3): the permutation w is **polished**.

Let (W, S) be a finite Coxeter system where W is the Coxeter group and S is the set of simple generators.

Definition (Gaetz and G. 2020)

We say that $w \in W$ is **polished** if there exist pairwise disjoint subsets $S_1, \ldots, S_k \subseteq S$ such that each S_i is a connected subset of the Dynkin diagram and coverings $S_i = J_i \cup J_i'$ for $i = 1, \ldots, k$ with $J_i \cap J_i'$ totally disconnected so that

$$w = \prod_{i=1}^{k} w_0(J_i) w_0(J_i \cap J_i') w_0(J_i')$$

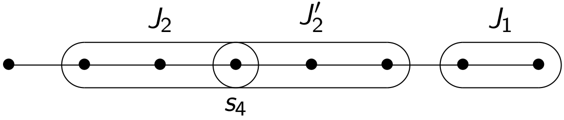
where the product is taken from left to right as $i = 1, 2, \ldots, k$.

Example of polished elements

The following element with $k = 2$, $J_1 = \{s_7, s_8\}$, $J_1' = \emptyset$, $J_2 = \{s_2, s_3, s_4\}$, $J_2' = \{s_4, s_5, s_6\}$, and multiplication in the order of

$$\begin{aligned} w = & \; w_0(J_1)w_0(J_2)w_0(J_2 \cap J_2')w_0(J_2') \\ = & \; 123456987 \cdot 154326789 \cdot 123546789 \cdot 123765489 \\ = & \; 154963287 \end{aligned}$$

is a polished element.



A top-heaviness result

The Bruhat interval $[\mathrm{id}, w]$ is top-heavy.

Theorem (Björner-Ekedahl 2009)

For $w \in \mathfrak{S}_n$, $|P_k^w| \leq |P_{\ell(w)-k}^w|$ for $k \leq \ell(w)/2$.

Let $\mathrm{udeg}_w(u)$ and $\mathrm{ddeg}_w(u)$ denote the up-degree and down-degree of u inside the interval $[\mathrm{id}, w]$.

Theorem (Gaetz and G. 2020)

Let $w \in \mathfrak{S}_n$ be smooth, then

$$\max_{u \in P_1^w} \mathrm{udeg}_w(u) \leq \max_{u \in P_{\ell(w)-1}^w} \mathrm{ddeg}_w(u),$$

with equality if and only if $[\mathrm{id}, w]$ is self-dual.

Background on Billey-Postnikov decomposition

For $w \in \mathfrak{S}_n$, its **right descents** and **left descnets** are

$$D_R(w) := \{i \mid w(i) > w(i{+}1)\}, \quad D_L(w) := \{i \mid w^{-1}(i) > w^{-1}(i{+}1)\}.$$

For $w \in \mathfrak{S}_n$, its **support** is

$$\mathrm{Supp}(w) := \{i \mid s_i \text{ appears in any/all reduced words of } w\}.$$

Definition (Parabolic decomposition)

For $J \subset S$, there is a length-additive factorization $w = w^J w_J$ such that $\mathrm{Supp}(w_J) \subset J$ and $D_R(w^J) \subset S \setminus J$.

Definition (Billey-Postnikov decomposition)

A parabolic decomposition $w = w^J w_J$ is called **Billey-Postnikov** if

$$\mathrm{Supp}(w^J) \cap J \subset D_L(w_J).$$

Background on Billey-Postnikov decomposition

Theorem (Richmond-Slofstra 2016)

The followings are equivalent for $w \in \mathfrak{S}_n$ and $J \subset S$:

- $\mathrm{Supp}(w^J) \cap J \subset D_L(w_J)$ *(the definition);*
- *the multiplication* $([\mathrm{id}, w^J] \cap W^J) \times [\mathrm{id}, w_J] \to [\mathrm{id}, w]$ *is a bijection;*
- *the Poincaré polynomials satisfy* $P_w(q) = P^J_{w^J}(q) P_{w_J}(q)$*;*
- w_J *is the maximum element of* $W_J \cap [\mathrm{id}, w]$*;*
- *the natural projection* $X(w) \to X^J(w^J)$ *restricted from* $G/B \to G/P_J$ *is a fiber bundle with fiber isomorphic to* $X(w_J)$*.*

