Balanced Shifted Tableaux

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Standard Young tableaux

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0)$ be a partition.

Definition

A **standard Young tableau** of shape λ is a filling of λ using $1, \ldots, |\lambda|$ such that each row and each column form increasing sequences.

For each box (i, j) in its Young diagram, let its **hook** $H_\lambda(i, j)$ consist of all the boxes directly to the right or the bottom of (i, j), including itself.

Theorem (Hook length formula)

The number of standard Young tableaux of shape λ equals

$$f^\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} |H_\lambda(i,j)|}.$$

Balanced tableaux

For a box $(i,j) \in \lambda$, let $\mathrm{rk}_\lambda(i,j)$ be the size of the right arm of $H_\lambda(i,j)$, i.e. the number of boxes to the right of (i,j), including itself.

Definition (Edelman-Greene 1987)

A **balanced tableau** of shape λ is a filling T of λ using $1, \ldots, |\lambda|$ such that $T(i,j)$ is the $\mathrm{rk}_\lambda(i,j)$-th largest entry in its hook.

3	7	4	2
5	8	6	
1	9		

Theorem (Edelman-Greene 1987)

For a partition λ, the number of balanced tableau of shape λ equals the number of standard Young tableaux of shape λ.

A counting problem

Fix λ. Let $\mathrm{rk}_\lambda : \lambda \to \mathbb{Z}_{>0}$ be a function on the boxes in λ.

Define a **balanced tableau with respect to** rk_λ to be a filling T of λ using $1, \ldots, |\lambda|$ such that $T(i,j)$ is the $\mathrm{rk}_\lambda(i,j)$-th largest entry in its hook.

- $\mathrm{rk}_\lambda =$ size of the right arm gives balanced tableaux
- $\mathrm{rk}_\lambda = \mathbf{1}$ gives standard Young tableaux

Question

Given λ, what are all the possible functions rk_λ such that the number of balanced tableaux with respect to rk_λ can be enumerated by the hook length formula?

Importance of Edelman-Greene

The Edelman-Greene insertion algorithm provides a bijection between reduced words of $w \in \mathfrak{S}_n$ and pairs of tableaux (P, Q) such that

- the **insertion tableau** P is increasing in rows and columns, whose reverse reading word is a reduced word of w, and
- the **recording tableau** Q is standard of the same shape.

Theorem (Edelman-Greene 1987)

The number of reduced words of the longest permutation $w_0 \in \mathfrak{S}_n$ equals the number of standard Young tableaux of the staircase shape $(n-1, n-2, \ldots, 1)$.

Theorem (Edelman-Greene 1987)

The Stanley symmetric function F_w is Schur-positive.

Example: $w = s_2 s_1 s_2 s_3 s_2$

$$\begin{array}{c|c|c|c|c|c|c}
P & \emptyset & \begin{array}{|c|}\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|c|}\hline 1 & 2\\\hline 2 & \multicolumn{1}{c}{}\\\cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{}\\\cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 2 & 3 & \multicolumn{1}{c}{}\\\cline{1-2}\end{array} \\
Q & \emptyset & \begin{array}{|c|}\hline 1\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & \multicolumn{1}{c}{}\\\cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1 & 3 & 4\\\hline 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{}\\\cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1 & 3 & 4\\\hline 2 & 5 & \multicolumn{1}{c}{}\\\cline{1-2}\end{array}
\end{array}$$

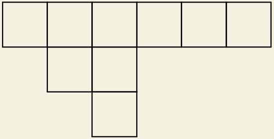
And we see that the reading word of P equals

$$s_2 s_3 s_1 s_2 s_3 = s_2 s_1 s_3 s_2 s_3 = s_2 s_1 s_2 s_3 s_2 = w.$$

Shifted shapes

Let $\lambda = (\lambda_1 > \cdots > \lambda_d)$ be a **strict** partition, which corresponds to a **shifted shape** by shifting the *i*-th row *i* steps to the right.

Example: the shifted shape $\lambda = (6, 2, 1)$



Definition

A **standard Young tableau** of shifted shape λ is a filling of λ using $1, \ldots, |\lambda|$ that is increasing in each row and column.

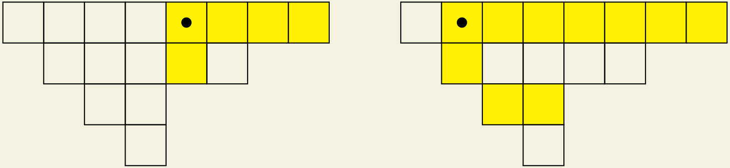
Let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of λ.

Hook length formula for shifted shapes

Let $\lambda = (\lambda_1 > \cdots > \lambda_d)$ be a shifted shape. The hook $H_\lambda(i,j)$ contains:

- boxes to the right and below, if $j \geq 0$;
- boxes to the right and below, and then turn again to the right with a "broken leg", if $j \leq 0$.

Hooks for shifted shapes



Theorem (Hook length formula for shifted shapes)

The number of standard Young tableaux of shifted shape λ equals

$$|\mathrm{SYT}(\lambda)| = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} |H_\lambda(i,j)|}.$$

Balanced Shifted Tableaux?

