The canonical bijection between pipe dreams and bumpless pipe dreams

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Overview

1. Schubert polynomials
   - Compatible sequences
   - Pipe dreams
   - Bumpless pipe dreams
2. The bijection
3. Monk's rule on PDs and BPDs
4. Why "canonical"?

The flag variety and Schubert polynomials

The flag variety is

$$Fl(\mathbb{C}^n) = \{\emptyset = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n \mid \dim V_i = i\}.$$

The Bruhat decomposition gives $Fl(\mathbb{C}^n) = \coprod_{\pi \in S_n} \Omega_\pi$ into open Schubert cells. The Schubert varieties are $X_\pi := \overline{\Omega_\pi}$.

The classes of Schubert varieties $\{[X_\pi] \mid \pi \in S_n\}$ form a $\mathbb{Z}$-linear basis of

$$H^*(Fl(\mathbb{C}^n), \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{I}$$

where $\mathfrak{I} = \langle \text{symmetric functions with no constant terms} \rangle$.

Schubert polynomial, $\{\mathfrak{S}_\pi \mid \pi \in S_n\} \subset \mathbb{Z}[x_1, \ldots, x_n]$ are nice polynomial representatives of Schubert classes.

Schubert polynomials

Schubert polynomials, $\{\mathfrak{S}_\pi \mid \pi \in S_n\}$, are polynomial representatives of the Schubert classes of the full flag variety.

The **divided difference operator** acts on polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$$

where s_i acts by swapping x_i and x_{i+1}.

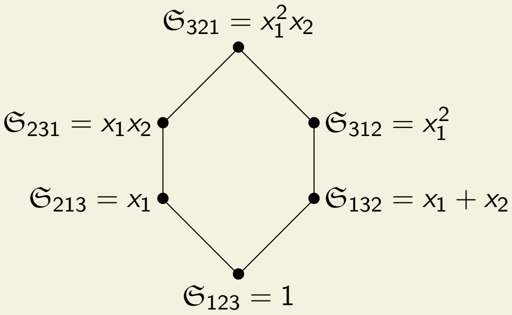
Definition (Schubert polynomials)

For $\pi \in S_n$, its **Schubert polynomial** is

$$\mathfrak{S}_\pi(x) := \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } \pi = n \; n{-}1 \; \cdots 2 \; 1, \\ \partial_i \mathfrak{S}_{\pi s_i}(x) & \text{if } \ell(\pi) < \ell(\pi s_i). \end{cases}$$

Schubert polynomials

Example: Schubert polynomials in S_3



Schubert polynomial has the **stability** property: for $m < n$ and $\pi \in S_m$, if we let $\iota : S_m \hookrightarrow S_n$ be the natural embedding, then $\mathfrak{S}_\pi = \mathfrak{S}_{\iota(\pi)}$.

From now on, assume all permutations live in

$$S_\infty := \{\text{bijections } \mathbb{Z}_{>0} \to \mathbb{Z}_{>0} \text{ with all but finitely many fixed points}\}.$$

Compatible sequences

Definition (Compatible sequence, Billey-Jockusch-Stanley 1993)

For $\pi \in S_\infty$, $(\mathsf{a} = (a_1, \ldots, a_\ell), \mathsf{r} = (r_1, \ldots, r_\ell)) \in \mathbb{Z}_{>0}^{2\times\ell}$ is a **(reduced) compatible sequence** of π, where $\ell = \ell(\pi)$, if

- $\mathsf{a} = (a_1, \ldots, a_\ell)$ is a reduced word of π;
- $r_1 \leq \cdots \leq r_\ell$ is weakly increasing;
- $r_j \leq a_j$ for $j = 1, \ldots, \ell$;
- if $a_j < a_{j+1}$, then $r_j < r_{j+1}$.

Let $\mathrm{RC}(\pi)$ denote all compatible sequences of π.

Theorem (Billey-Jockusch-Stanley 1993)

For $\pi \in S_\infty$, $\mathfrak{S}_\pi = \sum_{(\mathsf{a},\mathsf{r})\in\mathrm{RC}(\pi)} x_{r_1} \cdots x_{r_\ell}$.

Compatible sequences

Definition (Compatible sequence)

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- if $a_j < a_{j+1}$, then $r_j < r_{j+1}$.

