Journal of Algebra 634 (2023) 650–666

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Consecutive patterns in Coxeter groups

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A R T I C L E I N F O

Article history:
Received 28 November 2022
Available online 2 August 2023
Communicated by Tim Burness

Keywords:
Coxeter group
Parabolic decomposition
Consecutive pattern

A B S T R A C T

For an arbitrary Coxeter group element σ and a connected subset J of the Dynkin diagram, the parabolic decomposition $\sigma = \sigma^J \sigma_J$ defines σ_J as a consecutive pattern of σ, generalizing the notion of consecutive patterns in permutations. We then define the cc-Wilf-equivalence classes as an extension of the c-Wilf-equivalence classes for permutations, and identify non-trivial families of cc-Wilf-equivalent classes. Furthermore, we study the structure of the consecutive pattern poset in Coxeter groups and prove that its Möbius function is bounded by 2 when the arguments belong to finite Coxeter groups, but can be arbitrarily large otherwise.

1. Introduction

The concept of consecutive pattern containment is well known in the context of permutations as a way to characterize and generalize concepts such as peaks and runs. Specifically, we say that a permutation σ *consecutively contains* another permutation π if σ contains a contiguous subsequence with the same length and relative order as π. For

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https://doi.org/10.1016/j.jalgebra.2023.07.029
0021-8693/

example, a permutation σ consecutively contains the pattern 123 if there exists an index i such that $\sigma(i) < \sigma(i+1) < \sigma(i+2)$. Extensive research has been conducted into various aspects of this containment relation (see [8] for a survey). Most notably for this paper, some non-trivial classes of permutations π and τ that are consecutively contained inside the same number of size n permutations σ for any n, called *c-Wilf-equivalence* classes, have been identified (see [10,11,14]), and the structure of the poset formed by the partial ordering relation defined by consecutive containment has been well studied (see [9]).

This containment relation has applications in dynamical systems, where it is possible to prove that a sequence is not "random" if it never consecutively contains a certain pattern (see [7]). The values along an orbit of a discrete time dynamical system always avoid some *forbidden patterns*, which depend on exactly how the system operates, whereas completely random data almost surely consecutively contains all possible patterns as time approaches infinity. In fact, Brandt, Keller, and Pompe proved in [1] that the number of consecutive patterns that *can* be contained in the order of the values in an orbit grows exponentially at a rate proportional to the topological entropy of the dynamical system.

In algebraic combinatorics, consecutive patterns play a role in Robinson–Schensted recording tableaux and the study of box-ball systems [5]. Consecutive patterns have also been seen in Schubert calculus, as an important case of *interval patterns*, which is developed by Woo and Yong [18] to study singular locus of Schubert varieties.

Classical pattern containment, in which the subsequence in σ does not necessarily have to be contiguous, has been well-studied with clear applications to Schubert calculus. It is a famous result that a Schubert variety X_σ, indexed by a permutation σ, is smooth if and only if σ does not classically contain 3412 and 4231 [4,13]. In their seminal paper, Billey and Postnikov [2] introduced patterns in finite Weyl groups defined via inversion sets to characterize the smoothness of Schubert varieties in other Lie types. Billey-Postnikov patterns have since then been applied in numerous fruitful research (see for example [12,15–17]) to extend algebraic and combinatorics properties of permutations to Weyl group elements, often time allowing us to see more structures. However, consecutive patterns have not been studied systematically in other types.

In this paper, we generalize the notion of consecutive pattern containment in the symmetric group to all Coxeter groups in a natural way using parabolic decomposition.

Definition 1.1. Let (W, S) and (W', S') be irreducible Coxeter systems. Then $\sigma \in W'$ *consecutively contains* $\pi \in W$ if there exists some $J \subseteq S'$ and an isomorphism $\varphi_J : S \to J$ on the Dynkin diagram which induces an isomorphism $\varphi_J : W \to W'(J)$ on the Coxeter groups that sends π to σ_J. In this case, we say that (J, φ_J) is an *occurrence* of π in σ.

