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The weak Bruhat order on the symmetric group is Sperner

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Abstract. We construct a simple combinatorially-defined representation of $\mathfrak{sl}_2$ which respects the order structure of the weak order on the symmetric group. This is used to resolve the longstanding open problem of showing that the weak order has the strong Sperner property, and is therefore a Peck poset.

Keywords: poset, weak Bruhat order, Sperner, Peck

1 Introduction

1.1 The Sperner property

We refer the reader to [10] for basic facts and terminology about posets in what follows.

Let P be a finite ranked poset with rank decomposition

$$P = P_0 \sqcup \cdots \sqcup P_r.$$

We say that P is *k-Sperner* if no union of k antichains of P is larger than the union of the largest k ranks. If P is k-Sperner for $k = 1, ..., r+1$, we say that P is *strongly Sperner*. Let $p_i = |P_i|$, then we say P is *rank symmetric* if $p_i = p_{r-i}$ and *rank unimodal* if

$$p_0 \leq p_1 \leq \cdots \leq p_{j-1} \leq p_j \geq p_{j+1} \geq \cdots \geq p_r$$

for some j. If P is rank-symmetric, rank-unimodal, and strongly Sperner, then P is *Peck*.

The Sperner property has long been of interest in both extremal and algebraic combinatorics. For example, Sperner's Theorem, which asserts that the Boolean lattice B_n is Sperner, is central to extremal set theory. A wide variety of methods have been used to demonstrate that various posets have the Sperner property: Lubell gave an elegant probabilistic proof for the Boolean lattice, linear algebraic methods have been used (see Proposition 1), as have explicit combinatorial methods such as constructions of symmetric chain decompositions (see Section 3.2 for a definition).

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In [9], Stanley used the Hard Lefschetz Theorem from algebraic geometry to prove that a large class of posets are strongly Sperner and obtained the Erdős-Moser Conjecture as a corollary. This class includes the strong Bruhat order (see Section 1.2), but not the weak order considered here.

1.2 The weak order

Let S_n denote the symmetric group of permutations of n elements, viewed as a Coxeter group with respect to the simple transpositions $s_i = (i\,i+1)$ for $i = 1, \ldots, n-1$. The *weak order* $W_n = (S_n, \leq)$ is the poset structure on S_n whose cover relations are defined as follows: $u \lessdot w$ if and only if $w = us_i$ for some i and $\ell(w) = \ell(u) + 1$, where ℓ denotes Coxeter length. This poset is graded with the rank of w given by $\ell(w)$; the permutation of maximum length has one-line notation $n\,(n-1)\,(n-2)\ldots 1$ and length $\binom{n}{2}$.

This definition is in contrast to the *strong order* (or *Bruhat order*) on S_n which has cover relations corresponding to right multiplication by any transposition $t_{ij} = (i\,j)$, rather than just the simple transpositions s_i; it was proven in [9] that the strong order is Peck. The weak and strong orders share the same ground set and rank structure, so the weak order is rank-symmetric and rank-unimodal. The Sperner property of W_n, however, does not follow from that of the strong order, since W_n has fewer covering relations.

Whether W_n is Sperner has been investigated at least since Björner in 1984 [2], and a positive answer was conjectured by Stanley [8]. Our main result is a positive answer to this problem:

Theorem 1. *For all $n \geq 0$ the weak order W_n is strongly Sperner, and therefore Peck.*

1.3 Order raising operators

For $P = P_0 \sqcup \cdots \sqcup P_r$ a finite graded poset, and $S \subseteq P$, let $\mathbb{C}S$ denote the vector space of formal linear combinations of elements of S. A linear map $U : \mathbb{C}P \to \mathbb{C}P$ sending elements $x \in P$ to $\sum_y c_{xy} y$ is called an *order raising operator* if $c_{xy} = 0$ unless $x \lessdot y$. Any linear map $D : \mathbb{C}P \to \mathbb{C}P$ which maps each subspace $\mathbb{C}P_k$ into $\mathbb{C}P_{k-1}$ is called a *lowering operator*. We remark that a lowering operator does not need to respect the order.

