# Billey-Postnikov posets

Yibo Gao
Joint work with: Christian Gaetz (UCB)

Beijing International Center for Mathematical Research

Januarary 21, 2026

# Background on the Bruhat order

- Let $S_n$ be the symmetric group of permutations.
- We write a permutation in its **one-line** notation. For example, $w = 3412$ means $w(1) = 3$, $w(2) = 4$, $w(3) = 1$, $w(4) = 2$.
- The (strong) **Bruhat order** is generated by

$$w \leq wt_{ij} \text{ if } w(i) < w(j)$$

where $t_{ij} = (i\ j)$ swaps $i$ and $j$.

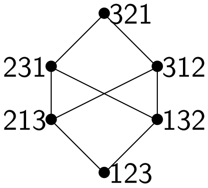



Figure: The Bruhat order on $S_3$

# Background on the Bruhat order

- The symmetric group $S_n$ has the following presentation:

$$S_n = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 = 1 & \text{for } i = 1, \ldots, n{-}1, \\ s_i s_j = s_j s_i & \text{for } |i-j| \geq 2, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i = 1, \ldots, n{-}2, \end{array} \right\rangle$$

  where $s_i = (i\ i{+}1)$ is called a **simple transposition**.
- The **Coxeter length** of $w \in S_n$ is the smallest $\ell = \ell(w)$ such that $w = s_{i_1} \cdots s_{i_\ell}$ is a product of $\ell$ simple transpositions.
- Such an expression is called a **reduced word**.
- It is a classical fact that $\ell(w) = |\mathrm{Inv}(w)|$, where

$$\mathrm{Inv}(w) := \{(i,j) \mid i < j,\ w(i) > w(j)\}.$$

- The **reflections** are $T = \{t_{ij} := (i\ j) \mid i < j\}$.
- The **Bruhat order** is generated by

$$w < wt_{ij} \text{ if } \ell(w) < \ell(wt_{ij}).$$

# Background on the Bruhat order

- Hilbert's fifteenth problem
- counting problems of projective geometry
- study cohomology theories
- The flag variety is

$$\begin{aligned}\mathrm{Fl}(\mathbb{C}^n) &= \{\emptyset = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n \mid \dim V_i = i\} \\ &= \mathrm{GL}(\mathbb{C}^n)/B\end{aligned}$$

where $B$ is the Borel subgroup of upper triangular matrices.

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \text{ where } V_i = \mathrm{span}(v_1, \ldots, v_i).$$

# Background on the Bruhat order

- The flag variety admits a **Bruhat decomposition**

$$\mathrm{Fl}(\mathbb{C}^n) = \bigsqcup_{w \in S_n} \Omega_w$$

  into **open Schubert cells**.
- The **Schubert variety** is $X_w := \overline{\Omega_w}$, also written as

$$X_w(E_\bullet) = \{F_\bullet \mid \dim(E_i \cap F_j) \geq \mathrm{rk}(w)[i,j] \text{ for all } 1 \leq i,j \leq n\}.$$

- $X_u \subset X_w$ if and only if $u \leq w$ in the Bruhat order.
- The **Schubert classes** $\sigma_w := [X_w]$'s form a linear basis of

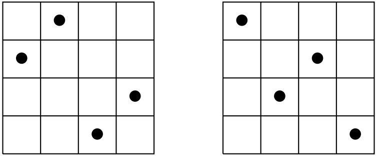
$$H^*(\mathrm{Fl}(\mathbb{C}^n), \mathbb{Z}) \simeq \mathbb{Z}[x_1, \ldots, x_n]/\mathrm{Sym}^+.$$

# Smooth permutations

## Theorem (Lakshmibai-Sandhya 1990, Carrell 1994)

*The followings are equivalent for $w \in S_n$:*

1. *the Schubert variety $X_w$ is smooth;*
2. *the interval $[\mathrm{id}, w]$ in the Bruhat order is rank-symmetric;*
3. *$w$ avoids 3412 and 4231;*
4. *The undirected Bruhat graph $\Gamma(w)$ is regular.*



## Definition

A permutation $w \in S_n$ **avoids** a pattern $\pi \in S_k$ if there does not exist $1 \leq a_1 < \cdots < a_k \leq n$ such that $w(a_i) < w(a_j)$ if and only if $\pi(i) < \pi(j)$.