The relevant background material on Coxeter groups and parabolic decomposition is covered in Section 2. The main results of this paper are as follows:

- In Section 3, we generalize the idea of c-Wilf-equivalence classes, where the "c" stands for "consectutive" to cc-Wilf-equivalence classes, where the other "c" stands

for "Coxeter", and identify families of cc-Wilf-equivalence classes (Theorem 3.7). We also provide conjectures (Conjecture 3.10 and Conjecture 3.11) for future research.
- In Section 4, we study the Möbius function on the consecutive pattern poset, extending the theory developed by Elizalde and McNamara [9]. We show that $\mu(\pi, \sigma)$ is bounded by a small absolute constant (2 is enough) in finite Coxeter groups (Theorem 4.7) and can be unbounded in infinite Coxeter groups (Theorem 4.8).

2. Preliminaries and background

2.1. Coxeter groups

We refer readers to [3] for a detailed exposition on Coxeter groups. A *Coxeter system* (of finite *rank n*) is a pair (W, S) consisting of a Coxeter group W and a set of generators $S = \{s_1, s_2, \ldots, s_n\}$, such that W has a group presentation of the form

$$\langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = e \text{ for } 1 \leq i, j \leq n \rangle$$

where the exponents $m_{i,j} \in \mathbb{Z}_{>0} \cup \{\infty\}$ satisfy the following relations:

- $m_{i,i} = 1$ for all $1 \leq i \leq n$,
- $m_{i,j} = m_{j,i}$ for all $1 \leq i, j \leq n$, and
- $m_{i,j} \geq 2$ for all $1 \leq i, j \leq n$ and $i \neq j$.

We use $m_{i,j} = \infty$ to mean that there is no relation between s_i and s_j. Note that $m_{i,j} = 2$ means that s_i and s_j commute.

A standard way of representing Coxeter systems visually is with *Dynkin diagrams* which consist of a graph with vertex set S and undirected edges between any two $r, s \in S$ satisfying $m_{r,s} \geq 3$, with edges where $m_{r,s} > 3$ being labeled with the corresponding value and edges where $m_{r,s} = 3$ being unlabeled for simplicity. Note that commuting elements do not have edges between them. We say that a Coxeter system (W, S) is *irreducible* if we cannot partition S into two sets $I \sqcup J$ such that W is the direct product of W_I and W_J. Equivalently, (W, S) is irreducible if its Dynkin diagram is connected.

Example 2.1. The archetypal example of a Coxeter system is $(\mathfrak{S}_n, S)$ where $\mathfrak{S}_n$ is the symmetric group on n elements and $S = \{s_1, s_2, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$ for all $1 \leq i \leq n-1$ is the set of adjacent transpositions. One can check that

- $s_i^2 = e$ for any $1 \leq i \leq n-1$,
- $(s_i s_{i+1})^3 = (s_{i+1} s_i)^3 = e$ for any $1 \leq i \leq n-2$, and
- $(s_i s_j)^2 = e$ for any $1 \leq i, j \leq n-1$ and $|i - j| > 1$,

so $(\mathfrak{S}_n, S)$ is in fact a Coxeter system. This system is commonly denoted type A_{n-1}.

$(\mathfrak{S}_6, S)$:

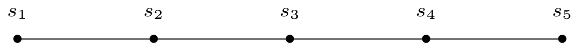

Fig. 1. Dynkin diagram for $(\mathfrak{S}_6, S)$.

Fig. 1 shows the Dynkin diagram for $n = 6$.

For each element $\sigma \in W$, we can write σ as a product of generators $s_{i_1} s_{i_2} \cdots s_{i_\ell}$. The minimal number of generators ℓ over all such ways to write σ is known as the *length* of σ and is denoted $\ell(\sigma)$. The corresponding product of generators $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is called a *reduced word*. Furthermore, the set of all $\{i_1, i_2, \ldots, i_\ell\}$ is called the *support* of σ (and is independent of the reduced word chosen) and is denoted $\mathrm{Supp}(\sigma)$. We say that a Coxeter system (W, S) is *finite* if W has finite size. Any finite W has a unique element $w_0(W)$ of maximal length.

2.2. Parabolic decompositions

For any $J \subseteq S$, let W_J (also denoted $W(J)$) be the subgroup of W generated by J; this is called the *parabolic subgroup* generated by J. Additionally, let $W^J = \{\sigma \in W \mid \ell(\sigma s) > \ell(\sigma) \text{ for all } s \in J\}$ be the *parabolic quotient* of J. For any $J \subset S$ and $\sigma \in W$, we have a unique factorization, called the *parabolic decomposition*, of the form $\sigma = \sigma^J \cdot \sigma_J$ where $\sigma^J \in W^J$ and $\sigma_J \in W_J$, and satisfying $\ell(\sigma) = \ell(\sigma^J) + \ell(\sigma_J)$. For our purposes, it is helpful to think of σ_J as the element of maximal length in W_J we can divide (multiply by the inverse of) to the right side of σ, and the leftover part is σ^J where multiplying by any $s \in J$ on its right increases its length.