- If we let $\mathrm{rk}_\lambda(i,j)$ be the size of the right arm of $H_\lambda(i,j)$ and define balanced shifted tableaux analogously, then they are **not** equinumerous to standard shifted tableaux.
- We make a definition for balanced shifted tableaux as close to this idea as possible, and show that they are equinumerous to standard shifted tableaux.

Extended shifted shapes

Definition (Gao, Gao and G. 2022)

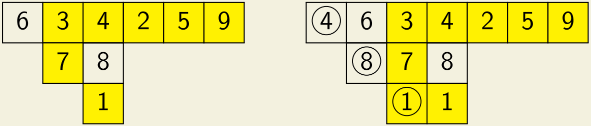
For a filling B of λ, its **extended filling** $\tilde{B}$ is a filling of the extended shape

$$\tilde{\lambda} = \lambda \cup \{(1, \bar{d}), (2, \overline{d-1}), \ldots (d, \bar{1})\}$$

which agrees with B on λ and equals $B(i, 0)$ on the newly added boxes $(i, -(d+1-i))$ for $i = 1, \ldots, d$.

The **extended hook** $\tilde{H}_\lambda(i,j) \subset \tilde{\lambda}$ is the hook $H_\lambda(i,j)$ for $j \geq 0$, and is $H_\lambda(i,j) \cup \{(d+1+j, j)\}$ if $j < 0$.

Example: extended filling and extended hooks



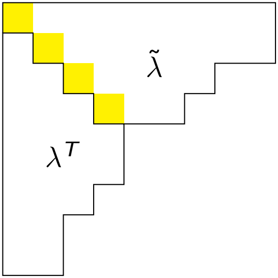
Intuitively, column 0 correspond to roots e_i's, while the extended boxes correspond to roots $2e_i$'s.

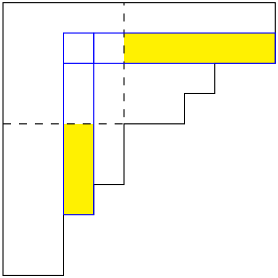
A balanced condition

Define the following rank function

- $\mathrm{rk}_\lambda(i,j) = \#$boxes in row i of $H_\lambda(i,j)$, $j \geq 0$,
- $\mathrm{rk}_\lambda(i,\bar{j}) = \#$boxes with non-negative column index of $H_\lambda(i,\bar{j})$, $j > 0$.

An alternative description of $\mathrm{rk}_\lambda(i,\bar{j})$:





Balanced Shifted Tableaux

Let $\mathrm{rk}_\lambda(i,j)$ be defined as above.

Definition (Gao, Gao and G. 2022)

A **balanced shifted tableau** of shape λ is a filling B of λ using $1,\ldots,|\lambda|$ such that for all $(i,j)\in\lambda$, $B(i,j)$ is the $\mathrm{rk}_\lambda(i,j)$-th largest entry in the extended hook $\tilde{H}_\lambda(i,j)$.

Let $\mathrm{BS}(\lambda)$ be the set of balanced shifted tableaux of shape λ.

Example: a balanced shifted tableau

4	6	3	4	2	5	9
	8	7	8			
		1	1			

Theorem (Gao, Gao and G. 2022)

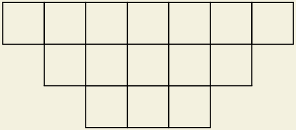
For a shifted shape λ, $|\mathrm{SYT}(\lambda)| = |\mathrm{BS}(\lambda)|$.

Proof sketch

Our proof is bijective, with the following strategy:

- We provide a bijection for the trapezoid $Z(d,r)$.
- For any λ, pad it to $Z(d,r)$ and apply the bijection for $Z(d,r)$.
- This framework is largely the same as the original arguments by Edelman and Greene, with the main difference that double staircases (the type B analogue of staircases) are not enough.

The trapezoid $Z(3,2)$



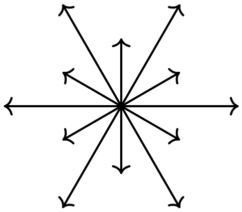
$$\begin{tikzcd}
\mathrm{SYT}(\lambda) \arrow[r, leftrightarrow] & \mathrm{SYT}(Z(d,r))|_\lambda \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{Red}(w^\lambda) \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{BS}(Z(d,r))|_\lambda \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{BS}(\lambda) \\
& \mathrm{SYT}(Z(d,r)) \arrow[r, leftrightarrow] & \mathrm{Red}(w^{(d,r)}) \arrow[r, leftrightarrow] & \mathrm{BS}(Z(d,r)) &
\end{tikzcd}$$

Root systems and Weyl groups

Definition (Root system)

Let $E = \mathbb{R}^d$. A **root system** $\Phi \subset E$ is a finite set of vectors, such that

- Φ spans E;
- for $\alpha \in \Phi$, $k\alpha \in \Phi$ iff $k \in \{\pm 1\}$;
- for $\alpha, \beta \in \Phi$, $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$;
- for $\alpha, \beta \in \Phi$,
$$\sigma_\alpha(\beta) := \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Phi.$$



Root systems and Weyl groups

Let $\Phi \subset E$ be a root system.

