Compatible Recurrent Identities of the Sandpile Group and Maximal Stable Configurations

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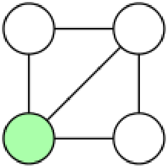
Chip-firing

Let G denote a simple and connected graph.

Definition (Sandpile)

A **sandpile** is a graph G that has a special vertex, called a **sink**.

Example: Diamond



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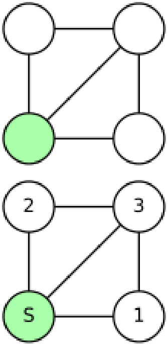

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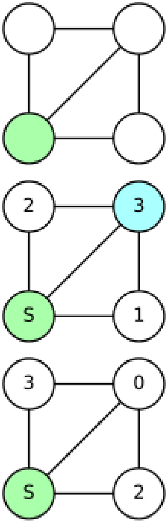

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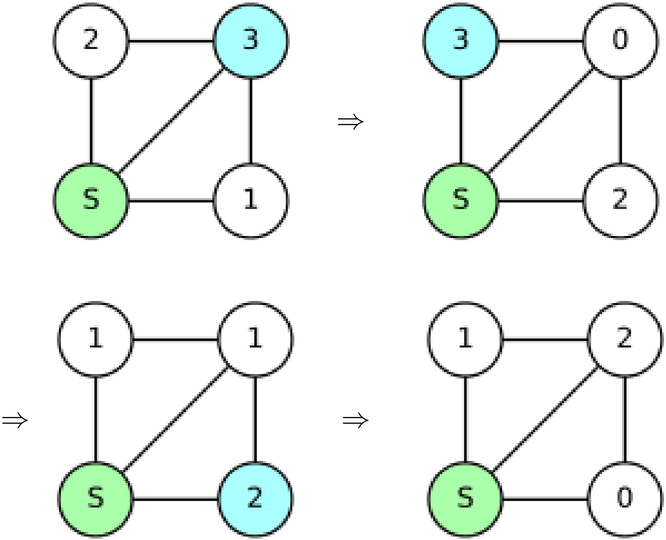
Definition (Chip-firing)

A non-sink vertex can **fire** if it has at least as many chips as its degree, sending one chip to each neighboring vertex. A chip configuration is **stable** if no vertex can fire.

Chip-firing Example: The Diamond Graph



Stabilization

Chip-firing displays **global confluence**, meaning:

- The chip-firing process will terminate at a stable configuration.
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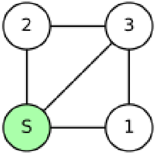
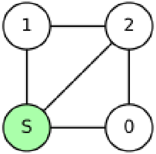
Definition (Stabilization)

The stable configuration that results from a chip configuration c is the **stabilization** of c, and denoted $\mathrm{Stab}(c)$.

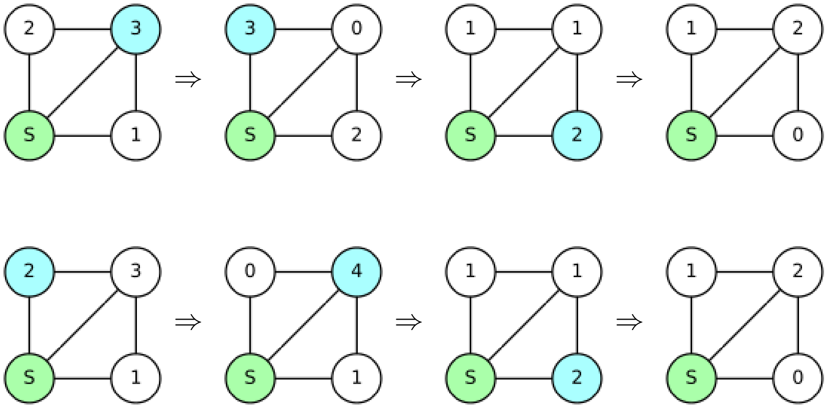


c



$\mathrm{Stab}(c)$

Stabilization Example: The Diamond Graph



An example of global confluence on the diamond graph

The Laplacian

Definition (Laplacian)

The **Laplacian** of a graph G with n vertices $v_1, \ldots, v_n$ is the $n \times n$ matrix Δ defined by

$$\Delta_{ij} = \begin{cases} -a_{ij} & \text{for } i \neq j, \\ d_i & \text{for } i = j, \end{cases}$$

where a_{ij} is the number of edges from vertex v_i to v_j, and d_i is the out-degree of v_i.