Now we are ready to define consecutive containment, the main object of study of this paper. The following definition is rewritten from Definition 1.1 in Section 1.

Definition 2.2. Let (W, S) and (W', S') be irreducible Coxeter systems. Then $\sigma \in W'$ *consecutively contains* $\pi \in W$ if there exists some $J \subseteq S'$ and an isomorphism $\varphi_J : S \to J$ on the Dynkin diagram which induces an isomorphism $\varphi_J \colon W \to W'(J)$ on the Coxeter groups that sends π to σ_J. In this case, we say that (J, φ_J) is an *occurrence* of π in σ.

Example 2.3. Consider the Coxeter system (W, S) of type A_5 with $S = \{s_1, s_2, \ldots, s_5\}$, constructed as described in Example 2.1, and consider $\sigma = 416253 \in W = \mathfrak{S}_6$, where the permutation is written in one-line notation. Let $J = \{s_2, s_3, s_4\}$. We then have the parabolic decomposition

$$\sigma = \sigma^J \sigma_J = (s_3 s_2 s_1 s_4 s_5) \cdot (s_4 s_3) = 412563 \cdot 125346.$$

Note that $\sigma_J = 125346$, which has the same relative order of values as σ in positions $2, 3, 4, 5$. If we consider another type A_3 Coxeter group $(\mathfrak{S}_4, \{r_1, r_2, r_3\})$, and an isomorphism of Dynkin diagrams φ_J which maps r_i to s_{i+1} for $i = 1, 2, 3$, then $\varphi_J(1423) = \sigma_J$. Thus, we say that σ consecutively contains 1423.

For the rest of this section, we present some facts about parabolic decomposition. The following propositions regarding values of σ_J for particular σ are useful.

Proposition 2.4. *Let (W, S) be a Coxeter system. Suppose $\sigma \in W$ and $J \subseteq S$. Then $(\sigma u)_J = \sigma_J u$ for any $u \in W_J$.*

Proof. Write σ in the form $\sigma^J \cdot \sigma_J$ where $\sigma^J \in W^J$ and $\sigma_J \in W_J$. Then $\sigma u = \sigma^J \cdot \sigma_J u$, but $\sigma^J \in W^J$ and $w_J u \in W_J$, so by the uniqueness of parabolic decomposition, $(\sigma u)_J = \sigma_J u$. $\square$

Proposition 2.5. *Let (W, S) be a Coxeter system. Suppose $\sigma \in W$ and $J \subseteq S$. If $r \in S \setminus J$ commutes with all $s \in J$, then $(\sigma r)_J = \sigma_J$.*

Proof. We have $\sigma r = \sigma^J \sigma_J r = \sigma^J r \sigma_J$. For any $s \in J$, the four elements $\{\sigma^J, \sigma^J r, \sigma^J s, \sigma^J rs\}$ form a diamond in the Bruhat order, with length $\ell, \ell+1, \ell+1, \ell+2$ for some ℓ, (see Theorem 1.4 of [6]). Since $\ell(\sigma^J) < \ell(\sigma^J s)$, we must have $\ell(\sigma^J r) < \ell(\sigma^J rs)$, meaning that $\sigma^J r$ does not have right descents in J and $\sigma r = (\sigma^J r)\sigma_J$ is the parabolic decomposition of σr. As a result, $(\sigma r)_J = \sigma_J$. $\square$

Corollary 2.6. *Let (W, S) be a Coxeter system. Suppose $\sigma \in W$ and $J \subseteq S$. If $\pi \in W$ such that any $r \in \mathrm{Supp}(\pi)$ commutes with any $s \in J$, then $(\sigma\pi)_J = \sigma_J$.*

Proposition 2.7. *Let (W, S) be a Coxeter system. Suppose $\sigma \in W$ and $J \subseteq S$. Then $(w_0(W)\sigma)_J = w_0(W_J)\sigma_J$.*