Example: compatible sequences

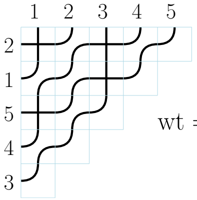
Let $\pi = 321 = s_1 s_2 s_1 = s_2 s_1 s_2$.
If $\mathrm{a} = (1, 2, 1)$, then there are no r's compatible with a.
If $\mathrm{a} = (2, 1, 2)$, then we can take $\mathrm{r} = (1, 1, 2)$.
Thus, $\mathfrak{S}_{321} = x_1^2 x_2$.

Pipe dreams (RC-graphs)

Definition (Pipe dream, RC-graph, Bergeron-Billey 1993)

A **(reduced) pipe dream** or **RC-graph** D of $\pi \in S_\infty$ is a filling of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by -tiles and ⊞-tiles, such that for any $i \in \mathbb{Z}_{>0}$, the pipe i that starts in column i ends at row $\pi(i)$, and that no two pipes cross twice.

The weight of a pipe dream D is $\mathrm{wt}(D) = \prod_{(i,j)\in\mathrm{cross}(D)} x_i$.



$\mathrm{wt} = x_1^2 x_2 x_3$

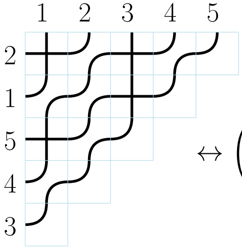
Theorem (Bergeron-Billey 1993)

For $\pi \in S_\infty$, $\mathfrak{S}_\pi = \sum_{D\in\mathrm{PD}(\pi)} \mathrm{wt}(D)$.

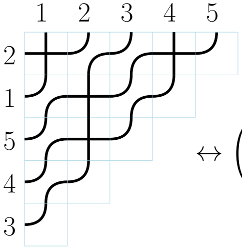
Pipe dreams and compatible sequences

There is a natural bijection from pipe dreams to compatible sequences:

- associate a simple transposition s_{i+j-1} to each ⊞-tile at (i,j) in D;
- from top to bottom, and right to left, read off the simple transposition and the row index of each ⊞-tile in D to obtain the **reading word** $\mathrm{a}(D)$ and $\mathrm{r}(D)$, respectively.



$$\leftrightarrow \begin{pmatrix} 3 & 1 & 4 & 3 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$



$$\leftrightarrow \begin{pmatrix} 4 & 1 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

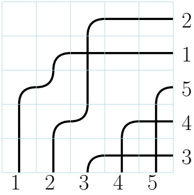
From now on, we consider pipe dreams and compatible sequences to be the same object, denoted as $\mathrm{PD}(\pi)$.

Bumpless pipe dreams

Definition (Bumpless pipe dream, Lam-Lee-Shimozono 2018)

A **(reduced) bumpless pipe dream** of $\pi \in S_\infty$ if a filling of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by , , , , ,  (but no ), such that for any $i \in \mathbb{Z}_{>0}$, the pipe i that starts in column i ends at row $\pi(i)$, and that no two pipes cross twice.

The weight of a bumpless pipe dream D is $\mathrm{wt}(D) = \prod_{(i,j)\in \mathrm{blank(D)}} x_i$.



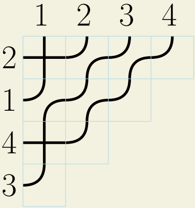
$\mathrm{wt} = x_1^2 x_2 x_3$

Theorem (Lam-Lee-Shimozono 2018)

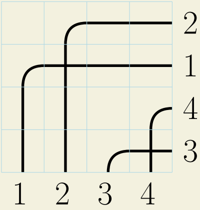
For $\pi \in S_\infty$, $\mathfrak{S}_\pi = \sum_{D\in \mathrm{BPD}(\pi)} \mathrm{wt}(D)$.

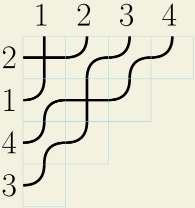
An example of a Schubert polynomial

Example: $\mathfrak{S}_{2143} = x_1x_3 + x_1x_2 + x_1^2$

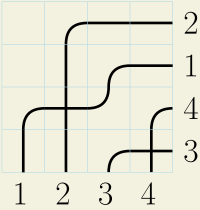


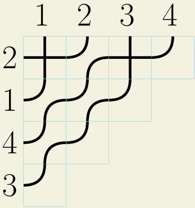
$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$



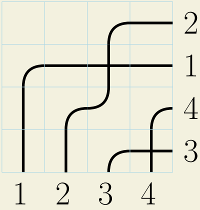


$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$$





$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$



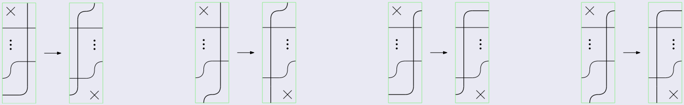
The bijection

We construct the bijection by interpreting compatible sequences on BPDs.