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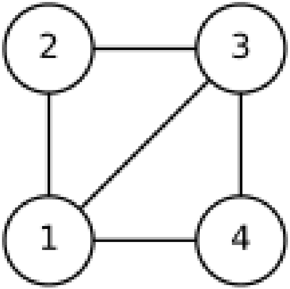
Definition (Reduced Laplacian)

The **reduced Laplacian** Δ' of a sandpile S on graph G is the matrix obtained by removing from Δ the row and column corresponding to the sink.

Firing a non-sink vertex v corresponds to the subtraction of the row of Δ' corresponding to v from the chip configuration.

Laplacian Example: The Diamond Graph



The Diamond Graph

$$\Delta = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

The Laplacian

$$\Delta' = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The Reduced Laplacian

The Sandpile Group

Definition (Sandpile Group)

The **sandpile group** of G with sink s is

$$\mathcal{S}(G) = \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}\Delta'(G).$$

This group is abelian, which is why chip-firing is also called the *abelian sandpile model*.

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Theorem (Matrix Tree Theorem)

$|\Delta'(G)|$ is equal to the number of spanning trees of G, or the number of trees that connect all vertices of G and are subgraphs of G.

Recurrent Configurations and the Sandpile Group

Definition (Recurrent)

A stable chip configuration c is called **recurrent** if for all stable configurations d, there exists a configuration e such that $\text{Stab}(d + e) = c$.

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Each equivalence class of the sandpile group has exactly one recurrent configuration.

The recurrent configurations of a sandpile form a group, under the operation $c + d = \mathrm{Stab}(c+d)$. This group is isomorphic under the inclusion map to the sandpile group $\mathcal{S}(G)$.

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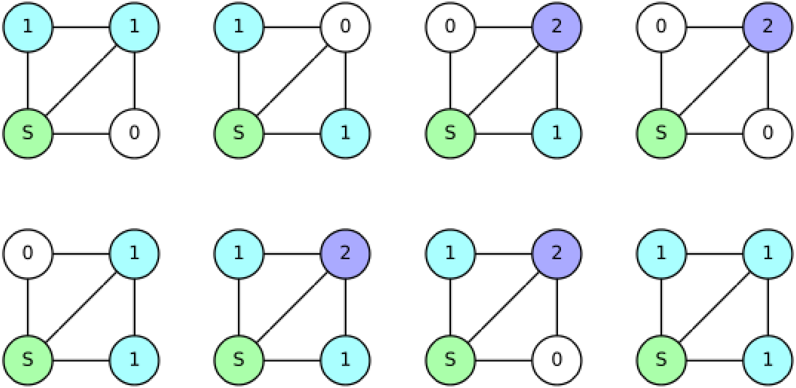
Definition (Recurrent Identity)

The **recurrent identity** is the identity element of the group of recurrent configurations, or the recurrent element in the same equivalence class as the all-zero configuration.

Sandpile Group Example

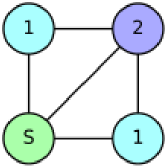
The sandpile group, represented by its recurrent elements, of the diamond graph with sink at one of the vertices of degree 3 is isomorphic to $\mathbb{Z}/8\mathbb{Z}$.



The Complete Maximal Identity Property

Definition (Maximal Stable Configuration)

The **maximal stable configuration** m_G is the chip configuration in which every non-sink vertex v has $d_v - 1$ chips, where d_v is the degree of vertex v (the number of edges incident to the vertex).