Proposition 1 (Stanley [9])**.** *Suppose there exists an order raising operator $U : \mathbb{C}P \to \mathbb{C}P$ such that if $0 \leq k < \frac{r}{2}$ then $U^{r-2k} : \mathbb{C}P_k \to \mathbb{C}P_{r-k}$ is invertible. Then P is strongly Sperner.*

In [8], Stanley suggested that the order raising operator $U : \mathbb{C}W_n \to \mathbb{C}W_n$ defined for $w \in W_n$ by

$$U \cdot w = \sum_{i:\, \ell(ws_i) = \ell(w)+1} i \cdot ws_i$$

and extended by linearity may have the desired property. He conjectured an explicit non-vanishing product formula for the determinants of the maps $U^{\binom{n}{2}-2k} : \mathbb{C}(W_n)_k \to \mathbb{C}(W_n)_{\binom{n}{2}-k}$ for $0 \leq k < \frac{1}{2}\binom{n}{2}$, which, by Proposition 1 would imply Theorem 1.

In Section 2, we prove that $U^{\binom{n}{2}-2k}$ is invertible by constructing a representation of $\mathfrak{sl}_2$ on $\mathbb{C}W_n$ with weight spaces $\mathbb{C}(W_n)_i$ such that the standard generator $e \in \mathfrak{sl}_2$ acts by U (a result of Proctor [6] implies that, if W_n is Peck, then there is *some* such representation in which e acts as an order raising operator).

Remark. In subsequent work [3], Hamaker, Pechenik, Speyer, and Weigandt interpret this $\mathfrak{sl}_2$-action in terms of derivatives of Schubert polynomials, and use this to prove Stanley's conjectured determinant.

2 An action of $\mathfrak{sl}_2$

We define a lowering operator $D : \mathbb{C}W_n \to \mathbb{C}W_n$ by

$$D \cdot w = \sum_{\substack{1 \leq i < j \leq n \\ \ell(wt_{ij}) = \ell(w) - 1}} \Big(2\big(w_i - w_j - a(w, wt_{ij})\big) - 1\Big) \cdot wt_{ij}$$

where $a(w, wt_{ij}) := \#\{k < i : w_j < w_k < w_i\}$. Here t_{ij} denotes the transposition of i and j, which is (i, j) in cycle notation. Note that the sum in our definition is over all covering relations in the strong Bruhat order (the fact that D does not respect the *weak* order will be immaterial to our argument). Combinatorial implications of the fact that the strong Bruhat order appears here are studied in [1]. See Figure 1 for a depiction of the order raising operator U and the lowering operator D in the case $n = 3$.

We also define a modified rank function $H : \mathbb{C}W_n \to \mathbb{C}W_n$ by

$$H(w) = \left(2 \cdot \ell(w) - \binom{n}{2}\right) \cdot w$$

for $w \in W_n$ and extending by linearity. Since H acts as a multiple of the identity on each rank, it is clear that for *any* raising operator U and lowering operator D we have

$$HU - UH = 2U \tag{2.1}$$

$$HD - DH = -2D. \tag{2.2}$$

In this section, we show that U, D together with H provide a representation of $\mathfrak{sl}_2$ on