Definition (The pop operation on BPDs)

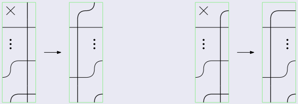
Let $D \in \mathrm{BPD}(\pi)$. Mark the "first" (top, right) □-tile in D by a label $\times$. Let r be its row index.

1. If $\times$ is not the rightmost in its horizontally contiguous block of □-tiles, move it to the rightmost □-tile in this block; let it be at coordinate (x, y). Consider the unique pipe p that passes through $(x, y+1)$. If $p \leq y$, go to step (2) and if $p = y+1$, go to step (3).
2. Do a column move to pipe p from its ⌜-tile in column $y+1$ to coordinate (x, y), and move the label $\times$ accordingly. Repeat step (1).



3. Remove the label $\times$ and resolve the ⊞-tile between pipe $p-1$ and pipe p.

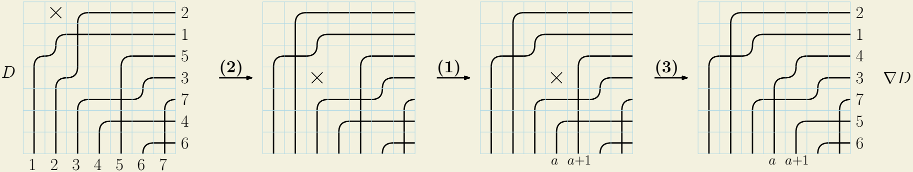
Let $a = p - 1 = y$.



We obtain $\nabla D \in \mathrm{BPD}(s_a\pi)$, and write $\mathrm{pop}(D) = (a, r)$.

The pop operation on BPDs

Example: ∇D



Here, $D \in \mathrm{BPD}(2153746)$, $\mathrm{pop}(D) = (4,1)$, $\nabla D \in \mathrm{BPD}(2143756)$.

Each step of the operation ∇ is invertible, so we can uniquely recover D (if it exists) from $\mathrm{pop}(D)$ and ∇D.

The bijection

Definition (G. and Huang 2021)

Given $D \in \mathrm{BPD}(\pi)$ with $\ell(\pi) = \ell$, let

$$\varphi(D) = (\mathrm{a} = (a_1, \ldots, a_\ell), \mathrm{r} = (r_1, \ldots, r_\ell))$$

where $\mathrm{pop}(\nabla^{i-1} D) = (a_i, r_i)$ for $i = 1, \ldots, \ell$.

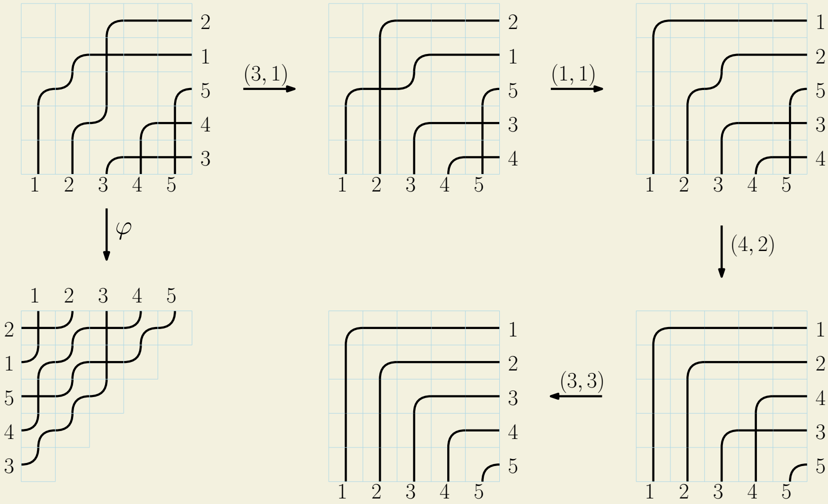
Theorem (G. and Huang 2021)

The map φ is a weight-preserving bijection between $\mathrm{BPD}(\pi)$ and $\mathrm{PD}(\pi)$, i.e. compatible sequences of π.