Proof. It suffices to prove that

$$w_0(W)\sigma \cdot (w_0(W_J)\sigma_J)^{-1} = w_0(W)\sigma\sigma_J^{-1}(w_0(W_J))^{-1} \in W^J.$$

Writing σ as $\sigma^J \sigma_J$ and using the fact that $(w_0(W_J))^2 = e$, the above is simplified to $w_0(W)\sigma^J w_0(W_J) \in W^J$, i.e. $\ell(w_0(W)\sigma^J w_0(W_J)s) > \ell(w_0(W)\sigma^J w_0(W_J))$ for all $s \in J$ which is equivalent to $\ell(\sigma^J w_0(W_J)s) < \ell(\sigma^J w_0(W_J))$. Note that $\sigma^J \cdot w_0(W_J)s$, and $\sigma^J \cdot w_0(W_J)$ are parabolic decompositions, so the inequality follows from the length-additivity of parabolic decompositions and the maximality of $w_0(W_J)$. $\square$

3. Wilf-equivalence classes

Our first goal is to analyze *Wilf-equivalence classes* for this definition (Definition 1.1) of consecutive containment, as a generalization of c-Wilf-equivalence classes in permutations.

Definition 3.1. Two Coxeter group elements $\pi, \tau \in W$ for some arbitrary finite irreducible Coxeter system (W, S) are said to be *cc-Wilf-equivalent* if for every finite irreducible

Coxeter system (W', S'), the number of elements $\sigma \in W'$ consecutively containing π is the same as the number of elements $\sigma \in W'$ containing τ.

We say that for a Coxeter system (W, S), a *diagram automorphism* is a graph automorphism of the Dynkin diagram, which induces a group automorphism on W that fixes S. The followings are two of the more straightforward cc-Wilf-equivalences that apply generally.

Proposition 3.2. *Let (W, S) be a finite irreducible Coxeter system, and let $\pi \in W$ be an arbitrary Coxeter group element. Then:*

(a) *π is cc-Wilf-equivalent to $w_0(W)\pi$, and*
(b) *if ϕ is a diagram automorphism of (W, S), then π is cc-Wilf-equivalent to $\phi(\pi)$.*

Proof. For (a), by Proposition 2.7, $(w_0(W')\sigma)_J = w_0(W'_J)\sigma_J$ for all $\sigma \in W'$, so for every occurrence (J, φ_J) of π in σ, we have that φ_J sends $w_0(W)\pi$ to

$$\varphi_J(w_0\pi) = \varphi_J(w_0(W)) \cdot \varphi_J(\pi) = w_0(W'_J) \cdot \sigma_J = (w_0(W')\sigma)_J$$

since the isomorphism φ_J sends the maximal element of W to the maximal element of W'_J; hence, every occurrence of π in σ corresponds to an occurrence of $w_0(W)\pi$ in $w_0(W')\sigma \in W'$, and vice versa. Therefore, π and $w_0(W)\pi$ are cc-Wilf-equivalent, as desired.

For (b), for every occurrence (J, φ_J) of π in σ, the isomorphism $\varphi_J \circ \phi^{-1}$ from W to $W'(J)$ sends $\phi(\pi)$ to σ_J, and since ϕ^{-1} fixes S and φ_J sends S to J, the isomorphism $\varphi_J \circ \phi^{-1}$ sends S to J. Thus, $(J, \varphi_J \circ \phi^{-1})$ is an occurrence of $\phi(\pi)$ in σ. As in above, every occurrence of π in σ corresponds to an occurrence of $\phi(\pi)$ in σ, and vice versa, so π and $\phi(\pi)$ are cc-Wilf-equivalent, as desired. □

In the case of the Coxeter group being the symmetric group $\mathfrak{S}_n$ (which is a Coxeter group of type A_{n-1}), the above proposition corresponds to the following corollary:

Corollary 3.3. *Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation in one-line notation. Then π is cc-Wilf-equivalent to*

(a) *its reverse: $\pi^R := \pi_n\pi_{n-1} \cdots \pi_1$,*
(b) *its complement: $\pi^C := (n+1-\pi_1)(n+1-\pi_2) \cdots (n+1-\pi_n)$, and*
(c) *its reverse complement: $\pi^{RC} := (n+1-\pi_n)(n+1-\pi_{n-1}) \cdots (n+1-\pi_1)$.*

Proof. We know that $w_0(A_{n-1}) = n(n-1)\dots 21$. Hence, $w_0(A_{n-1})\pi = \pi^C$, which proves (b) by Proposition 3.2 part (a). Furthermore, using the canonical set of generators for the symmetric group $\mathfrak{S}_n$, the adjacent transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$, the automorphism ϕ sending $s_i \mapsto s_{n-i}$ fixes S and sends π to π^{RC}. Hence, (c) is proved by Proposition 3.2 part (b).