- We can partition Φ into **positive roots** Φ^+ and negative roots Φ^-.
- Given $\Phi = \Phi^+ \sqcup \Phi^-$, there is a unique choice of **simple roots** $\Delta = \{\alpha_0, \ldots, \alpha_{d-1}\} \subset \Phi^+$ such that each $\alpha \in \Phi^+$ can be written as a unique non-negative integral linear combination of Δ.
- The **Weyl group** $W(\Phi) \subset \mathrm{GL}(E)$ is generated by reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$, or equivalently, by $\{\sigma_\alpha \mid \alpha \in \Delta\}$.
- $\{s_i := \sigma_{\alpha_i} \mid \alpha_i \in \Delta\}$ is the set of **simple reflections**.

For $w \in W(\Phi)$,

- let $\ell(w)$ be its **Coxeter length**, i.e. $\ell(w)$ is the minimal ℓ such that $w = s_{a_1} \cdots s_{a_\ell}$ is a product of ℓ simple reflections;
- such a word $\mathbf{a} = (a_1, \ldots, a_\ell)$ is called a **reduced word** of w;
- the **inversion set** is $\mathrm{Inv}(w) := \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}$;
- it's a classical fact that $\ell(w) = |\,\mathrm{Inv}(w)|$.

Root systems and Weyl groups

We adopt the following convention for type A, B, C root systems.

- Type A_{n-1}:
  - $\Phi(A_{n-1}) = \{e_j - e_i \mid 1 \le i \neq j \le n\}$,
  - $\Phi^+(A_{n-1}) = \{e_j - e_i \mid 1 \le i < j \le n\}$,
  - $\Delta(A_{n-1}) = \{e_{i+1} - e_i \mid 1 \le i \le n-1\}$,
  - $W(A_{n-1}) = \mathfrak{S}_n$.
- Type B_n:
  - $\Phi(B_n) = \{\pm e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{\pm e_i \mid 1 \le i \le n\}$,
  - $\Phi^+(B_n) = \{e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{e_i \mid 1 \le i \le n\}$,
  - $\Delta(B_n) = \{\alpha_0 = e_1\} \cup \{\alpha_i = e_{i+1} - e_i \mid 1 \le i \le n-1\}$,
  - $W(B_n) = \{\text{permutations on } 1, \ldots, n, \bar{1}, \ldots, \bar{n} \mid w(i) = -w(\bar{i}), \forall i\}$.
- Type C_n:
  - $\Phi(C_n) = \{\pm e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{\pm 2e_i \mid 1 \le i \le n\}$,
  - $\Phi^+(C_n) = \{e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{2e_i \mid 1 \le i \le n\}$,
  - $\Delta(C_n) = \{2e_1\} \cup \{e_{i+1} - e_i \mid 1 \le i \le n-1\}$,
  - $W(C_n) = W(B_n)$.

Reflection order of positive roots

Definition (Reflection order)

Given a reduced word $\mathbf{a} \in \mathrm{Red}(w)$, its corresponding **reflection order** is an ordering $\mathrm{ro}(\mathbf{a}) = \gamma_1, \ldots, \gamma_{\ell(w)}$ of $\mathrm{Inv}(w)$ where $\gamma_j = s_{a_1} \cdots s_{a_{j-1}} \alpha_j \in \Phi^+$.

A reduced word $\mathbf{a} \in \mathrm{Red}(w)$ can be viewed as a chain

$$w^{(0)} = \mathrm{id} \to w^{(1)} \to \cdots \to w^{(\ell)} = w$$

where $w^{(i)} = w^{(i-1)} s_{a_i}$. Then the root γ_i satisfies $w^{(i)} = \sigma_{\gamma_i} w^{(i-1)}$.

Example: reflection order for $\mathbf{a} = (2,1,2,3,2)$ in $\mathfrak{S}_4$

$$1234 \xrightarrow{e_3-e_2} 1324 \xrightarrow{e_3-e_1} 3124 \xrightarrow{e_2-e_1} 3214 \xrightarrow{e_4-e_1} 3241 \xrightarrow{e_4-e_2} 3421 = w.$$

We have $\gamma_1 = e_3 - e_2$, $\gamma_2 = e_3 - e_1$, $\gamma_3 = e_2 - e_1$, $\gamma_4 = e_4 - e_1$, $\gamma_5 = e_4 - e_2$.

Example: reflection order for $\mathbf{a} = (2,1,0,3,1)$ in $W(B_3)$

$$1234 \xrightarrow{e_3-e_2} 1324 \xrightarrow{e_3-e_1} 3124 \xrightarrow{e_3} \bar{3}124 \xrightarrow{e_4-e_2} \bar{3}142 \xrightarrow{e_3+e_1} 1\bar{3}42 = w.$$

Reflection order of positive roots

The following proposition is classical and well-known, which is basically equivalent to the **biconvexity** classification of inversion sets.