For consistency, for $D \in \mathrm{PD}(\pi)$ whose **first** (top, right) ⊞-tile is at coordinate $(r, a - r + 1)$, we write $\nabla D \in \mathrm{PD}(s_a \pi)$ to be the pipe dream by removing D's first ⊞-tile, and write $\mathrm{pop}(D) = (a, r)$.

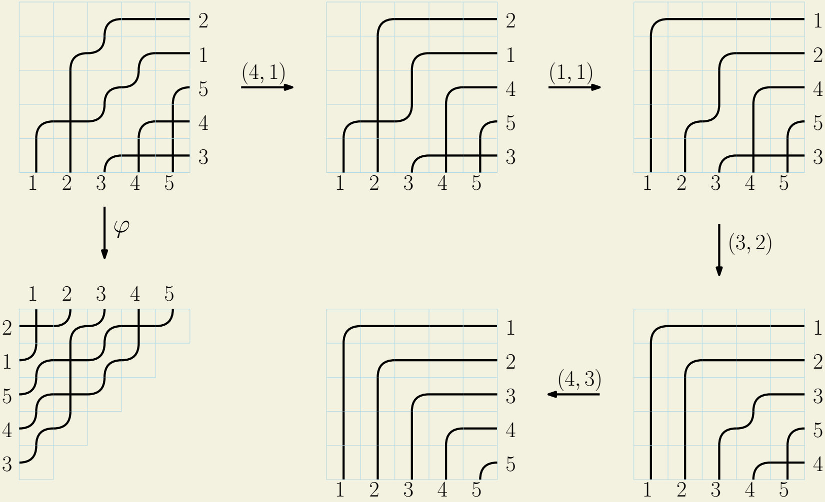
The bijection

Example: the bijection φ



The bijection

Example: the bijection φ



Monk's rule

Theorem (Monk's rule)

For $\pi \in S_\infty$ and $\alpha \geq 1$,

$$\mathfrak{S}_\pi \mathfrak{S}_{s_\alpha} = \sum_{k \leq \alpha < l,\ \ell(\pi t_{kl}) = \ell(\pi)+1} \mathfrak{S}_{\pi t_{kl}}.$$

Recall $\mathfrak{S}_{s_\alpha} = x_1 + x_2 + \cdots + x_\alpha$.

Theorem (Monk's rule)

For $\pi \in S_\infty$ and $\alpha \geq 1$,

$$x_\alpha \mathfrak{S}_\pi + \sum_{k < \alpha,\ \ell(\pi t_{k\alpha}) = \ell(\pi)+1} \mathfrak{S}_{\pi t_{k\alpha}} = \sum_{l > \alpha,\ \ell(\pi t_{\alpha l}) = \ell(\pi)+1} \mathfrak{S}_{\pi t_{\alpha l}}.$$

If RHS is a single term, this is the **transition formula**.
If the sum in LHS is empty, this is the **cotransition formula**.

Why "canonical"?

Theorem (Monk's rule)

For $\pi \in S_\infty$ and $\alpha \geq 1$,

$$x_\alpha \mathfrak{S}_\pi + \sum_{k<\alpha,\ \ell(\pi t_{k\alpha})=\ell(\pi)+1} \mathfrak{S}_{\pi t_{k\alpha}} = \sum_{l>\alpha,\ \ell(\pi t_{\alpha l})=\ell(\pi)+1} \mathfrak{S}_{\pi t_{\alpha l}}.$$

Monk's rule can be proved combinatorially on both PDs and BPDs.

$$\begin{cases} x_\alpha \rightsquigarrow : & \mathrm{PD}(\pi) \to \bigcup_{\substack{l>\alpha \\ \pi t_{\alpha l} \gtrdot \pi}} \mathrm{PD}(\pi t_{\alpha l}) \\ m_{k,\beta} : & \mathrm{PD}(\pi t_{k\beta}) \to \bigcup_{\substack{l>\beta \\ \pi t_{\beta l} \gtrdot \pi}} \mathrm{PD}(\pi t_{\beta l}),\ k<\beta, \pi t_{k\beta} \gtrdot \pi \end{cases}$$

Our main theorem is the following.

Theorem (G. and Huang 2021)

The bijection φ intertwines with $x_\alpha \rightsquigarrow$, $m_{k,\beta}$'s.

Why "canonical"?

Our main theorem unifies all approaches to define the bijection via any versions of Monk's rule. Let's define some bijections via **cotransition**.