Finally, note that $\pi^R = \left(\pi^C\right)^{RC}$, hence (a) is proved as well. $\square$

Furthermore, since there is only one automorphism $\phi \neq \mathrm{id}$ that preserves S type A (symmetric group), there are only two possible φ_J for a particular J corresponding to the consecutive pattern containment of either π or π^J. In other words, if π and σ are permutations, then σ consecutively contains π in the Coxeter group sense if and only if σ consecutively contains either π or π^{RC} in the permutation pattern sense.

We call the cc-Wilf-equivalences in Proposition 3.2 *trivial*. We shall demonstrate the non-trivial equivalence of families of Coxeter group elements.

Let (W, S) be a finite irreducible Coxeter system, and let $\pi, \tau \in W$. Let $\beta = \pi^{-1}\tau$.

Definition 3.4. For a fixed Coxeter group element $\sigma \in W'$ of an arbitrary finite irreducible Coxeter system (W', S'), define

$$\mathcal{O}_\pi(\sigma) := \{(J, \varphi_J(\beta)) \mid (J, \varphi_J) \text{ is an occurrence of } \pi \text{ in } \sigma\},$$
$$\mathcal{O}_\tau(\sigma) := \{(J, \varphi_J(\beta^{-1})) \mid (J, \varphi_J) \text{ is an occurrence of } \tau \text{ in } \sigma\}.$$

Suppose that for any two $(J, b_J), (J', b_{J'}) \in \mathcal{O}_\pi(\sigma) \cup \mathcal{O}_\tau(\sigma)$ such that $J \neq J'$, any element of $\mathrm{Supp}(b_J)$ commutes with and is distinct from any element of J'. Then, we say that π and τ are *difference-disjoint* with respect to σ. If π and τ are difference-disjoint with respect to all $\sigma \in W'$ for any choice of (W', S'), then we say that π, τ are *strongly difference-disjoint*.

Definition 3.4 essentially means that if we right multiply some $\varphi_J(\beta)$ where (J, φ_J) is an occurrence of π in σ (or symmetrically for τ), we do not affect (commute with the elements of) every other occurrence of π or τ.

Example 3.5. Consider the two permutations of length 6 defined by $\pi = 163425$ and $\tau = 164325$, and the permutation of length 10 defined by $\sigma = 1\ 10\ 3\ 4\ 2\ 9\ 7\ 6\ 5\ 8$, all in one-line notation. Then π, τ are group elements of the Coxeter system $(\mathfrak{S}_6, S)$ where $S = \{s_1, s_2, \ldots, s_5\}$ and $s_i = (i, i+1)$ for $1 \leq i \leq 5$, and σ is a group element of the Coxeter system $(\mathfrak{S}_{10}, R)$ where $R = \{r_1, r_2, \ldots, r_9\}$ and $r_i = (i, i+1)$ for $1 \leq i \leq 9$.

It can be checked, either through direct parabolic decomposition or the relative order of elements in the permutation, that σ contains π with occurrence at $(\{r_1, r_2, \ldots, r_5\}, \varphi)$ where $\varphi(s_i) = r_i$ for all $1 \leq i \leq 5$, and σ contains τ with occurrence at $(\{r_5, r_6, \ldots, r_9\}, \varphi')$ where $\varphi'(s_i) = r_{4+i}$ for all $1 \leq i \leq 5$. Furthermore, these are the only occurrences.

Thus, we can compute

$$\mathcal{O}_\pi(\sigma) = \{(\{r_1, r_2, \ldots, r_5\}, \varphi(\beta))\},$$

$$\mathcal{O}_\tau(\sigma) = \{(\{r_5, r_6, \ldots, r_9\}, \varphi'(\beta^{-1}))\}.$$

The condition that π and τ are difference-disjoint with respect to σ is then equivalent to any element of $\operatorname{Supp}\varphi(\beta)$ commuting with all $\{r_5, r_6, \ldots, r_9\}$, and any element $\operatorname{Supp}\varphi'(\beta)$ commuting with all $\{r_1, r_2, \ldots, r_5\}$.