Non-example: $w = 312$, $J = \{2\}$

- $w^J = 312$ and $w_J = 123$ so $\mathrm{Supp}(w^J) \cap J = \{2\} \not\subseteq D_L(w_J) = \emptyset$.
- $([\mathrm{id}, w^J] \cap W^J) = \{123, 213, 312\}$, $[\mathrm{id}, w_J] = \{id\}$, while $[\mathrm{id}, w] = \{123, 213, 132, 312\}$.
- the maximum element of $W_J \cap [\mathrm{id}, w] = 132$ which is not $w_J = 123$.

Fun with Billey-Postnikov decomposition

Theorem (Gaetz and G. 2022+)

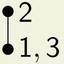
Suppose that a permutation $w \in \mathfrak{S}_n$ is J-BP and K-BP, for some $J, K \subset [n-1]$. Then w is $(J \cup K)$-BP and $(J \cap K)$-BP.

Conjecture

The above holds for any Coxeter groups.

By the fundamental theorem of finite distributive lattices, this says that for a permutation $w \in \mathfrak{S}_n$, there exists a poset P_w on S such that $\{J \mid w \text{ is } J\text{-BP}\} = \mathcal{L}(P_w)$, the lattice of order ideals of P_w.

Examples

- $w = 3412$, $J \in \{\emptyset, \{2\}, \{1,2,3\}\}$ so $P_w =$ 

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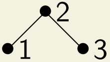
- $w = 4231$, $J \in \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{1,2,3\}\}$ so $P_w =$ 

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Fun with Billey-Postnikov decomposition

Definition (Richmond-Slofstra 2016)

Let $K \subset J$. We say that w^K is (K, J)-**BP** if the projection

$$\pi : X^K(w^K) \to X^J(w^J) \quad \text{from} \quad \pi : G/P_K \to G/P_J$$

is a fiber bundle with fiber isomorphic to $X(w_J^K)$.

This is a generalization of the previous story where $K = \emptyset$.

Proposition (Gaetz and G. 2022+)

If w is (K, J)-BP, then w is $(K \setminus L,\ J \setminus L)$-BP for $L \subset K \subset J$.

Fun with Billey-Postnikov decomposition

Proposition (Gaetz and G. 2022+)

If w is (K, J)-BP, then w is $(K \setminus L,\ J \setminus L)$-BP for $L \subset K \subset J$.

Proof.

We have the following pullback **and** pushout diagram, for $A, B \subset S$.

$$\begin{array}{ccc} X^{A\cap B}(w^{A\cap B}) & \longrightarrow & X^{A}(w^{A}) \\ \downarrow & & \downarrow \\ X^{B}(w^{B}) & \longrightarrow & X^{A\cup B}(w^{A\cup B}) \end{array}$$

Pullback preserves fiber bundles. □

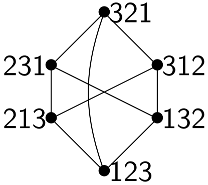
Remark

One easily constructs examples in this way to show that pushout does not preserve fiber bundles.

Background on the (undirected) Bruhat graph

Definition

The **undirected Bruhat graph** Γ has vertex set $\mathfrak{S}_n$ and edges $w \sim wt$ for reflections T. Let $\Gamma(u, v)$ be its restriction to a Bruhat interval $[u, v]$.



Theorem (Lakshmibai-Sandhya 1990, Carrell 1994)

A permutation $w \in \mathfrak{S}_n$ is smooth if and only if the undirected Bruhat graph $\Gamma(\mathrm{id}, w)$ is regular.

Background on the directed Bruhat graph

Definition

The **directed Bruhat graph** $\widehat{\Gamma}$ has vertex set $\mathfrak{S}_n$ and edges $w \to wt$ for reflections T if $\ell(w) < \ell(wt)$.

The directions of the edges, and sometimes additional edge labels, are centrally important in geometrical context.

The directed Bruhat graph has very few symmetry.