For each $\pi \in S_\infty$, fix a choice $\alpha = \alpha_\pi \in \mathbb{Z}_{>0}$ such that there are no $k < \alpha$ with $\ell(\pi t_{k\alpha}) = \ell(\pi) + 1$. Then

$$x_\alpha \mathfrak{S}_\pi = \sum_{l>\alpha, \ell(\pi t_{\alpha l}) = \ell(\pi)+1} \mathfrak{S}_{\pi t_{\alpha l}}.$$

For example, for $\pi = \underline{42}5\underline{1}6873$, we can choose $\alpha_\pi \in \{1, 2, 4\}$.

For a fixed family of choices $\{\alpha_\pi\}_{\pi \in S_\infty}$, one can define a weight-preserving bijection between BPD(π) and PD(π) via reverse induction on $\ell(\pi)$, i.e. from RHS to LHS of the above equation.

Our main theorem implies that these bijections are all the same, which do not depend on the choices $\{\alpha_\pi\}_{\pi \in S_\infty}$.

Monk's rule on pipe dreams

$$\begin{cases} x_\alpha \rightsquigarrow : & \mathrm{PD}(\pi) \to \bigcup_{\substack{l>\alpha \\ \pi t_{\alpha l} \gtrdot \pi}} \mathrm{PD}(\pi t_{\alpha l}) \\ m_{k,\beta} : & \mathrm{PD}(\pi t_{k\beta}) \to \bigcup_{\substack{l>\beta \\ \pi t_{\beta l} \gtrdot \pi}} \mathrm{PD}(\pi t_{\beta l}),\ k < \beta, \pi t_{k\beta} \gtrdot \pi \end{cases}$$

Definition (Bergeron-Billey 1993, Billey-Holroyd-Young 2019)

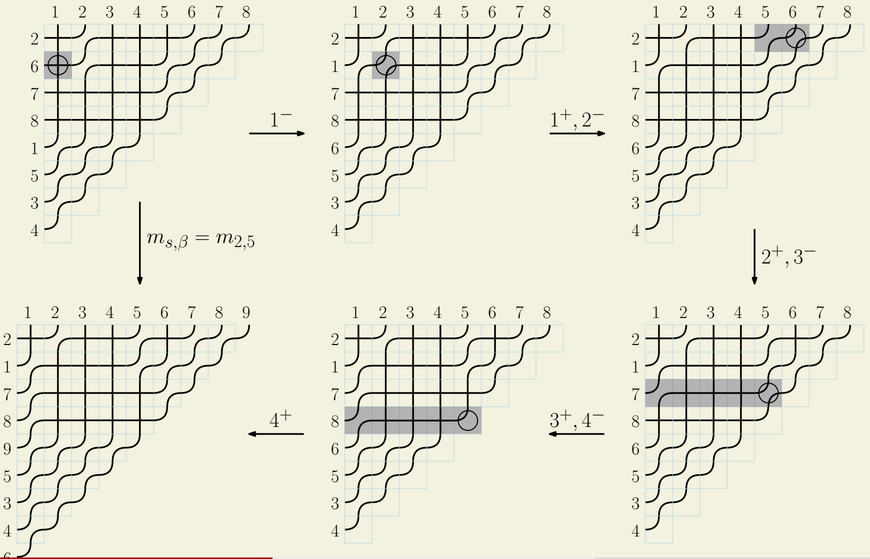
Let $D \in \mathrm{PD}(\pi t_{k\beta})$. The following procedure produces $m_{k,\beta}(D)$.

1. Locate the $\boxplus$-tile between pipe $\pi^{-1}(k)$ and $\pi^{-1}(\beta)$, make it ◜◞. Search towards the right and make the first ◜◞-tile into a $\boxplus$-tile.
2. If the newly added $\boxplus$-tile creates a double crossing with a $\boxplus$-tile at coordinate (x, y), make the tile at (x, y) into an ◜◞-tile, search towards its right and turn the first ◜◞-tile into a $\boxplus$-tile. Repeat.

For $D \in \mathrm{PD}(\pi)$ and $x_\alpha \rightsquigarrow D$, replace the first step above by inserting a $\boxplus$-tile into the leftmost ◜◞-tile in row α.