Note that $\operatorname{Supp}\varphi(\beta) \subseteq \{r_1, r_2, \ldots, r_5\}$, so the first condition implies $\operatorname{Supp}\varphi(\beta) \subseteq \{r_1, r_2, r_3\}$. But $\operatorname{Supp}\varphi(\beta) = \varphi(\operatorname{Supp}\beta)$, where φ is applied element-wise. Thus $\operatorname{Supp}\beta \subseteq \{s_1, s_2, s_3\}$. Similarly, the second condition implies $\operatorname{Supp}\beta \subseteq \{s_3, s_4, s_5\}$.

Thus it is necessary for $\operatorname{Supp}\beta \subseteq \{s_3\}$ for π and τ to be difference-disjoint with respect to σ. Conveniently, $\beta = \pi^{-1}\tau = s_3$, so this is satisfied. However, we can see that the difference disjoint condition is quite restrictive, even when only looking at one σ.

Definition 3.6. We say that π, τ are *automorphic-equivalent* if for every automorphism ϕ of W fixing S, the automorphism ϕ fixes π if and only if it fixes τ.

Together, we have the following,

Theorem 3.7. *If π and τ are both strongly difference-disjoint and automorphic-equivalent, then they are cc-Wilf-equivalent.*

Proof. Let (W', S') be an arbitrary finite irreducible Coxeter system. Let C_π be the set of elements of W' that contain π, and let C_τ be the set of elements of W' that contain τ.

The idea is to construct a function $f\colon W' \to W'$ that bijectively sends C_π to C_τ by "applying" β to all occurrences of π and β^{-1} to all occurrences of τ.

Formally, define f in the following way: for any $\sigma \in W'$, using the notation from Definition 3.4, we define $\mathcal{O}(\sigma)$ as the set obtained by removing elements of $\mathcal{O}_\pi(\sigma) \cup \mathcal{O}_\tau(\sigma)$ with the same J, and picking a "canonical" choice for the second element if there are duplicates (which we can do in a well defined way for both π and τ since they are automorphic-equivalent). Let

$$f(\sigma) := \sigma \cdot \left(\prod_{(J,\, b_J) \,\in\, \mathcal{O}(\sigma)} b_J \right),$$

where the order of the product does not matter since by the definition of strongly difference disjoint, the terms all mutually commute.

Note that if (J_1, φ_{J_1}) is an occurrence of π in σ, then we can write the parabolic decomposition $\sigma = \sigma^{J_1}\sigma_{J_1} = \sigma^{J_1}\varphi_{J_1}(\pi)$. Thus

$$f(\sigma) = \sigma^{J_1}\varphi_{J_1}(\pi) \cdot \varphi_{J_1}(\beta) \left(\prod_{(J,\, b_J) \,\in\, \mathcal{O}(\sigma),\ J \neq J_1} b_J \right) = \sigma^{J_1} \left(\prod_{(J,\, b_J) \,\in\, \mathcal{O}(\sigma),\ J \neq J_1} b_J \right) \cdot \varphi_{J_1}(\tau),$$

since $\varphi_{J_1}(\pi\beta) = \varphi_{J_1}(\tau) \in W_J$ and every b_J (for $J \neq J_1$) commutes with every element of J_1 by definition.

By Proposition 2.4, since $\varphi_{J_1}(\tau) \in W_{J_1}$, we have

$$f(w)_{J_1} = \left[\sigma^{J_1}\left(\prod_{(J,b_J)\in\mathcal{O}(\sigma),\ J\neq J_1} b_J\right)\right]_{J_1} \cdot \varphi_{J_1}(\tau),$$

so by repeatedly applying Proposition 2.5, which is valid since any element of $\mathrm{Supp}(b_J)$ commutes with any element of J_1 for $J \neq J_1$, we have

$$f(w)_{J_1} = \left[\sigma^{J_1}\right]_{J_1} \cdot \varphi_{J_1}(\tau) = \varphi_{J_1}(\tau),$$

since $\sigma^{J_1} \in W^{J_1}$ by definition. Therefore, if (J_1, φ_{J_1}) is an occurrence of π in σ, then it is also an occurrence of τ in $f(\sigma)$. Similarly, we can prove that if (J_2, φ_{J_2}) is an occurrence of τ in σ, then it is also an occurrence of π in $f(\sigma)$.

