Boolean elements in the Bruhat order

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Boolean Weyl group elements

For a Weyl group W, an element $w \in W$ is called *boolean* if the interval $[\mathrm{id}, w]$ in the (strong) Bruhat order is isomorphic to a Boolean lattice. The following characterization of boolean elements is a simple consequence of the subword property of the strong Bruhat order.

Proposition

An element $w \in W$ is boolean if and only if any reduced expression (or equivalently, all reduced expressions) of w does not contain repeated letters.

For type $A_n = \mathfrak{S}_{n+1}$, Tenner [1] proved that a permutation $w \in A_n$ is boolean iff it avoids 321 and 3412, and gave a signed pattern avoidance characterization of boolean elements in types B and D, with the number of patterns being 10 and 20, respectively. By analyzing inversion sets, we obtain a more succinct **pattern avoidance characterization of boolean elements**. In our most concise version, boolean elements are characterized by avoiding just 3 *linear patterns*:

- $321 = s_1s_2s_1 \in W(A_2)$, $3412 = s_2s_1s_3s_2 \in W(A_3)$,
- and the new $s_2s_1s_3s_4s_2 \in W(D_4)$.

This will be made precise after discussing some background.

Background on the Bruhat order and inversion sets

The *(strong) Bruhat order* on a Weyl group W is the transitive closure of all

$$w \lessdot wt, \quad \ell(w) = \ell(wt) - 1$$

where t is some reflection and ℓ is the Coxeter length. The identity Weyl group element e is the minimum of this partial order.

We let Φ be a root system, with positive roots $\Phi^+ \subset \Phi$ and simple roots $\Delta \subset \Phi^+$. Let $W = W(\Phi)$ be its Weyl group. For $w \in W$, the *inversion set* is

$$I_\Phi(w) = \{\beta \in \Phi^+ \mid w\beta \in \Phi^-\}.$$

The next proposition is useful and well-known (see for example [2]).

Proposition

The inversion set uniquely characterizes a Weyl group element. In other words, $I_\Phi : W \to 2^{\Phi^+}$ is injective. Moreover, a subset $I \subset \Phi^+$ is the inversion set of some Weyl group element if and only if it is *biconvex*; that is, if and only if:

1. if $\alpha, \beta \in I$, $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in I$ and,
2. if $\alpha, \beta \notin I$, $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \notin I$.

Patterns in Weyl group elements

We use a pattern restriction map defined by Billey and Postnikov [3]. Let $E' \subset E$ be a subspace; $\Phi' = \Phi \cap E'$ is then a root system with positive roots $(\Phi')^+ = \Phi^+ \cap E'$. It follows from the previous proposition that $I_\Phi(w) \cap E'$ is biconvex, and that there is a unique element $w' \in W(\Phi')$ such that $I_{\Phi'}(w') = I_\Phi(w) \cap E'$. We call this w' the *restriction* of w to Φ', denoted $w|_{\Phi'}$.

Definition of BP pattern containment

We say that $w \in W(\Phi)$ contains the BP (Billey-Postnikov) pattern $\pi \in W(R)$ if there exists a subspace $E' \subset E$ such that there is an isomorphism between root systems R and $\Phi' := \Phi \cap E$ that preserves positive roots and maps π to $w|_{\Phi'}$.

We introduce a new notion of *linear patterns*, which will enable a very efficient characterization of boolean elements.

Definition of linear pattern containment

We say that $w \in W(\Phi)$ contains the *linear pattern* $\pi \in W(R)$ if there exists a linear transformation $R \to \Phi$ that maps positive roots to positive roots, inversions $I_R(\pi)$ of π to inversions $I_\Phi(w)$ of w, and non-inversions $R^+ \setminus I_R(\pi)$ to non-inversions $\Phi^+ \setminus I_\Phi(w)$.

Linear patterns can be seen as a relaxation of BP patterns.

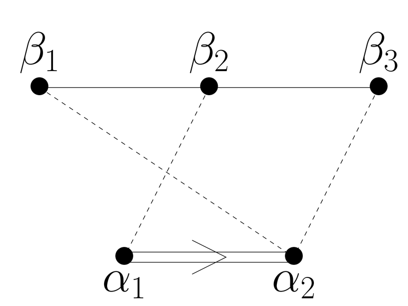

Figure 1: The linear pattern $s_2s_1s_3s_2 \in W(A_3)$ in $s_1s_2s_1 \in W(B_2)$

A linear pattern avoidance characterization of boolean elements

The first version of our main theorem is a characterization of boolean elements in terms of linear pattern avoidance.