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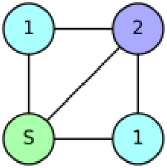

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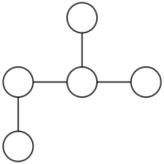
Definition (Complete Maximal Identity Property)

A graph G is said to have the **complete maximal identity property** if for all vertices $v \in G$, the recurrent identity of the sandpile group with graph G and sink v is equal to the maximal stable configuration.

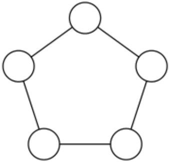
Graphs with the Complete Maximal Identity Property

Proposition (Gao and L.)

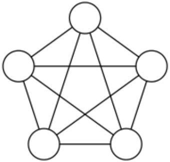
All trees, odd cycle graphs C_{2n+1}, and complete graphs K_n have the complete maximal identity property; moreover, the sandpile group of any tree is the trivial group, so the maximal stable configuration is the only recurrent configuration.



A Tree



C_5

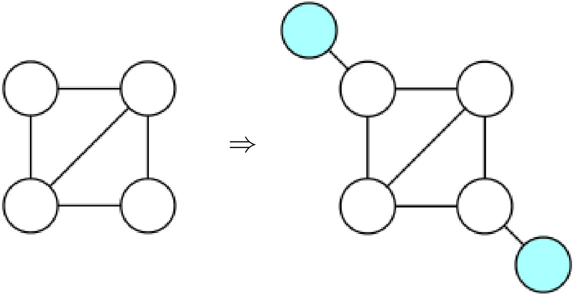


K_5

Creating Graphs with the Complete Maximal Identity Property by Adding Trees

Theorem (Gao and L.)

Given any connected graph G, there exists infinitely many graphs derived from adding trees to G that have the complete maximal identity property.



Biconnected Graphs

Because we may add trees to graphs to give them the complete maximal identity property, we wish to have a notion of irreducibility that eliminates such graphs which have trees added to them.

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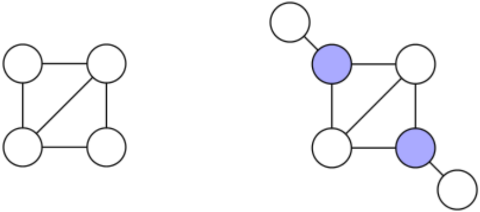
Definition (Biconnected)

A **biconnected** graph is a graph that remains connected even if you remove any single vertex and its incident edges.

In other words, a biconnected graph must have two completely different (share no edges) paths from any vertex to another.



Biconnected Not biconnected

Biconnected Graphs with the Complete Maximal Identity Property

Odd cycles C_{2n+1}, complete graphs K_n.

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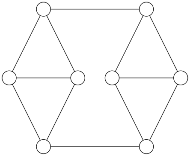
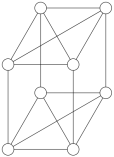
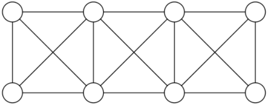
Computer search on all biconnected graphs with 11 vertices or less:



2-Diamond Ring



$K_4 \square P_2$



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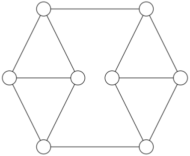

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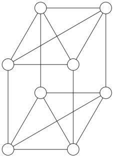

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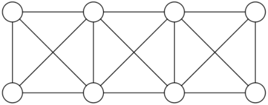

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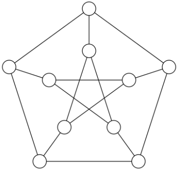
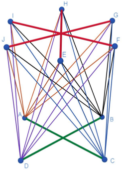


The Petersen Graph



$(P_2 \sqcup P_2 \sqcup P_2) + (P_2 \sqcup P_2)$

Conjectures and Further Research

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The only graph of the form $K_i \,\square\, P_j$ for $i, j > 1$ that has the complete maximal identity property is $K_4 \,\square\, P_2$.

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We are also working on generalizing the CMIP to the complete identity property, where the recurrent identities are compatible across sinks but not necessarily the maximal stable configuration.

Acknowledgements

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- The MIT Math Department

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