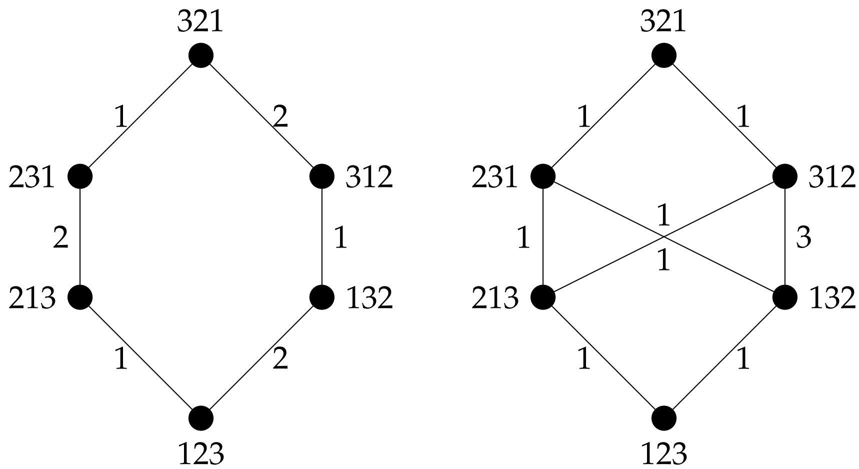

Figure 1: The edge weights for the order raising operator U (left) and the lowering operator D (right).

$\mathbb{C}W_n$. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has a standard linear basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and is determined by the relations $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. Here $[,]$ denotes the standard Lie bracket: $[X, Y] := XY - YX$.

In light of (2.1) and (2.2), all that remains is to check that $[U, D] = H$. We can view $[U, D] = UD - DU$ and H as matrices of size $n! \times n!$ with rows and columns indexed by permutations, and we show that they are equal by comparing entries via Lemmas 1 and 2 below.

Lemma 1. *For every $w \in W_n$, $(UD - DU)_{w,w} = 2 \cdot \ell(w) - \binom{n}{2}$.*

Proof. Assume $w \in (W_n)_k$, meaning $\ell(w) = k$. We have that, by definition,

$$\begin{aligned} UD_{w,w} &= \sum_{u \in (W_n)_{k-1}} D_{u,w} \cdot U_{w,u} = \sum_{u \lessdot_{W_n} w} D_{u,w} \cdot U_{w,u} \\ &= \sum_{i:\ w_i > w_{i+1}} i \cdot \Big(2\big(w_i - w_{i+1} - a(w, ws_i)\big) - 1 \Big) \\ &= \sum_{i:\ w_i > w_{i+1}} 2i(w_i - w_{i+1}) + \sum_{i:\ w_i > w_{i+1}} \big(-2i \cdot \#\{j < i : w_{i+1} < w_j < w_i\} - i \big) \end{aligned}$$

where $\lessdot_{W_n}$ denotes the covering relations in the weak order W_n. Similarly,

$$-DU_{w,w} = \sum_{i:\ w_i < w_{i+1}} 2i(w_i - w_{i+1}) + \sum_{i:\ w_i < w_{i+1}} \Big(2i \cdot \#\{j < i : w_i < w_j < w_{i+1}\} + i \Big)$$

Putting them together, we obtain

$$\begin{aligned} (UD - DU)_{w,w} =& \sum_i 2i(w_i - w_{i+1}) + A \\ =& 2(w_1 - w_2) + 4(w_2 - w_3) + \cdots + (2n-2)(w_{n-1} - w_n) + A \\ =& n^2 + n - 2nw_n + A \end{aligned}$$

where by switching the order of summation, we can write A as a sum over j's instead of i's as above:

$$A = \sum_{j=1}^{n-1} \left(\mathrm{sgn}(w_{j+1} - w_j) \cdot j - 2 \left(\sum_{\substack{i:\ i > j \\ w_{i+1} < w_j < w_i}} i \right) + 2 \left(\sum_{\substack{i:\ i > j \\ w_i < w_j < w_{i+1}}} i \right) \right).$$

Here $\mathrm{sgn} : \mathbb{R}^\times \to \{\pm 1\}$ is the sign function. Denote the summand above by A_j so that $A = \sum_{j=1}^{n-1} A_j$ and let's see how each A_j simplifies.