Theorem (Waterhouse 1989)

Let W be an irreducible Coxeter group that is not dihedral. Then $\mathrm{Aut}(\widehat{\Gamma})$*, or equivalently* $\mathrm{Aut}((W, \leq))$*, is generated by the graph automorphism of the Dynkin diagram and the group inversion map on W.*

Background on the (undirected) Bruhat graph

Definition

An **automorphism** φ of a graph $G = (V, E)$ is a bijection $\varphi : V \to V$ on the vertices such that $(a, b) \in E$ if and only if $(\varphi(a), \varphi(b)) \in E$. Let $\mathrm{Aut}(G)$ be the **automorphism group** of the graph G.

The undirected Bruhat graph has many more automorphisms.

Question

Can we describe $\mathrm{Aut}(\Gamma(u, v))$, *or* $\mathrm{Aut}(\Gamma(\mathrm{id}, w))$*?*

There are some automorphism $\varphi \in \mathrm{Aut}(\Gamma(\mathrm{id}, w))$:

- multiplication on the left by s_i, where $i \in D_L(w)$;
- multiplication on the right by s_i, where $i \in D_R(w)$;
- **middle multiplication**.

Middle multiplication

Proposition (Gaetz and G. 2022)

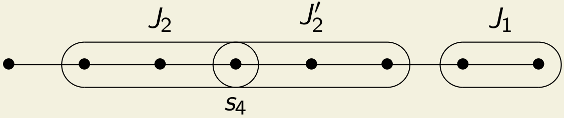
Suppose $w = w^J w_J$ is a Billey-Postnikov decomposition, and $\mathrm{Supp}(w^J) \cap \mathrm{Supp}(w_J) = \{s\} \subset J$, *then the middle multiplication map*

$$\phi : x \mapsto x^J s x_J$$

is an automorphism of the Bruhat graph $\Gamma(\mathrm{id}, w)$.

Recall polished elements

$$w = w_0(J_1) w_0(J_2) w_0(J_2 \cap J_2') w_0(J_2').$$



The identity orbit under $\mathsf{Aut}(\Gamma(\mathrm{id}, w))$

Conjecture (Gaetz and G. 2022)

Let $w \in \mathfrak{S}_n$ *and* $\mathcal{O} = \{\varphi(\mathrm{id}) \mid \varphi \in \mathsf{Aut}(\Gamma(\mathrm{id}, w))\}$ *be the orbit of the identity under graph automorphisms of* $\Gamma(\mathrm{id}, w)$. *Then*

$$\mathcal{O} = [\mathrm{id}, v], \text{ for some } v \leq w.$$

- We have a conjectural formula for v.
- Equivalently, we conjecture that the identity orbit $\mathcal{O}$ is "essentially" generated by left, right and middle multiplications.
- We cannot prove that $\mathcal{O}$ is downwards closed, or it has a unique maximum.
- Not true in other types.

Vertex-transitive elements

Definition (Gaetz and G. 2022)

A permutation $w \in \mathfrak{S}_n$ is **vertex-transitive** if $\mathrm{Aut}(\Gamma(\mathrm{id}, w))$ acts transitively on the vertex set $[\mathrm{id}, w]$.

If w is vertex-transitive, then every vertex in $\Gamma(\mathrm{id}, w)$ has the same degree, meaning that w is smooth.

Theorem (Gaetz and G. 2022)

A permutation w is vertex-transitive if and only if it avoids the smooth patterns 3412 *and* 4231*, as well as* 34521 *and* 54123*.*

- The patterns here form a subset of those avoided by self-dual permutations!
- This is a special case of the identity orbit conjecture.

Background on special matchings

Special matchings are introduced by Brenti-Caselli-Marietti to study the combinatorial invariance conjecture of Kazhdan-Lusztig polynomials.

Definition (Brenti-Caselli-Marietti 2006)

A **special matching** M of a poset P is a perfect matching of the Hasse diagram of P such that $u \lessdot v \Rightarrow M(u) \le M(v)$ for all $u, v \in P$ such that $M(u) \neq v$.