Proposition (Björner 1984)

Let $\gamma = \gamma_1, \ldots, \gamma_{\ell(w)}$ be an ordering of $\mathrm{Inv}(w)$. Then γ is a reflection order if and only if for all the triples $\alpha, \beta, \alpha+\beta \in \Phi^+$ such that $\alpha, \alpha+\beta \in \mathrm{Inv}(w)$,

1. *if $\beta \notin \mathrm{Inv}(w)$, then α appears before $\alpha+\beta$ in this sequence;*
2. *and if $\beta \in \mathrm{Inv}(w)$, then $\alpha+\beta$ appears in the middle of α and β.*

$\mathrm{BS}(Z(d,r)) \leftrightarrow \mathrm{Red}(w^{(d,r)})$ via reflection order

Recall that $d = \ell(\lambda)$ is the number of parts. Let $r \geq 0$. The **trapezoid** is

$$Z(d,r) := (r+2d-1, r+2d-3, \ldots, r+3, r+1)$$

with height d and base lengths $r+2d-1$ and $r+1$. We provide an extended label of $Z(d,r)$ by roots:

Example: label of $Z(3,2)$ by roots

$2e_3$	e_3+e_2	e_3+e_1	e_3	e_4+e_3	e_5+e_3	e_3-e_1	e_3-e_2
	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$2e_1$	e_1	e_4+e_1	e_5+e_1		

$\mathrm{BS}(Z(d,r)) \leftrightarrow \mathrm{Red}(w^{(d,r)})$ via reflection order

Write a signed permutation $w \in W(B_n)$ by its one-line notation $w(1)w(2)\cdots w(n)$. Define a signed permutation $w^{(d,r)} \in W(B_{d+r})$ by

$$w^{(d,r)}(i) := \begin{cases} d+i & \text{if } 0 < i \leq r, \\ \overline{i-r} & \text{if } i > r. \end{cases}$$

For example, $w^{(3,2)} = 45\bar{1}\bar{2}\bar{3}$.

Since $\mathrm{Inv}(w^{(d,r)})$ is the set of root labels of $Z(d,r)$, the following proposition makes sense:

Proposition (Gao, Gao and G. 2022)

The reflection order bijects $\mathrm{Red}(w^{(d,r)})$ *to* $\mathrm{BS}(Z(d,r))$.

$\mathrm{BS}(Z(d,r)) \leftrightarrow \mathrm{Red}(w^{(d,r)})$ via reflection order

Example: label of $Z(3,2)$ by roots

$2e_3$	e_3+e_2	e_3+e_1	e_3	e_4+e_3	e_5+e_3	e_3-e_1	e_3-e_2
	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$2e_1$	e_1	e_4+e_1	e_5+e_1		

Consider a reduced word $\mathbf{a} = (2, 0, 1, 0, \ldots$ with reflection order

$$12345 \xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3+e_1} 3\bar{1}245 \xrightarrow{e_3} \bar{3}\bar{1}245 \longrightarrow \cdots$$

	3	4				1
		2				

$\mathrm{BS}(Z(d,r)) \leftrightarrow \mathrm{Red}(w^{(d,r)})$ via reflection order

It is easy to see that a reflection order gives a balanced shifted tableaux. Recall that in a reflection order, $\alpha+\beta$ appears between α and β.

Example: label of $Z(3,2)$ by roots

$2e_3$	e_3+e_2	e_3+e_1	e_3	e_4+e_3	e_5+e_3	e_3-e_1	e_3-e_2
	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$2e_1$	e_1	e_4+e_1	e_5+e_1		

The other direction $\mathrm{BS}(Z(d,r)) \to \mathrm{Red}(w^{(d,r)})$ is weirdly very technical.

Kraśkiewicz's insertion

Kraśkiewicz's insertion is the type B analogue of Edelman-Greene.

For a unimodal sequence of non-negative integers
$\mathbf{R} = (r_1 > r_2 > \ldots > r_k < r_{k+1} < \ldots < r_m)$,

- the decreasing part is $\mathbf{R}^{\downarrow} = (r_1 > r_2 > \ldots > r_k)$, and
- the increasing part is $\mathbf{R}^{\uparrow} = (r_{k+1} < r_{k+2} < \ldots < r_m)$.

Let $w \in W(B_n)$. Kraśkiewicz's insertion maps $\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathrm{Red}(w)$ to a pair of shifted tableaux $(P(\mathbf{a}), Q(\mathbf{a}))$ of the same shape by

$$(P^{(i)}, Q^{(i)}) := (P^{(i-1)}, Q^{(i-1)}) \leftarrow a_i,$$

starting with $(P^{(0)}, Q^{(0)}) = (\emptyset, \emptyset)$.

Kraśkiewicz's insertion

Step 1: Set $\mathbf{R}$ to be the first row of $P^{(i-1)}$ and $a = a_i$.
Step 2: Insert a into $\mathbf{R}$:

- Case 0 ($\mathbf{R} = \emptyset$): Insert a into the left-most box. Stop.
- Case 1 ($\mathbf{R}a$ is unimodal): Append a_i to the right of $\mathbf{R}$. Stop.
- Case 2 ($\mathbf{R}a$ is not unimodal): Let b be the smallest number in $\mathbf{R}^{\uparrow}$ such that $b \geq a$.
  - Case 2.0 ($a = 0$ and $\mathbf{R}$ contains 101 as a subsequence): We leave $\mathbf{R}$ unchanged and return to start of Step 2 with $a = 0$ and $\mathbf{R}$ equals the next row.
  - Case 2.1.1 ($b \neq a$): Replace b with a and set $c = b$.
  - Case 2.1.2 ($b = a$): Keep $\mathbf{R}^{\uparrow}$ unchanged and set $c = a + 1$.