# Root systems and Weyl groups

- Let $\Phi$ be a finite crystallographic root system.
- Let $\Delta \subset \Phi^+$ be the set of **simple roots** in a choice of **positive roots**.
- For $\alpha \in \Phi$, write $s_\alpha$ for the **reflection** across $\alpha$.
- The **Weyl group** $W = W(\Phi)$ is generated by $\{s_\alpha \mid \alpha \in \Phi\}$, or equivalently, by $\{s_\alpha \mid \alpha \in \Delta\}$.
- For $w \in W$, its **Coxeter length** is the smallest $\ell = \ell(w)$ such that $w = s_{i_1} \cdots s_{i_\ell}$ is a product of $\ell$ simple reflections.
- For $w \in W$, its **(right) inversion set** is

$$\mathrm{Inv}(w) := \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}.$$

- It is a classical fact that $\ell(w) = |\mathrm{Inv}(w)|$.
- The **Bruhat order** is generated by

$$w < ws_\alpha \text{ if } \ell(w) < \ell(ws_\alpha) \text{ for } \alpha \in \Phi^+.$$

# Root systems and Weyl groups

- Let $G$ be a complex reductive group and $B$ a Borel subgroup. Then $G/B$ is the generalized **flag variety**.
- Similar to type $A$, we have a **Bruhat decomposition**

$$G/B = \bigsqcup_{w \in W} BwB/B$$

where $W = N_G(T)/T$ is the Weyl group.
- The **Schubert cell** is $BwB/B \simeq \mathbb{C}^{\ell(w)}$ and the **Schubet variety** is $X_w := \overline{BwB/B}$.
- $X_u \subset X_w$ if and only if $u \leq w$ in the Bruhat order.

# Parabolic decompositions

- For $w \in W$, its **descents** are

$$D_R(w) = \{\alpha \in \Delta \mid w\alpha \in \Phi^-\}, \quad D_L(w) = \{\alpha \in \Delta \mid w^{-1}\alpha \in \Phi^-\}.$$

- For $w \in W$, its **support** is

$$\mathrm{Supp}(w) := \{\alpha \in \Delta \mid s_\alpha \leq w\}.$$

- For $w \in W$ and $J \subset \Delta$, there is a length-additive factorization $w = w^J w_J$ such that $\mathrm{Supp}(w_J) \subset J$ and $D_R(w^J) \subset \Delta \setminus J$, called the **parabolic decompositions**.
- $w^J$ is the minimal coset representative of $w$ in $W/W(J)$.

**Example of a parabolic decomposition**

Let $w = 892367541$ and $J = \{1, 2, 4, 5, 6\}$, then

$$w^J = 289|3567|4|1, \quad w_J = 231|4675|8|9.$$

# Parabolic decompositions

- Parabolic subgroups $P_J \supset B$ are indexed by $J \subset \Delta$.
- The **partial flag variety** $G/P_J$ has a **Bruhat decomposition**

$$G/P_J = \bigsqcup_{w \in W^J} BwP_J/P_J.$$

- The (parabolic) Schubert variety is $X^J_{w^J} = \overline{Bw^J P_J/P_J}$.
- There is a natural projection $\pi_J : G/B \to G/P_J$.
- The image of $X_w \subset G/B$ under $\pi_J$ is $X^J_{w^J} \subset G/P_J$.

**Definition (Billey-Postnikov 2005)**

The parabolic decomposition $w = w^J w_J$ is a **Billey-Postnikov decomposition** if and only if $\pi_J : X_w \to X^J_{w^J}$ is a fiber bundle.

- Motivation: inductively study (rational) smoothness.

# Billey-Postnikov decompositions

- This theory is further developed by Richmond-Slofstra.
- See the chapter "Coxeter groups and Billey–Postnikov decompositions" by Oh-Richmond in *Handbook of Combinatorial Algebraic Geometry*, for a nice survey.