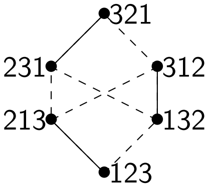


Figure: A special matching on $\mathfrak{S}_3$

Special matchings and Kazhdan-Lusztig polynomials

Definition

The Kazhdan-Lusztig R-**polynomial** is defined via

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } s \in D_R(u), \\ qR_{us,vs}(q) + (q-1)R_{u,vs} & \text{if } s \notin D_R(u), \end{cases}$$

where $R_{u,v}(q) = 0$ if $u \not\leq v$ and $R_{u,u}(q) = 1$.

The following is a long standing conjecture, which is usually refered to as the combinatorial invariance conjecture.

Conjecture (Lusztig, Dyer)

Let (W, S) be a Coxeter system and $u, v \in W$. Then the polynomial $P_{u,v}(q)$ (or equivalently $R_{u,v}(q)$) depends only on the isomorphism type of the interval $[u, v]$ as a poset.

Special matchings and Kazhdan-Lusztig polynomials

Theorem (Brenti-Caselli-Marietti 2006)

Let (W,S) be a Coxeter system and M be a special matching of $[\mathrm{id}, w]$. Then

$$R_{u,w}(q) = q^c R_{M(u),M(w)}(q) + (q^c - 1)R_{u,M(w)}(q)$$

where $c = 1$ if $M(u) \gtrdot u$, and $c = 0$ otherwise.

This theorem solves the combinatorial invariance conjecture of Kazhdan-Lusztig polynomials for **lower intervals** $[\mathrm{id}, w]$:

- special matchings for $[\mathrm{id}, w]$ always exist, and can be classified;
- special matchings only depend on the combinatorial type of the poset;
- special matchings may not exist for general intervals.

Question

Do we have a similar recurrence for general intervals when a special matching exists?

Special matchings and Bruhat automorphisms

Theorem (Gaetz and G. 2022, implicit in Brenti-Caselli-Marietti 2006)

Let $u \leq v$ in any Coxeter group W. Any special matching M of the Hasse diagram of $[u, v]$ is an automorphism of $\Gamma(u, v)$.

We can prove some cases of the converse.

Theorem (Gaetz and G. 2022)

Let $u \leq v$ in a Coxeter group W which is right-angled or the symmetric group. Then any perfect matching of the Hasse diagram of $[u, v]$ that is an aumorphism of $\Gamma(u, v)$ is a special matching.

Question

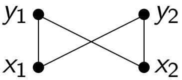
Is this true for general Coxeter groups?

The butterfly structures

Our proof relies heavily on analyzing **butterflies**.

Definition (Gaetz and G. 2022)

Elements $x_1, x_2, y_1, y_2 \in W$ form a **butterfly** if $x_i \lessdot y_j$ for all $i, j \in \{1, 2\}$.



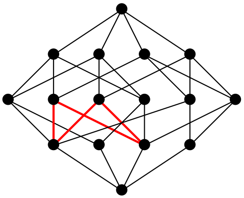
The following technical lemma implies the theorem on the last slide.

Lemma (Gaetz and G. 2022)

Assume that W is a Coxeter group which is right-angled or the symmetric group, and $x_1, x_2, y_1, y_2 \in [u, v]$ form a butterfly. Then there is an element $z \in [u, v]$ with $y_1, y_2 \lessdot z$.

The butterfly structures

It is not true in general that a butterfly in $[u, v]$ has an upper cover in $[u, v]$. The following counterexample is from type F_4.



Conjecture (Gaetz and G. 2022)

Let W be any Coxeter group and $x_1, x_2, y_1, y_2 \in [u, v]$ form a butterfly. Then there exists $z \in [u, v]$ such that $z \lessdot x_1, x_2$ or $z \gtrdot y_1, y_2$.

Lemma (Gaetz and G. 2022)

Let W be of finite simply-laced types and x_1, x_2, y_1, y_2 form a butterfly. Then there exists $z, z' \in W$ such that $z \lessdot x_1, x_2$ and $z' \gtrdot y_1, y_2$.

Thanks

Thank you for listening!