  Now insert c into $\mathbf{R}^{\downarrow}$. Let d be the largest integer such that $d \leq c$.
  - Case 2.1.3 ($d \neq c$): Replace d with c and set $a' = d$.
  - Case 2.1.4 ($d = c$): Keep $\mathbf{R}^{\downarrow}$ unchanged and set $a' = c - 1$.

Step 3: Repeat Step 2 with $a = a'$ and $\mathbf{R}$ the next row.

Kraśkiewicz's insertion

Let $\mathbf{a} = (3, 1, 2, 1, 0, 3, 4, 3)$.

$$P^{(0)} = \emptyset \qquad Q^{(0)} = \emptyset$$

$$P^{(3)} = \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline \end{array} \qquad Q^{(3)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

$$P^{(4)} = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} \qquad Q^{(4)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array}$$

$$P^{(7)} = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 2 & 1 & 0 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{4}{|c}{} \\ \cline{2-2} \end{array} \qquad Q^{(7)} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 7 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{4}{|c}{} \\ \cline{2-2} \end{array}$$

$$P^{(8)} = \begin{array}{|c|c|c|c|c|c|}\hline 4 & 2 & 1 & 0 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 1 & 3 & \multicolumn{3}{|c}{} \\ \cline{2-3} \end{array} \qquad Q^{(8)} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 7 \\ \hline \multicolumn{1}{c|}{} & 4 & 8 & \multicolumn{3}{|c}{} \\ \cline{2-3} \end{array}$$

Kraśkiewicz's insertion

Definition (Kraśkiewicz 1989)

A shifted tableaux T with d rows is a **standard decomposition tableaux** of $w \in W(B_n)$ if

- $\pi(T) = T_d T_{d-1}, \ldots, T_1$ is a reduced word of w,
- T_i is a unimodal subsequence of maximal length in $T_d T_{d-1} \ldots T_i$.

Theorem (Kraśkiewicz 1989)

The Kraśkiewicz's insertion gives a bijection between $\{\mathbf{a} \in \mathrm{Red}(w)\}$ *and the pairs of tableaux* $(P(\mathbf{a}), Q(\mathbf{a}))$ *where* $P(\mathbf{a})$ *is a standard decomposition tableaux of* w *and* $Q(\mathbf{a})$ *is a standard tableaux of the same shape.*

Corollary (Kraśkiewicz 1989)

For any $w \in W(B_n)$,

$$|\,\mathrm{Red}(w)| = \sum_{P \in \mathrm{SDT}(w)} f^{\mathrm{sh}(P)}.$$

Kraśkiewicz's insertion

Definition

A signed permutation $w \in W(B_n)$ is **vexillary** if $\mathrm{SDT}(w)$ consists of exactly one shifted tableau. We denote this tableau as $P(w)$.

In this case, the **type** C **Stanley symmetric function** of w equals a single **Schur-**Q **function**.

Theorem (Billey-Lam 1998)

A signed permutation $w \in W(B_n)$ is vexillary if and only if w pattern avoids the following:

$$\begin{array}{ccccccccc} \bar{3}2\bar{1} & \bar{3}21 & 32\bar{1} & 321 & 3\bar{1}2 & \bar{2}31 & \bar{1}32 & \bar{4}\bar{1}\bar{2}3 & \bar{4}1\bar{2}3 \\ \bar{3}\bar{4}\bar{1}\bar{2} & \bar{3}\bar{4}1\bar{2} & 3\bar{4}\bar{1}\bar{2} & 3\bar{4}1\bar{2} & 3142 & \bar{2}\bar{3}4\bar{1} & 2413 & 2\bar{3}4\bar{1} & 2143 \end{array}.$$

This is equivalent to w avoiding 2143 as a permutation in $\mathfrak{S}_{2n}$.

An enumeration problem interlude

For $\pi \in \mathfrak{S}_k$, write $B_n(\pi)$ as the set of signed permutations $w \in B_n$ which avoid π as if $w \in \mathfrak{S}_{2n}$.

Warning

This is not the same as Billey-Postnikov pattern avoidance.

Theorem (G. and Hänni 2020)

For $n \geq 1$, $|B_n(2143)| = |B_n(1234)|$.

This settled a conjecture by Anderson and Fulton.

Our technique also shows that, in the Weyl groups of type D_n,

Corollary (G. and Hänni 2020)

For $n \geq 1$, $|D_n(2143)| = |D_n(1234)|$.

An enumeration problem interlude

These numbers have very nice enumeration formula.

Theorem (Egge 2010)

$$|B_n(1234)| = \sum_j \binom{n}{j}^2 C_j$$

where C_j is the j-th Catalan number.

There should be more Wilf-equivalent families in this sense:

Conjecture (G. and Hänni 2020)

For $m \geq 2k$,

$$|B_n(12\cdots m)| = |B_n\big(k\cdots 1\ (k{+}1)\cdots(m{-}k)\ m\cdots(m{-}k{+}1)\big)|.$$

$\mathrm{SYT}(Z(d,r)) \leftrightarrow \mathrm{Red}(w^{(d,r)})$ via Kraśkiewicz's insertion

It's straightforward to check that $w^{(d,r)}$ is vexillary. The shifted tableau $P(w^{(d,r)})$ has shape $Z(d,r)$ and can be nicely described.