Assume first that $w_j < w_{j+1}$. Then $\mathrm{sgn}(w_{j+1} - w_j) = 1$. Let $j < j_1 < j_2 < \cdots < j_p$ be all the indices such that $w_{j_m} - w_j$ and $w_{j_m+1} - w_j$ have different signs, for $m \geq 1$. In this case of $w_j < w_{j+1}$, we know $w_j < w_{j_1}$, $w_j > w_{j_1+1}$, $w_j > w_{j_2}$, $w_j < w_{j_2+1}$ and so on. As a result,

$$\begin{aligned} A_j =& j - 2(j_1 + j_3 + \cdots) + 2(j_2 + j_4 + \cdots) \\ =& -(j_1 - j) + (j_2 - j_1) - (j_3 - j_2) + \cdots \pm j_p \\ =& -(j_1 - j) + (j_2 - j_1) - (j_3 - j_2) + \cdots \pm (n - j_p) \pm n \\ =& \#\{i > j : w_i < w_j\} - \#\{i > j : w_i > w_j\} \pm n \end{aligned}$$

where the last sign is $+$ if $w_j < w_n$ and is $-$ if $w_j > w_n$. The case $w_j > w_{j+1}$ yields the exact same formula with the same argument.

Once we consider all the A_j's together, the last terms $\pm n$ will appear as $+n$ for $w_n - 1$ times and will appear as $-n$ for $n - w_n$ times. Therefore,

$$\begin{aligned}(UD-DU)_{w,w} =& n^2 + n - 2nw_n + \sum_{j=1}^{n-1} A_j \\ =& n^2 + n - 2nw_n + \#\{i > j : w_i < w_j\} - \#\{i > j : w_i > w_j\} \\ &+ n(w_n - 1) - n(n - w_n) \\ =& k - \left(\binom{n}{2} - k\right) = 2k - \binom{n}{2}.\end{aligned}$$

□

Lemma 2. *For $w \neq u \in W_n$, $(UD - DU)_{w,u} = 0$.*

Proof. It suffices to check cases where $(UD)_{w,u} \neq 0$ or $(DU)_{w,u} \neq 0$. Let $k = \ell(w) = \ell(u)$. Let's first say that $(DU)_{w,u} \neq 0$, in which case there exists $v = us_b$ with $\ell(v) = \ell(u) + 1$ and $v = wt_{ij}$ $(i < j)$ with $\ell(v) = \ell(w) + 1$. We view W_n as a directed graph where there are up edges corresponding to covering relations of the weak Bruhat order W_n and down edges corresponding to covering relations in the strong Bruhat order. There are a few cases as follows. We will see that in each case there are exactly two directed paths of length 2 from u to w: one goes up then down and the other one goes down then up. Moreover, the edge weights of these two paths will be same and thus give $(UD - DU)_{w,u} = 0$.

Case 1: $\{b, b+1\} \cap \{i, j\} = \emptyset$. It is clear that there are exactly two directed path from u to w of length 2, which are $u \to v \to w$ and $u \to ut_{ij} \to w$. By definition, $U_{v,u} = U_{w,ut_{ij}} = b$ and $D_{w,v} = D_{ut_{ij},u}$.

Case 2: $b = i$. As $w \neq u$, we must have $j > b + 1$. By our condition on the path $u \to v \to w$, we know that $u_b < u_j < u_{b+1}$ and therefore there exists one more path of length 2 from u to w, which is $u \to x \to w$ where $x = ut_{i+1,j}$ and $x = ws_b$. The up edges of these two paths both have weight b and for the down edges, $D_{w,v} = 2(w_j - w_i - a(v,w)) - 1$ and $D_{x,u} = 2(u_{i+1} - u_j - a(u,x)) - 1$. We have $w_j = u_{i+1}$ and $w_i = u_j$ and since u_i is less than both u_{i+1} and u_j, we conclude that $a(v,w) = a(u,x)$.