**Theorem (Richmond-Slofstra 2016)**

*The followings are equivalent for $w \in W$ and $J \subset S$:*

1. $\mathsf{Supp}(w^J) \cap J \subset D_L(w_J)$;
2. *the multiplication* $([\mathrm{id}, w^J] \cap W^J) \times [\mathrm{id}, w_J] \to [\mathrm{id}, w]$ *is a bijection;*
3. *the Poincaré polynomials satisfy* $P_w(q) = P^J_{w^J}(q) P_{w_J}(q)$;
4. $w_J$ *is the maximum element of* $W_J \cap [\mathrm{id}, w]$;
5. $\pi_J : X_w \to X^J_{w^J}$ *is a fiber bundle, in which case the fiber is* $X_{w_J}$.

In this case, we say $w$ is BP at $J$. Write $J \in \mathrm{BP}(w)$.

# Billey-Postnikov decompositions

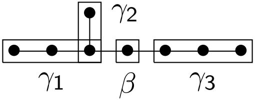
**Non-example: $w = 312 = s_2 s_1,\ J = \{2\}$**

- $w^J = 312 = s_2 s_1$ and $w_J = 123 = \mathrm{id}.$
- $\mathrm{Supp}(w^J) \cap J \not\subseteq D_L(w_J)$
- $[\mathrm{id}, w^J]^J = \{123, 213, 312\}$, $[\mathrm{id}, w_J] = \{\mathrm{id}\}$, while $[\mathrm{id}, w] = \{123, 213, 132, 312\}$.
- The $\times$ map $[\mathrm{id}, w^J]^J \times [\mathrm{id}, w_J] \to [\mathrm{id}, w]$ is a strict injection.
- The maximum element of $W_J \cap [\mathrm{id}, w] = 132$.
- The maximum element of $W_J \cap [\mathrm{id}, w]$ is not $w_J = 123$.
- The projection $\pi_J$ forgets $F_2$.
  $X_w(E_\bullet) = \{F_\bullet = (F_1 \subset F_2 \subset \mathbb{C}^3) \mid E_1 \subset F_2\} \subset G/B$.
  $X^J_{w^J}(E_\bullet) = \{F_\bullet = (F_1 \subset \mathbb{C}^3) \mid E_1 \subset F_2 \text{ for some } F_2 \supset F_1\} = G/P_J$.
- The fiber at $(E_1 \subset \mathbb{C}^3)$ is $\mathbb{P}^1$, but the fiber at any other points is a single point. So it's not a fiber bundle.

# Pattern characterization of BP decompositions

**Definition (Gaetz-G. 2025)**

$(\beta, c_1\gamma_1, \cdots, c_k\gamma_k)$ forms a ***J*-star** pattern if $\beta \in \Phi_J^+$, $c_1, \ldots, c_k \in \mathbb{Z}_{>0}$, $\gamma_1, \ldots, \gamma_k \in \Phi^+ \setminus \Phi_J^+$ such that $\beta + \sum_{i \in I} c_i\gamma_i \in \Phi^+$ for all $I \subset [k]$.
We say that $w \in W$ **contains the *J*-star** $(\beta, c_1\gamma_1, \cdots, c_k\gamma_k)$ if $\beta \notin \mathrm{Inv}(w)$ and $\beta + \sum_{i=1}^k c_i\gamma_i \in \mathrm{Inv}(w)$.