It follows that if $\sigma \in C_\pi$, then $f(\sigma) \in C_\tau$, so $f(C_\pi) \subseteq C_\tau$. Similarly $f(C_\tau) \subseteq C_\pi$. But note that for any $\sigma \in W$, we have $f(f(\sigma)) = \sigma$, hence f is its own two sided inverse, so it is bijective. Therefore, $|C_\pi| = |C_\tau|$ as desired. $\square$

This can be seen as a rough generalization of the fact that "minimally overlapping" permutations (patterns that when they appear, can share at most one element) are c-Wilf-equivalent if they have the same first and last elements (see [8]).

The following is a quick example of how we can use Theorem 3.7 to establish some particular families of cc-Wilf-equivalent classes.

Proposition 3.8. *Let $n \geq 8$. Then $\pi = 1\ n-1\ \sigma_1\sigma_2\cdots\sigma_{n-4}\ 2\ n$ and $\tau = 1\ n-1\ \sigma'_1\sigma'_2\cdots\sigma'_{n-4}\ 2\ n$, where $\sigma_1\sigma_2\cdots\sigma_{n-4}$ and $\sigma'_1\sigma'_2\cdots\sigma'_{n-4}$ are permutations of $\{3, 4, \ldots, n-2\}$, are strongly difference-disjoint.*

Proof. Since $n \geq 8$, the elements π and τ can only be consecutively contained in some Coxeter group element living in a Coxeter system of type A or D. We can check, through the relative order of elements (specifically by looking at the position of the largest element, which will almost always be one of n and $n-1$), that for any σ, for any two $(J_1, \varphi_{J_1}), (J_1, \varphi_{J_1}) \in \mathcal{O}_\pi(\sigma) \cup \mathcal{O}_\tau(\sigma)$ with $J_1 \neq J_2$, we have that J_1 and J_2 share at most one element, corresponding to an adjacent transposition on the first or last two elements of an occurrence of π or τ. But this commutes with any β or β^{-1} which only permute inside the $\sigma_1\sigma_2\cdots\sigma_{n-4}$ and $\sigma'_1\sigma'_2\cdots\sigma'_{n-4}$, as desired. $\square$

Recall that there is a only one automorphism $\phi \neq \mathrm{id}$ that preserves the type A Dynkin diagram, and that $\phi(\pi) = \pi^{RC}$. Thus, π and τ are automorphic-equivalent if and only if both or neither of $\pi = \pi^{RC}$ and $\tau = \tau^{RC}$ are true. We then arrive at the following new application:

Corollary 3.9. *Let $n \geq 8$ and consider the permutations π and τ described in Proposition 3.8. If both or neither of $\pi = \pi^{RC}$ and $\tau = \tau^{RC}$ are true, then π and τ are cc-Wilf-equivalent.*

To finish, we provide some conjectures about the strength of Theorem 3.7.

Conjecture 3.10. *If π, τ are cc-Wilf-equivalent, then π and τ are automorphic-equivalent.*

Conjecture 3.11. *If π, τ are cc-Wilf-equivalent, then π and τ are difference-disjoint.*

4. The Möbius function of the consecutive pattern poset

Definition 1.1 suggests the following natural definition of a poset.

Definition 4.1. The *consecutive pattern poset* is defined by $\pi \leq \sigma$ if σ consecutively contains π. We also identify $\pi = \sigma$ if $\pi \leq \sigma$ and $\sigma \leq \pi$, meaning that there exists a diagram automorphism that identifies these two elements. By convention, the identity element e of the trivial group satisfies $e \leq \tau$ for any Coxeter group element τ.

This poset can be defined for all Coxeter group elements simultaneously, but as we focus on the intervals in this poset, we typically start with an ambient Coxeter system (W, S). It is a graded poset, with the rank of σ being equal to the rank of the Coxeter group to which it belongs. We denote this $|\sigma|$. Furthermore, if $\sigma \in W$ for some irreducible Coxeter system (W, S), then S has finite size, so there are finitely many $J \subseteq S$, hence the closed interval $[\pi, \sigma] := \{\tau \mid \pi \leq \tau \leq \sigma\}$ is finite. We can similarly define the intervals $[\pi, \sigma)$, $(\pi, \sigma]$, and (π, σ).