The insertion tableau for $w^{(3,2)}$

$$P(w^{(3,2)}) = \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 3 & 0 & 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 0 & 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 0 & 1 & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array}$$

Corollary

By restricting to the recording tableaux, Kraśkiewicz's insertion gives a bijection between $\mathrm{Red}(w^{(d,r)})$ *and* $\mathrm{SYT}(Z(d,r))$.

Proof sketch

We have now finished the second row of

$$
\begin{tikzcd}
\mathrm{SYT}(\lambda) \arrow[r, leftrightarrow] & \mathrm{SYT}(Z(d,r))|_\lambda \arrow[r, longleftrightarrow] \arrow[d, hook, "\subseteq"] & \mathrm{Red}(w^\lambda) \arrow[r, longleftrightarrow] \arrow[d, hook, "\subseteq"] & \mathrm{BS}(Z(d,r))|_\lambda \arrow[r, leftrightarrow] \arrow[d, hook, "\subseteq"] & \mathrm{BS}(\lambda) \\
& \mathrm{SYT}(Z(d,r)) \arrow[r, longleftrightarrow] & \mathrm{Red}(w^{(d,r)}) \arrow[r, longleftrightarrow] & \mathrm{BS}(Z(d,r)) &
\end{tikzcd}
$$

For an arbitrary λ, choose r large enough such that $\lambda \subset Z(d,r)$.
The choice of r will not matter for the bijection.
We now describe $\mathrm{SYT}(\lambda) \leftrightarrow \mathrm{SYT}(Z(d,r))|_\lambda$ and $\mathrm{BS}(\lambda) \leftrightarrow \mathrm{BS}(Z(d,r))|_\lambda$.

$\mathrm{SYT}(\lambda) \leftrightarrow \mathrm{SYT}(Z(d,r))|_\lambda$

For $T \in \mathrm{SYT}(\lambda)$, we pad it to obtain $T^+ \in \mathrm{SYT}(Z(d,r))$.

Example: padding a standard shifted tableau

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 5 & 6 & 9 \\ & 4 & 7 & & & \\ & & 8 & & & \end{array}, \text{ then } T^+ = \begin{array}{ccccccc} 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ & 4 & 7 & 11 & 12 & 13 & \\ & & 8 & 14 & 15 & & \end{array}.$$

Define $\mathrm{SYT}(Z(d,r))|_\lambda$ to be the set of all such T^+ obtained from some $T \in \mathrm{SYT}(\lambda)$.

$\mathrm{SYT}(Z(d,r))|_\lambda \leftrightarrow \mathrm{Red}(w^\lambda)$

For every $T^+ \in \mathrm{SYT}(Z(d,r))|_\lambda$,

- entries $|\lambda|+1, \ldots, |Z(d,r)|$ are at fixed positions;
- so the first $|Z(d,r)| - |\lambda|$ steps of inverse Kraśkiewicz's insertion are the same.

Warning

Inverse Kraśkiewicz's insertion may not be well-defined.

Inverse Kraśkiewicz's insertion

insertion tableau P

4	3	0	1	2	3	4
	3	0	1	2	3	
		0	1	2		

4	2	0	1	2	3	4
	2	0	1	2	3	
		0	1			

recording tableau Q

						10
			11	12	13	
			14	15		

						10
			11	12	13	
			14			

letter a_i's

$a_{15} = 2$

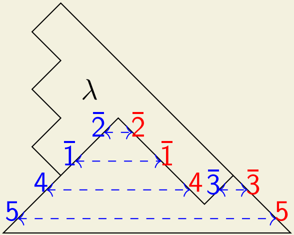
$\mathrm{SYT}(Z(d,r))|_\lambda \leftrightarrow \mathrm{Red}(w^\lambda)$

Once the padded entries are gone, we have a fixed insertion tableaux $P(w^\lambda)$ which determines w^λ.

We can read off w^λ explicitly:

- draw a right triangle with sidelength $d+r$ over λ at $(1,1)$;
- rotate by 45° and consider the Dyck path bordering λ;
- label the upsteps in order by $d+r, d+r-1, \ldots, d+1, \bar{1}, \bar{2}, \ldots, \bar{d}$;
- read the corresponding downsteps in order to obtain w^λ

Example: $w^\lambda = \bar{2}\bar{1}4\bar{3}5$ for $\lambda = (6,2,1) \subset Z(3,2)$



$\mathrm{BS}(\lambda) \leftrightarrow \mathrm{BS}(Z(d,r))|_\lambda$

There exists a way to pad balanced shifted tableaux as well.

Lemma (Gao, Gao and G. 2022)

Let $B \in \mathrm{BS}(\lambda)$ and $i \in [d]$ such that $i = 1$ or $\lambda_{i-1} - \lambda_i \geq 3$. Let $\lambda^{\#}$ be obtained from λ by adding a box to the i-th row and let j be the column index of the added box. Define $B^{\#}$ obtained from B by

- *interchange column j and $j+1$ of B,*
- *and set $B^{\#}(i,j) = |\lambda| + 1$.*

Then $B^{\#} \in \mathrm{BS}(\lambda^{\#})$. Moreover, this is a bijection between $\mathrm{BS}(\lambda)$ and $\{T \in \mathrm{BS}(\lambda^{\#}) \mid T(i,j) = |\lambda| + 1\}$.

Repeatedly applying this lemma from top to bottom, we can pad any $B \in \mathrm{BS}(\lambda)$ to $B^{+} \in \mathrm{BS}(Z(d,r))$.