Note that in finite crystallographic types, the multiset $\{c_1, \ldots, c_k\}$ can only be $\{1\}$, $\{2\}$, $\{3\}$, $\{1,1\}$, $\{1,2\}$, $\{1,1,1\}$,

**Theorem (Gaetz-G. 2025)**

*$w$ is BP at $J$ if and only if $w$ does not contain any $J$-stars.*

# Pattern characterization of BP decompositions

**Corollary (Gaetz-G. 2025)**

*Let $w \in S_n$ and $J = \{a-1, \ldots, b\} \subset S$ be connected. Then $w$ is BP at $J$ if and only if $w$ avoids $\underline{23}1$, $3\underline{12}$ and $3\underline{14}2$.*

**Example of pattern condition for BP decompositions in $S_n$**

- The permutation $w = 62\underline{3147}589$ contains $3\underline{14}2$ with $J = \{3, 4, 5\}$. Thus $w$ is not BP at $J$.
- The permutation $3\underline{14}2$ contains the $J$-star with $J = \{2\}$, $\beta = \alpha_2$, $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_3$, as $\beta = e_2 - e_3 \notin \mathrm{Inv}(w)$ and $\beta_1 + \gamma_1 + \gamma_2 = e_1 - e_4 \in \mathrm{Inv}(w)$.

Our result immediately recovers one of the main theorems of Alland-Richmond 2018 which says the restriction of $\mathrm{Fl}(n) \to \mathrm{Gr}(k, n)$ to $X_w$ is a fiber bundle if and only if $w$ avoids $23|1$ and $3|12$.

# Billey-Postnikov posets

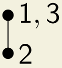
We work in the generality of finite crytallographic types.

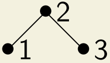
**Theorem (Gaetz-G. 2025)**

*If $w$ is BP at both $J$ and $K$, then $w$ is BP at $J \cup K$ and $J \cap K$.*

By the fundamental theorem of finite distributive lattices, this says that there exists a poset $\mathrm{bp}(w)$ on $\Delta$, which we call the **Billey-Postnikov poset**, such that $\mathrm{BP}(w)$ is the lattice of order ideals of $\mathrm{bp}(w)$.

**Examples**

- $w = 3412$, $J \in \big\{\emptyset, \{2\}, \{1,2,3\}\big\}$ so $\mathrm{bp}(w) =$ 

.
- $w = 4231$, $J \in \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{1,2,3\}\}$ so $\mathrm{bp}(w) =$ 

.

# Billey-Postnikov posets

**Theorem (Gaetz-G. 2025)**

*If $w$ is BP at both $J$ and $K$, then $w$ is BP at $J \cup K$ and $J \cap K$.*

**Question**

*Is there a geometric proof?*

$$\begin{array}{ccc} X^{J\cap K}_{w^{J\cap K}} & \longrightarrow & X^{J}_{w^{J}} \\ \downarrow & & \downarrow \\ X^{K}_{w^{K}} & \longrightarrow & X^{J\cup K}_{w^{J\cup K}} \end{array}$$

This is not a pushforward or pullback diagram.

**Conjecture (Gaetz-G. 2025)**

*The above is true in any Coxeter groups.*

It holds for finite crystallographic types and rank 3 Coxeter groups.

# Billey-Postnikov posets: properties

**Proposition (Gaetz-G. 2025)**

*In the poset* bp$(w)$*, each element is covered by at most* $l$ *elements, where* $l$ *is the number of leaves in the Dynkin diagram.*

**Conjecture**

*If* $i < j$ *in* bp$(w)$*, then there is exactly one path from* $i$ *to* $j$ *in the Hasse diagram of* bp$(w)$*.*

**Proposition (Gaetz-G. 2025)**

*For* $w \in S_n$*, there is a polynomial time algorithm to construct* bp$(w)$*.*

**Theorem (Alland-Richmond 2018)**

*For* $w \in S_n$*,* bp$(w)$ *is indexed by singletons if and only if* $w$ *avoids* 3412*,* 52341 *and* 635241 *(in particular containing all smooth permutations).*

# Applications: a "canonical" bijection

**Theorem (Lakshmibai-Sandhya 1990, Carrell 1994)**

*$X_w$ is rationally smooth if and only if $[\mathrm{id}, w]$ is rank-symmetric.*

In finite simply-laced types, smooth = rationally smooth.
As combinatorialists, we seek **bijections** for equinumerous sets.
Write $[\mathrm{id}, w]_k := \{u \le w \mid \ell(u) = k\}$.