Monk's rule on pipe dreams

Example: basic Monk steps on pipe dreams



Monk's rule on bumpless pipe dreams

$$\begin{cases} x_\alpha \rightsquigarrow : & \mathrm{BPD}(\pi) \to \bigcup_{\substack{l>\alpha \\ \pi t_{\alpha l} \gtrdot \pi}} \mathrm{BPD}(\pi t_{\alpha l}) \\ m_{k,\beta} : & \mathrm{BPD}(\pi t_{k\beta}) \to \bigcup_{\substack{l>\beta \\ \pi t_{\beta l} \gtrdot \pi}} \mathrm{BPD}(\pi t_{\beta l}),\ k<\beta, \pi t_{k\beta} \gtrdot \pi \end{cases}$$

Definition (Huang 2020)

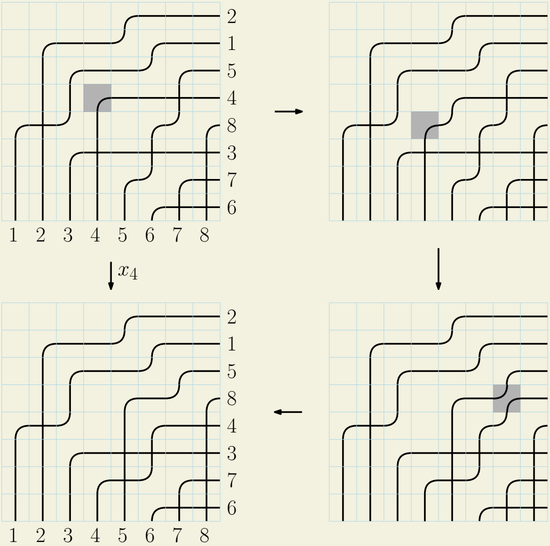
Let $D \in \mathrm{BPD}(\pi t_{k\beta})$. The following procedure produces $m_{k,\beta}(D)$.

1. Locate the $\boxplus$-tile at coordinate (x,y) between pipe $\pi^{-1}(k)$ and $\pi^{-1}(\beta)$ and turn it into .
2. Do a min-droop for  at (x,y) into coordinate (x',y').
   (a) If (x',y') was $\square$, let (x,y) be the coordinate of the -tile in row x' and repeat step (2).
   (b) If (x',y') was , turn it into $\boxplus$. If these two pipes cross somewhere else at (x,y), make it into  and repeat step (2), otherwise stop.

For $x_\alpha \rightsquigarrow$, replace the first step above by initializing (x,y) to be the rightmost -tile in row α.

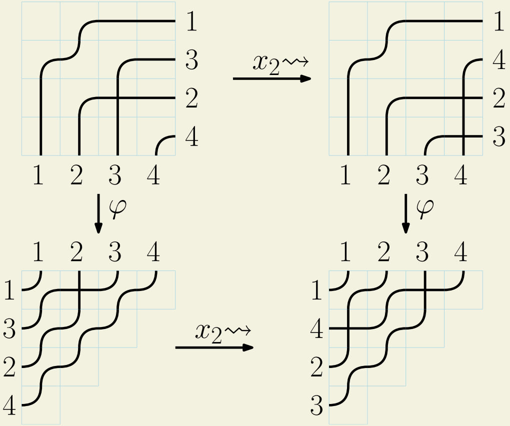
Monk's rule on bumpless pipe dreams

Example: basic Monk steps on bumpless pipe dreams



Basic Monk steps on pipe dreams and bumpless pipe dreams look similar, but we note that they don't correspond.

Example: basic Monk steps on PDs and BPDs



The main theorem

Recall Monk's rule

$$x_\alpha \mathfrak{S}_\pi + \sum_{k<\alpha,\ \ell(\pi t_{k\alpha})=\ell(\pi)+1} \mathfrak{S}_{\pi t_{k\alpha}} = \sum_{l>\alpha,\ \ell(\pi t_{\alpha l})=\ell(\pi)+1} \mathfrak{S}_{\pi t_{\alpha l}}.$$

Theorem (G. and Huang 2021)

The following diagrams commute for $\pi \in S_\infty$.