$\mathrm{BS}(\lambda) \leftrightarrow \mathrm{BS}(Z(d,r))|_\lambda$

Here is an example.

Example: padding a balanced shifted tableau

$$B = \begin{array}{|c|c|c|c|c|c|c|}\hline 6 & 3 & 4 & 5 & 9 & 10 & 1 \\ \hline \multicolumn{1}{c|}{} & 7 & 8 & 11 & 12 & 13 & \multicolumn{1}{|c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 2 & \multicolumn{4}{|c}{} \\ \cline{3-3} \end{array}, \text{ then } B^{\#} = \begin{array}{|c|c|c|c|c|c|c|}\hline 6 & 3 & 4 & 9 & 5 & 10 & 1 \\ \hline \multicolumn{1}{c|}{} & 7 & 8 & 12 & 11 & 13 & \multicolumn{1}{|c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 2 & 14 & \multicolumn{3}{|c}{} \\ \cline{3-4} \end{array}.$$

Similarly, let $\mathrm{BS}(Z(d,r))|_\lambda$ be the set of balanced shifted tableaux B^+ that can be obtained in this way from some $B \in \mathrm{BS}(\lambda)$.

$\mathrm{BS}(Z(d,r))|_\lambda \leftrightarrow \mathrm{Red}(w^\lambda)$

For every $B^+ \in \mathrm{BS}(Z(d,r))|_\lambda$,

- entries $|\lambda|+1, \ldots, |Z(d,r)|$ are at fixed positions;
- so the last $|Z(d,r)| - |\lambda|$ roots of the corresponding reflection order are fixed;
- this means $\mathrm{BS}(Z(d,r))|_\lambda$ is in bijection with some $\mathrm{Red}(u^\lambda)$;
- we can check that $w^\lambda = u^\lambda$.

Now all steps of bijection are completed.

Let's go the other way and start with a balanced shifted tableau

$$B = \begin{array}{|c|c|c|c|c|c|}
\hline
6 & 3 & 4 & 1 & 5 & 9 \\
\hline
\multicolumn{1}{c|}{} & 7 & 8 & \multicolumn{3}{c}{} \\
\cline{2-3}
\multicolumn{2}{c|}{} & 2 & \multicolumn{3}{c}{} \\
\cline{3-3}
\end{array}.$$

We have $\lambda = (6, 2, 1)$ and choose $Z(3, 2)$.

An example: $\mathrm{BS}(\lambda) \to \mathrm{BS}(Z(d,r))|_\lambda$

We pad it to $B^+ \in \mathrm{BS}(Z(d,r))|_\lambda$:

$$\begin{array}{cccccc} 6 & 3 & 4 & 1 & 5 & 9 \\ & 7 & 8 & & & \\ & & 2 & & & \end{array} \to \begin{array}{ccccccc} 6 & 3 & 4 & 1 & 5 & 9 & 10 \\ & 7 & 8 & & & & \\ & & 2 & & & & \end{array} \to$$

$$\begin{array}{ccccccc} 6 & 3 & 4 & 5 & 1 & 9 & 10 \\ & 7 & 8 & 11 & & & \\ & & 2 & & & & \end{array} \to \begin{array}{ccccccc} 6 & 3 & 4 & 5 & 9 & 1 & 10 \\ & 7 & 8 & 11 & 12 & & \\ & & 2 & & & & \end{array} \to$$

$$\begin{array}{ccccccc} 6 & 3 & 4 & 5 & 9 & 10 & 1 \\ & 7 & 8 & 11 & 12 & 13 & \\ & & 2 & & & & \end{array} \to \begin{array}{ccccccc} 6 & 3 & 4 & 9 & 5 & 10 & 1 \\ & 7 & 8 & 12 & 11 & 13 & \\ & & 2 & 14 & & & \end{array} \to$$

$$\begin{array}{ccccccc} 6 & 3 & 4 & 9 & 10 & 5 & 1 \\ & 7 & 8 & 12 & 13 & 11 & \\ & & 2 & 14 & 15 & & \end{array} = B^+$$

An example: $\mathrm{BS}(Z(d,r))|_\lambda \to \mathrm{Red}(w^{(d,r)})$

$$B^+ = \begin{array}{|c|c|c|c|c|c|c|}\hline 6 & 3 & 4 & 9 & 10 & 5 & 1 \\ \hline \multicolumn{1}{c|}{} & 7 & 8 & 12 & 13 & 11 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 2 & 14 & 15 & \multicolumn{2}{c}{} \\ \cline{3-5} \end{array}$$

gives a reflection order

$$12345 \xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3+e_1} 3\bar{1}245 \xrightarrow{e_3} \bar{3}\bar{1}245 \xrightarrow{e_3-e_1} \bar{1}\bar{3}245$$
$$\xrightarrow{e_3+e_2} \bar{1}2\bar{3}45 \xrightarrow{e_2+e_1} 2\bar{1}\bar{3}45 \xrightarrow{e_2} \bar{2}\bar{1}\bar{3}45 \xrightarrow{e_4+e_3} \bar{2}\bar{1}4\bar{3}5 \xrightarrow{e_5+e_3} \bar{2}\bar{1}45\bar{3}$$
$$\xrightarrow{e_2-e_1} \bar{1}\bar{2}45\bar{3} \xrightarrow{e_4+e_2} \bar{1}4\bar{2}5\bar{3} \xrightarrow{e_5+e_2} \bar{1}45\bar{2}\bar{3} \xrightarrow{e_4+e_1} 4\bar{1}5\bar{2}\bar{3} \xrightarrow{e_5+e_1} 45\bar{1}\bar{2}\bar{3},$$