**Theorem (Gaetz-G. 2025)**

*Fix $X_w$ (rationally) smooth in a finite simply-laced Weyl group $W$. Then the matrix $(c_{uv}^w)$ with rows indexed by $u \in [\mathrm{id}, w]_k$ and columns indexed by $v \in [\mathrm{id}, w]_{\ell(w)-k}$* **is** *upper triangular with 1's on the diagonal.*

Here, $\sigma_u \sigma_v = \sum_w c_{uv}^w \sigma_w$, $\sigma_w = [X_{w_0 w}] \in H^*(G/B)$.

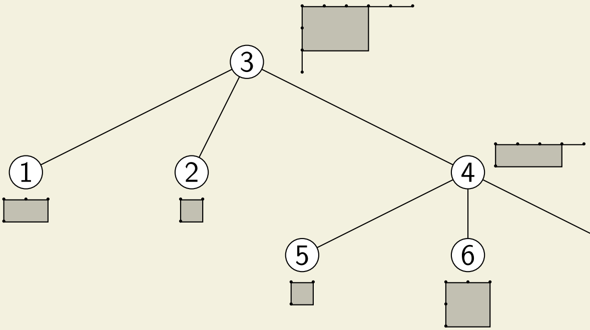
**Corollary (Gaetz-G. 2025)**

*There is a* **canonical** *bijection between $[\mathrm{id}, w]_k$ and $[\mathrm{id}, w]_{\ell(w)-k}$.*

# Applications: a "canonical" bijection

## Example of the bijection

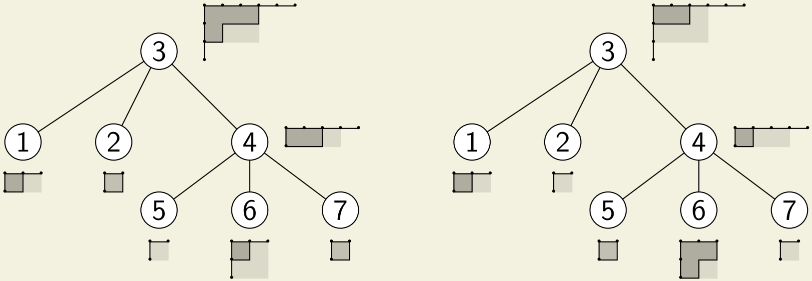
- Let $w = 65178432 \in S_8$ be smooth, and let $\mathrm{bp}(w)$ be as shown.
- $w$ is BP at $J = S \setminus \{3\}$, with $w^J = 15623478$. Decorate this data at node 3. Then continue downwards for $w_J = 321|78654$.
- Each node in $\mathrm{bp}(w)$ is decorated with a rectangle.



# Applications: a "canonical" bijection

## Example of the bijection

- For each $u \leq w$, $u^J$ is a partition shape inside $w^J$, where $J = S \setminus \{3\}$.
- Its image $v$ under this bijection should have $v^J$ being the complement shape of $u^J$ inside the rectangle $w^J$.
- Continue downwards to the parabolic subgroups.



# Remarks on the bijection

- This "canonical nature" fails in non-simply-laced types. Let $w = s_2s_3s_1s_2s_1s_3s_2s_4s_3$ in $F_4$, which is smooth. The matrix of structure constants in $G/P_J$ for $J = \{1,2,4\}$ contains

| $u \backslash v$ | $s_2s_3s_1s_2s_3$ | $s_3s_4s_1s_2s_3$ | $s_2s_3s_4s_2s_3$ |
|---|---|---|---|
| $s_4s_1s_2s_3$ | 1 | 1 | 1 |
| $s_3s_1s_2s_3$ | 1 | 1 | 0 |
| $s_3s_4s_2s_3$ | 0 | 1 | 1 |

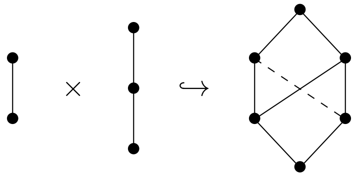
.

- Gasharov 1998 showed that for smooth $w \in S_n$, $[\mathrm{id}, w]$ admits a **Lehmer code**. One can artificially construct (many, but likely bad) bijections between $[\mathrm{id}, w]_k$ and $[\mathrm{id}, w]_{\ell(w)-k}$.