$$\begin{array}{ccc} \mathrm{BPD}(\pi) & \xrightarrow{x_\alpha \rightsquigarrow} & \bigcup_{\substack{l>\alpha \\ \pi t_{\alpha,l} \gtrdot \pi}} \mathrm{BPD}(\pi t_{\alpha,l}) \\ \downarrow \varphi & & \downarrow \varphi \\ \mathrm{PD}(\pi) & \xrightarrow{x_\alpha \rightsquigarrow} & \bigcup_{\substack{l>\alpha \\ \pi t_{\alpha,l} \gtrdot \pi}} \mathrm{PD}(\pi t_{\alpha,l}) \end{array}, \qquad \begin{array}{ccc} \mathrm{BPD}(\pi t_{k,\beta}) & \xrightarrow{m_{k,\beta}} & \bigcup_{\substack{l>\beta \\ \pi t_{\beta,l} \gtrdot \pi}} \mathrm{BPD}(\pi t_{\beta,l}) \\ \downarrow \varphi & & \downarrow \varphi \\ \mathrm{PD}(\pi t_{k,\beta}) & \xrightarrow{m_{k,\beta}} & \bigcup_{\substack{l>\beta \\ \pi t_{\beta,l} \gtrdot \pi}} \mathrm{PD}(\pi t_{\beta,l}) \end{array}.$$

We use induction on $\ell(\pi)$. For $B \in \mathrm{BPD}(\pi t_{k\beta})$ with $D = \varphi(B)$, to show that $m_{k,\beta}(B)$ and $m_{k,\beta}(D)$ are in bijection, we need

- $\mathrm{pop}(m_{k,\beta}(B)) = \mathrm{pop}(m_{k,\beta}(D))$,
- $\nabla(m_{k,\beta}(B))$ and $\nabla(m_{k,\beta}(D))$ are in bijection.

Proof idea of the main theorem

Recall that $B \in \mathrm{BPD}(\pi t_{k\beta})$ and $D \in \mathrm{PD}(\pi t_{k\beta})$ are in bijection.
We want to show that $\nabla(m_{k,\beta}(B))$ and $\nabla(m_{k,\beta}(D))$ are in bijection.

If ∇ and $m_{k,\beta}$ commute, then $\nabla(m_{k,\beta}(B)) = m_{k,\beta}(\nabla B)$ and $\nabla(m_{k,\beta}(D)) = m_{k,\beta}(\nabla D)$ are in bijection by induction hypothesis.

In general, ∇ and $m_{k,\beta}$ don't commute, in which case $\nabla(m_{k,\beta}(D))$ is the same as applying Monk's rule twice to ∇D.

We argue separately on PDs and BPDs how well ∇ commutes with $m_{k,\beta}$. This is dependent on a "global" criterion $\mathrm{Des}_L(\rho)$ where

$$\rho := \mathrm{perm}(m_{k,\beta}(\nabla B)) = \mathrm{perm}(m_{k,\beta}(\nabla D)).$$

The main technical lemma

Lemma (G. and Huang 2021)

1. *Suppose* $D \in \mathrm{PD}(\pi t_{k\beta})$ *or* $\mathrm{BPD}(\pi t_{k\beta})$. *Let* $\mathrm{pop}(D) = (i, r)$.
   1. *If* $(s, \beta) \neq (\pi^{-1}(i+1), \pi^{-1}(i))$, *let* $\rho := \mathrm{perm}(m_{k,\beta}(\nabla D))$. *Then*
   $$\mathrm{pop}(m_{k,\beta}(D)) = \begin{cases} (i+1, r) & \text{if } i \in \mathrm{Des}_L(\rho) \\ (i, r) & \text{otherwise} \end{cases}.$$
   $$\nabla(m_{k,\beta}(D)) = \begin{cases} m_{\rho^{-1}(i+2),\rho^{-1}(i+1)}(m_{k,\beta}(\nabla D)) & \text{if } i, i+1 \in \mathrm{Des}_L(\rho) \\ m_{k,\beta}(\nabla D) & \text{otherwise} \end{cases}.$$
   2. *If* $(s, \beta) = (\pi^{-1}(i+1), \pi^{-1}(i))$, *then* $\mathrm{pop}(m_{k,\beta}(D)) = (i+1, r)$. *Furthermore, let* $\rho := \mathrm{perm}(\nabla(D))$. *Then,*
   $$\nabla(m_{k,\beta}(D)) = \begin{cases} m_{\rho^{-1}(i+2),\rho^{-1}(i+1)}(\nabla D) & \text{if } i+1 \in \mathrm{Des}_L(\rho) \\ \nabla D & \text{otherwise} \end{cases}.$$
2. *Suppose* $D \in \mathrm{PD}(\pi)$ *and* $\mathrm{pop}(D) = (i, r)$. *Then...*

Thanks

Thank you for listening!