for which we read off

$$\mathbf{a} = 201012103412312 \in \mathrm{Red}(w^{(3,2)}).$$

An example: $\mathrm{Red}(w^{(d,r)}) \to \mathrm{SYT}(Z(d,r))|_\lambda$

Kraśkiewicz's insertion of $\mathbf{a} = 201012103412312$ gives

$$P^{(0)} = \emptyset \qquad Q^{(0)} = \emptyset$$

$$P^{(3)} = \begin{array}{|c|c|c|}\hline 2 & 0 & 1 \\ \hline \end{array} \qquad Q^{(3)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

$$P^{(4)} = \begin{array}{ccc} \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} \\ \hline & \multicolumn{1}{|c|}{0} & \\ \cline{2-2} \end{array} \qquad Q^{(4)} = \begin{array}{ccc} \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline & \multicolumn{1}{|c|}{4} & \\ \cline{2-2} \end{array}$$

$$P^{(5)} = \begin{array}{cccc} \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline & \multicolumn{1}{|c|}{0} & & \\ \cline{2-2} \end{array} \qquad Q^{(5)} = \begin{array}{cccc} \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline & \multicolumn{1}{|c|}{4} & & \\ \cline{2-2} \end{array}$$

$$\vdots \qquad \vdots$$

$$P^{(15)} = \begin{array}{ccccccc} \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \\ \cline{2-6} & & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & & \\ \cline{3-5} \end{array} \qquad Q^{(15)} = \begin{array}{ccccccc} \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{10} \\ \hline & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{11} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} & \\ \cline{2-6} & & \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{14} & \multicolumn{1}{c|}{15} & & \\ \cline{3-5} \end{array}$$

An example: $\mathsf{SYT}(Z(d,r))|_\lambda \to \mathsf{SYT}(\lambda)$

Delete the largest entries from $Q(\mathbf{a})$ until $|\lambda|$ to get

$$T^+ = \begin{array}{ccccccc} 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ & 4 & 7 & 11 & 12 & 13 & \\ & & 8 & 14 & 15 & & \end{array}, \quad T = \begin{array}{cccccc} 1 & 2 & 3 & 5 & 6 & 9 \\ & 4 & 7 & & & \\ & & 8 & & & \end{array}.$$

We now completed the bijection

Example: the bijection $\mathsf{BS}(\lambda) \to \mathsf{SYT}(\lambda)$

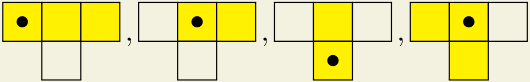
$$B = \begin{array}{cccccc} 6 & 3 & 4 & 1 & 5 & 9 \\ & 7 & 8 & & & \\ & & 2 & & & \end{array} \quad T = \begin{array}{cccccc} 1 & 2 & 3 & 5 & 6 & 9 \\ & 4 & 7 & & & \\ & & 8 & & & \end{array}$$

Some remarks

There are other notions of "balanced tableaux" in the literature, including

- "standard w-tableau" by Kraśkiewicz,
- "balanced labeling" by Fomin-Greene-Reiner-Shimozono,
- and its type B analogue by Hamaker.

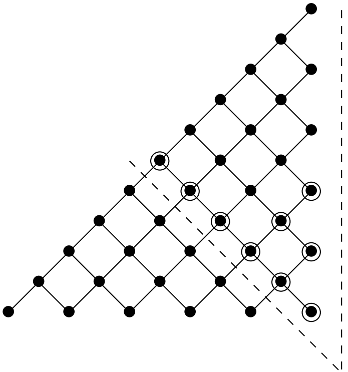
Hooks and corners of shifted shape $(3, 1)$



- Our definition focuses on hook length formula.
- The above definitions focus on Rothe diagram.
- The difference is analogous to dominant permutations v.s. Grassmannian permutations.

Some remarks

A shifted shape λ can also be thought of as an order ideal in the principal order filter of the (co)minuscule node in the type B_n root poset:



We have $\mathrm{SYT}(Z(d,r)) = \#\,\mathrm{Red}(45\bar{3}6\bar{2}7\bar{1})$.
But showing $\#\,\mathrm{Red}(45\bar{1}\bar{2}\bar{3}) = \#\,\mathrm{Red}(45\bar{3}6\bar{2}7\bar{1})$ is far from trivial!

Some remarks

What about other root systems?
Unfortunately, $\#\,\mathrm{Red}(w_0(D_4)) = 2316$ and the number of linear extensions of the root poset is $e(\Phi(D_r)^+) = 2400$.
These two quantities also fail to be equal in F_4.

Conjecture (Stanley 1984)

For any Coxeter group W and $J \subset S$,

$$\#\,\mathrm{Red}(w_0^J) \leq e(\Phi_J^+).$$

Thanks!

We thank Alex Yong and Jianping Pan for helpful conversations.

Thank you for listening!