# Applications: Lehmer codes

**Definition**

A poset $P$ admits a **Lehmer code** if there exists an order-preserving bijection $L : C_{a_1} \times \cdots \times C_{a_k} \to P$ for some $a_1, \ldots, a_k \in \mathbb{Z}_{>0}$.



- The map $L$ is almost never an order isomorphism.
- The usual **Lehmer code** on the symmetric group is given by

$$L(w)_i = \#\{j > i \mid w(j) < w(i)\}.$$

For example, $L(635241) = (5, 2, 3, 1, 1)$.

# Applications: Lehmer codes

**Theorem (Carrell-Peterson 1994)**

*The followings are equivalent for any $w$ in any Coxeter group:*

- *the Kazhdan-Lusztig polynomial $P_{\mathrm{id},w}(q) = 1$;*
- *$[\mathrm{id}, w]$ is rank-symmetric in the Bruhat order.*

We call such elements **rationally smooth**.

- If $[\mathrm{id}, w]$ admits a Lehmer code, then $w$ is rationally smooth.
- Gasharov 1998: for $w \in S_n$ smooth, $[\mathrm{id}, w]$ admits a Lehmer code.
- Billey-Fan-Losonczy 1999: conjecture that $[\mathrm{id}, w]$ admits a Lehmer code for any rationally smooth $w$ in finite Weyl groups.
- Billey 1999: affirmative answer in type $A$ and $B$.
- Bolognini-Sentinelli 2025: affirmative answer for $[\mathrm{id}, w_0]$ in type $D_n$ and $H_3$.
- Bishop–Milićević–Thomas 2025: $[\mathrm{id}, w_0]$ does not have a Lehmer code in $F_4$, $H_4$ and $E_6$; more computations.
- Sentinelli-Zatti 2025: no Lehmer code for $[\mathrm{id}, w_0(F_4)]$.

# Applications: Lehmer codes

**Theorem (Gaetz-G. 2025)**

- *In types $A_n$, $B_n$, $D_n$, $I_2(m)$, $H_3$, every rationally smooth elements admit a Lehmer code.*
- *In types $H_4$, $F_4$, a rationally smooth element $w$ admits a Lehmer code if and only if $w \neq w_0$.*

- The proof heavily relies on Billey-Postnikov decompositions.
- Other Coxeter groups? e.g. affine type $A$.

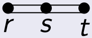
# Grassmannian BP decompositions in infinite types

**Conjecture (Richmond-Slofstra 2016; Oh-Richmond 2024)**

*Let $W$ be any Coxeter group and $w \in W$ be rationally smooth. Then $w$ is BP at some $J = S \setminus \{\alpha\}$, i.e. has a Grassmannian BP decomposition.*

- This is a fundamental step in many previous results.

**Theorem (Gaetz-G. 2025)**

*This fails for $w = srstrsr$ in affine type $\widetilde{C_2}$.* 


- This conjecture might still be true if we allow either $w$ or $w^{-1}$ to have a Gassmannian BP decomposition.

# Implicit applications: self-dual permutations

- A poset $P$ is **self-dual** if it admits an order-reversing involution.
- A permutation $w$ is **self-dual** if $[\mathrm{id}, w]$ is self-dual.
- This is a strictly stronger condition than smoothness.