Characterization theorem Linear pattern version

Let Φ be an irreducible root system. An element $w \in W(\Phi)$ is boolean if and only if w avoids the linear patterns $s_1s_2s_1 \in W(A_2)$, $s_2s_1s_3s_2 \in W(A_3)$, and $s_2s_1s_3s_4s_2 \in W(D_4)$.

A BP pattern avoidance characterization of boolean elements

From the linear pattern version, one can deduce a BP pattern characterization of boolean elements, which is the second version of our main theorem.

Characterization theorem BP pattern version

An element $w \in W$ is boolean if and only if w avoids the BP patterns in Table 1.

type	forbidden patterns	# patterns
A_2	$s_1s_2s_1 = s_2s_1s_2$ (321)	1
A_3	$s_2s_1s_3s_2$ (3412)	1
$B_2 = C_2$	$s_1s_2s_1$, $s_2s_1s_2$, $s_1s_2s_1s_2 = s_2s_1s_2s_1$	3
D_4	$s_2s_1s_3s_4s_2$	1
G_2	all patterns of Coxeter length at least 3	5

Table 1: Forbidden patterns for boolean elements in Weyl groups

k-boolean elements

We say a permutation $w \in \mathfrak{S}_n$ is *k-boolean* if for any reduced word of w, there is no simple transposition s_i that appears strictly more than k times. This generalizes boolean elements, as w is boolean iff it is 1-boolean. It turns out that 2-boolean elements of $\mathfrak{S}_n$ can also be characterized in terms of pattern avoidance.

Pattern avoidance characterization of 2-boolean elements

A permutation $w \in \mathfrak{S}_n$ is 2-boolean if and only if w avoids 3421, 4312, 4321 and 456123.

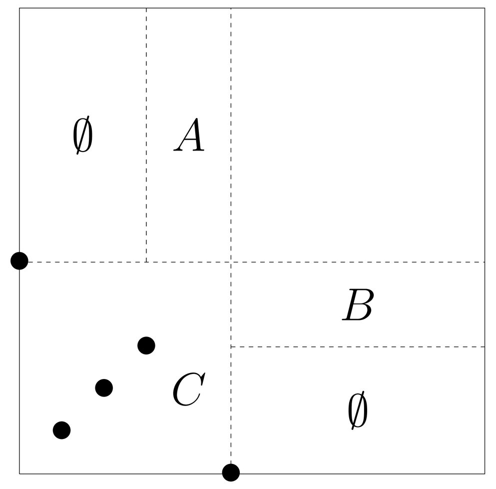

Figure 2: Structure of a 2-boolean permutation

We suspect that there is no pattern avoidance characterization of k-boolean elements for any $k \geq 3$. For $k = 3$, $436512 = s_3s_2s_3s_4s_5s_1s_2s_3s_4s_3$ is not 3-boolean. However, 4357612, which contains 436512 as a pattern, is 3-boolean.

Enumeration of 2-boolean elements

The pattern avoidance characterization makes it possible to derive a recurrence relation for the number of 2-boolean elements in $\mathfrak{S}_n$, which we present in the form of the following generating function. This is sequence A124292 in OEIS [4].

Enumeration theorem

Let $f(n)$ be the number of 2-boolean permutations in $\mathfrak{S}_n$. Then

$$\sum_{n\geq 0} f(n)q^n = \frac{1-5q+5q^2}{1-6q+9q^2-3q^3}.$$

Open problems

- Is there a pattern avoidance characterization of 2-boolean elements in any Weyl group?
- Is there a way to enumerate 2-boolean elements without going through pattern avoidance?
- Can one enumerate k-boolean elements for $k > 2$?
- Are there examples showing that k-boolean elements are not characterized by pattern avoidance for any $k \geq 4$?

References

[1] Bridget Eileen Tenner. Pattern avoidance and the Bruhat order. *J. Combin. Theory Ser. A*, 114(5):888–905, 2007.

[2] Christophe Hohlweg and Jean-Philippe Labbé. On inversion sets and the weak order in Coxeter groups. *European J. Combin.*, 55:1–19, 2016.

[3] Sara Billey and Alexander Postnikov. Smoothness of Schubert varieties via patterns in root subsystems. *Adv. in Appl. Math.*, 34(3):447–466, 2005.

[4] Neil JA Sloane et al. The on-line encyclopedia of integer sequences, 2020.

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