Case 3: $b + 1 = i$. Similarly, we know that $u_j < u_b < u_i = u_{b+1}$ and the only other directed path is $u \to x \to w$ with $x = ut_{b,j} = ws_b$. The up edges both have weight b; for the down edges the key parameters $a(v,w)$ and $a(u,x)$ are equal, since u_{b+1} is greater than both u_b and u_j.

Case 4: $b = j$. Here $u_b = u_j < u_{b+1} < u_i$ and the other path is $u \to x \to w$ with $x = ut_{i,j+1} = ws_b$. The up edges have the same weight b and the down edges have the same weight since the transposition $w = vt_{i,j}$, $x = ut_{i,j+1}$ swaps entries with the same values and all four permutations have the same values at indices $1, 2, \ldots, i-1$.

Case 5: $b+1=j$. As $w \neq u$, we know $i<b$. First, $u_b < u_{b+1}$ and since $w = vt_{ij}$ with $\ell(w) = \ell(v) - 1$, we must have $u_b < u_i < u_j = u_{b+1}$. The other path is $u \to x \to w$ with $x = ut_{i,j-1} = ws_b$. The up edges have the same weight b and the down edges have the same weight since the transposition $w = vt_{i,j}$, $x = ut_{i,j-1}$ swaps entries with the same values and all four permutations have the same values at indices $1, 2, \dots, i-1$.

Beginning instead with the assumption that $(UD)_{w,u} \neq 0$, it is easy to see that all cases are already included above. □

We now complete the proof of the main theorem.

Proof of Theorem 1. Lemmas 1 and 2 together show that the map sending $e \mapsto U$, $f \mapsto D$, and $h \mapsto H$ defines a representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}W_n$ with weight spaces $\mathbb{C}(W_n)_k$ of weight $2k - \binom{n}{2}$. It is an immediate consequence of the theory of highest weight representations (see, for example, Theorem 4.60 of [4]) that $U^{\binom{n}{2}-2k} : \mathbb{C}(W_n)_k \to \mathbb{C}(W_n)_{\binom{n}{2}-k}$ is an isomorphism. Since U is an order raising operator by definition, Proposition 1 implies the desired result. □

3 Further directions

3.1 Other Coxeter types

The weak and strong Bruhat orders generalize naturally to any finite Coxeter group C, with the role of the simple transpositions $(i\,i+1)$ replaced by any choice of simple reflections, and the set of all transpositions $(i\,j)$ replaced by the set of all reflections in C. Stanley's result [9] that the strong order is strongly Sperner applies to any finite Weyl group. An easy argument proves the same for the dihedral groups, and computer checks verify that the strong orders on the exceptional Coxeter groups of types H_3 and H_4 are also strongly Sperner. Since strong orders for all Coxeter types are known to be rank-symmetric and rank-unimodal, and since products of Peck posets are Peck [7], it follows that Stanley's result can be extended to all finite Coxeter groups. As our results for the weak order apply only to the symmetric group, it is natural to ask:

Problem 1. *Is the weak order on any finite Coxeter group strongly Sperner?*

An easy argument answers this question in the affirmative for the dihedral groups, and computer checks have also verified it for all Coxeter groups of rank at most four.

3.2 Symmetric chain decompositions

A ranked poset P has a *symmetric chain decomposition* if P can be decomposed (as a set) into a disjoint union of saturated chains, each of which occupies a set of ranks which is symmetric about the middle rank of P. For example, a symmetric chain decomposition of the posets W_3 and S_3 in Figure 1 is given by $\{123, 213, 231, 321\} \sqcup \{132, 312\}$.

It is well-known that any poset with a symmetric chain decomposition is Peck. In [5], Leclerc observed that the weak order on the Coxeter group of type H_3 is Peck, but does not admit a symmetric chain decomposition. Stanley observes [9] that the strong orders for each of the infinite families of finite Coxeter groups admit symmetric chain decompositions. We ask:

Problem 2. *Which Coxeter group weak orders admit a symmetric chain decomposition? Do all Coxeter group strong orders admit one?*

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