**Theorem (Gaetz-G. 2020)**

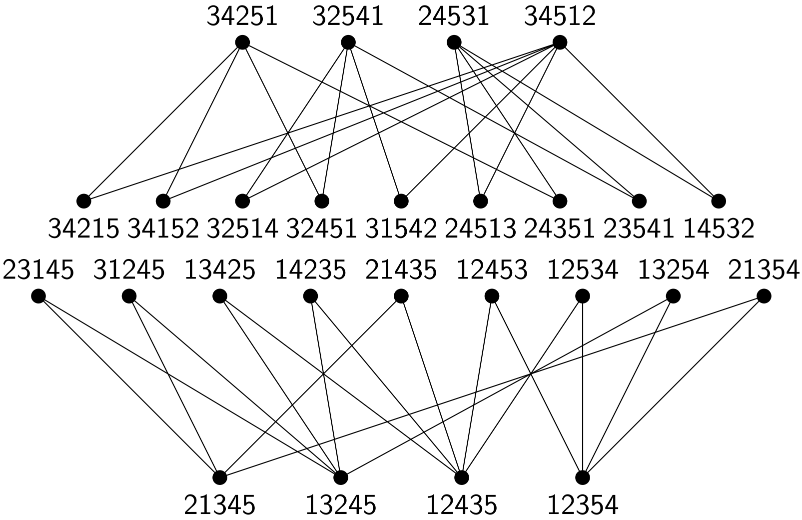
*The followings are equivalent for $w \in S_n$:*

1. *$w$ is self-dual;*
2. *the bipartite graphs $\Gamma_w$ and $\Gamma^w$ are isomorphic;*
3. *$w$ avoids the smooth patterns* 3412 *and* 4231 *as well as* 34521, 45321, 54123, *and* 54312*;*
4. *$w$ is polished.*

- Question: what about other types?

# Example of $\Gamma_w$ and $\Gamma^w$ for $w = 34521$

Let $\Gamma_w$ and $\Gamma^w$ be the bipartite graphs on $[\mathrm{id}, w]_1 \sqcup [\mathrm{id}, w]_2$ and $[\mathrm{id}, w]_{\ell-1} \sqcup [\mathrm{id}, w]_{\ell-2}$ respectively with edges given by cover relations.



# Implicit applications: self-dual permutations

**Definition (Gaetz-G. 2020)**

Let $W$ be any Coxeter group. We say that $w \in W$ is **polished** if there exist pairwise disjoint subsets $S_1, \ldots, S_k \subseteq S$ such that each $S_i$ is a connected subset of the Dynkin diagram and coverings $S_i = J_i \cup J_i'$ for $i = 1, \ldots, k$ with $J_i \cap J_i'$ totally disconnected so that

$$w = \prod_{i=1}^{k} w_0(J_i) w_0(J_i \cap J_i') w_0(J_i')$$

where the product is taken from left to right as $i = 1, 2, \ldots, k$.

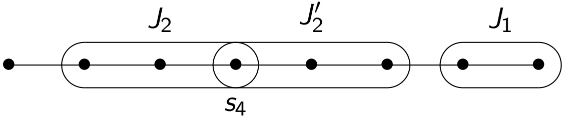
This is adding restrictions to the iterated BP decomposition of $w$ (or $w^{-1}$).

# Example of polished elements

The following element with $k = 2$, $J_1 = \{s_7, s_8\}$, $J_1' = \emptyset$, $J_2 = \{s_2, s_3, s_4\}$, $J_2' = \{s_4, s_5, s_6\}$, and multiplication in the order of

$$\begin{aligned} w = & \quad w_0(J_1)w_0(J_2)w_0(J_2 \cap J_2')w_0(J_2') \\ = & \quad 123456987 \cdot 154326789 \cdot 123546789 \cdot 123765489 \\ = & \quad 154963287 \end{aligned}$$

is a polished element.



# A top-heaviness result

The Bruhat interval $[\mathrm{id}, w]$ is top-heavy.

**Theorem (Björner-Ekedahl 2009)**

*For $w \in S_n$, $\#[\mathrm{id}, w]_k \leq \#[\mathrm{id}, w]_{\ell(w)-k}$ for $k \leq \ell(w)/2$.*

Let $\mathrm{udeg}_w(u)$ and $\mathrm{ddeg}_w(u)$ denote the up-degree and down-degree of $u$ inside the interval $[\mathrm{id}, w]$.

**Theorem (Gaetz-G. 2020)**

*Let $w \in S_n$ be smooth, then*

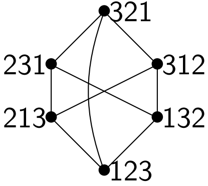
$$\max_{u \in [\mathrm{id}, w]_1} \mathrm{udeg}_w(u) \leq \max_{u \in [\mathrm{id}, w]_{\ell-1}} \mathrm{ddeg}_w(u),$$

*with equality if and only if $[\mathrm{id}, w]$ is self-dual.*

# Implicit applications: automorphisms of $\Gamma(w)$

**Definition**

The **undirected Bruhat graph** $\Gamma$ has vertex set $S_n$ and edges $w \sim wt$ for reflections $T$. Let $\Gamma(u, v)$ be its restriction to a Bruhat interval $[u, v]$.




**Theorem (Lakshmibai-Sandhya 1990, Carrell 1994)**

*A permutation $w \in S_n$ is smooth if and only if the undirected Bruhat graph $\Gamma(\mathrm{id}, w) =: \Gamma(w)$ is regular.*

# Implicit applications: automorphisms of $\Gamma(w)$

- The **directed Bruhat graph** has no interesting automorphisms: Waterhouse 1989 showed that $\mathrm{Aut}(W, \leq)$ is generated by automorphisms of the Dynkin diagram and the group inversion map.
- The undirected Bruhat graph has many more automorphisms.

**Question**

*Can we describe* $\mathrm{Aut}(\Gamma(u, v))$*, or* $\mathrm{Aut}(\Gamma(w))$*?*

We can write down some automorphisms $\varphi \in \mathrm{Aut}(\Gamma(w))$:

- multiplication on the left by $s_i$, where $i \in D_L(w)$;
- multiplication on the right by $s_i$, where $i \in D_R(w)$;
- **middle multiplication**.

# Implicit applications: automorphisms of $\Gamma(w)$

## Proposition (Gaetz-G. 2022)

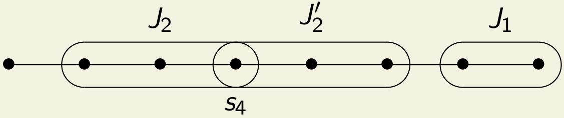
*Suppose* $w = w^J w_J$ *is a Billey-Postnikov decomposition, and* $\mathrm{Supp}(w^J) \cap \mathrm{Supp}(w_J) = \{s\} \subset J$, *then the middle multiplication map*

$$\phi : x \mapsto x^J s x_J$$

*is an automorphism of the Bruhat graph* $\Gamma(w)$.

## Recall polished elements

$$w = w_0(J_1) w_0(J_2) w_0(J_2 \cap J_2') w_0(J_2').$$



# The identity orbit under $\mathrm{Aut}(\Gamma(w))$

**Conjecture (Gaetz-G. 2022)**

*Let $w \in S_n$ and $\mathcal{O} = \{\varphi(\mathrm{id}) \mid \varphi \in \mathrm{Aut}(\Gamma(w))\}$ be the orbit of the identity under graph automorphisms of $\Gamma(w)$. Then*

$$\mathcal{O} = [\mathrm{id}, v], \text{ for some } v \leq w.$$

- We have a conjectural formula for $v$.
- Equivalently, we conjecture that the identity orbit $\mathcal{O}$ is "essentially" generated by middle multiplications.
- We cannot prove $\mathcal{O}$ is downwards closed, or it has a maximum.
- Not true in other types.

# Vertex-transitive permutations

**Definition (Gaetz-G. 2022)**

A permutation $w \in S_n$ is **vertex-transitive** if $\mathrm{Aut}(\Gamma(w))$ acts transitively on the vertex set $[\mathrm{id}, w]$.

- This is a strictly stronger condition than smoothness: if $w$ is vertex-transitive, every vertex in $\Gamma(w)$ has the same degree.

**Theorem (Gaetz-G. 2022)**

*A permutation $w$ is vertex-transitive if and only if it avoids the smooth patterns* 3412 *and* 4231*, as well as* 34521 *and* 54123*.*

- $\{$vertex-transitive patterns$\} \subsetneq \{$self-dual patterns$\}$!
- This is a special case of the identity orbit conjecture.

# Thanks

Thank you for listening!