Recall that the Möbius function $\mu(\pi, \sigma)$ can be defined recursively as

$$\mu(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ -\displaystyle\sum_{\pi \leq \tau < \sigma} \mu(\pi, \tau) & \text{if } \pi < \sigma \end{cases}.$$

We can rewrite the second condition, which says if $\pi < \sigma$, then $\sum_{\tau \in [\pi, \sigma]} \mu(\pi, \tau) = 0$.

We prove bounds on the size of the Möbius function when τ is an element of a finite irreducible Coxeter group, which has been classified (see for example Appendix A1 of [3]).

First we prove some structural facts.

Definition 4.2. Suppose (W, S) is a Coxeter system. For some $s \in S$, we say that a saturated chain $\mathcal{C}$ in the consecutive pattern poset with maximum element $\sigma \in W$ is *s-anchored* if for each $\tau \in \mathcal{C}$, there exists an occurrence $(J_\tau, \varphi_{J_\tau})$ of τ in σ such that $s \in J_\tau$.

The following lemma is useful.

Lemma 4.3. *Suppose (W, S) is a Coxeter system whose Dynkin diagram is a path graph, and suppose $S = \{s_1, s_2, \dots, s_n\}$ is a labeling of S such that s_1 corresponds to a degree 1 vertex on the Dynkin diagram of (W, S), and s_i is connected to s_{i+1} for all $i = 1, 2, \dots, n-1$. Then if $\mathcal{C}$ is an s_1-anchored saturated chain, then*

$$\sum_{\tau \in \mathcal{C}} \mu(\pi, \tau) \in \{-1, 0, 1\},$$

for any Coxeter group element π such that $\pi \le \tau$ for all $\tau \in \mathcal{C}$.

Proof. We assume $\mathcal{C}$ to be nonempty. Suppose $\mathcal{C}$ has maximum element σ. Proceed with strong induction on $|\sigma|$.

We can manually check the base cases $|\sigma| = |\pi|$, where we must have $\sigma = \pi$, so the sum is 1, and $|\sigma| = |\pi| + 1$, where we must have $\mu(\pi, \sigma) = -1$, so

$$\sum_{\tau \in \mathcal{C}} \mu(\pi, \tau) \in \{0, -1\},$$

depending on whether $\pi \in \mathcal{C}$.

Now assume that, for some fixed π, the lemma is true for all s-anchored saturated chains $\mathcal{C}$ with maximum element σ satisfying $|\sigma| < n$ (where $n > |\pi| + 1$ is the rank of (W, S)). We will show that it is true for all $\mathcal{C}$ with maximum element $\sigma \in W$, i.e. $|\sigma| = n$ since W is arbitrary.

The key is to look at the set $\mathcal{C}' = [\pi, \sigma] \setminus \left[\pi, \sigma_{S \setminus \{s_1\}}\right]$. In particular, for any $\tau \in \mathcal{C}'$ with occurrence (J, φ_J) in σ, we must have $J = \{s_i, s_{i+1}, \dots, s_j\}$ for some integers $i \le j$ by the restrictions on the Dynkin diagram. But if $i \ne 1$, then $\tau \le \sigma_{S \setminus \{s_1\}}$, so we must have $i = 1$. It follows that $\mathcal{C}'$ is also an s_1-anchored saturated chain, i.e. it is a saturated chain of the form

$$\sigma > \sigma_{S \setminus \{s_n\}} > \sigma_{S \setminus \{s_{n-1}, s_n\}} > \cdots,$$

ending at some Coxeter group element. But we know that

$$\sum_{\tau \in \mathcal{C}'} \mu(\pi, \tau) = \sum_{\tau \in [\pi, \sigma]} \mu(\pi, \tau) - \sum_{\tau \in \left[\pi, \sigma_{S \setminus \{s_1\}}\right]} \mu(\pi, \tau) = 0 - 0 = 0,$$

since $|\sigma| > |\pi| + 1$.

Now, $\mathcal{C}$ also has the form

$$\sigma > \sigma_{S \setminus \{s_n\}} > \sigma_{S \setminus \{s_{n-1}, s_n\}} > \cdots,$$

but may end at a different element. We have three cases: $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' \supseteq \mathcal{C}$, and $\mathcal{C}' = \mathcal{C}$ (see Fig. 2).

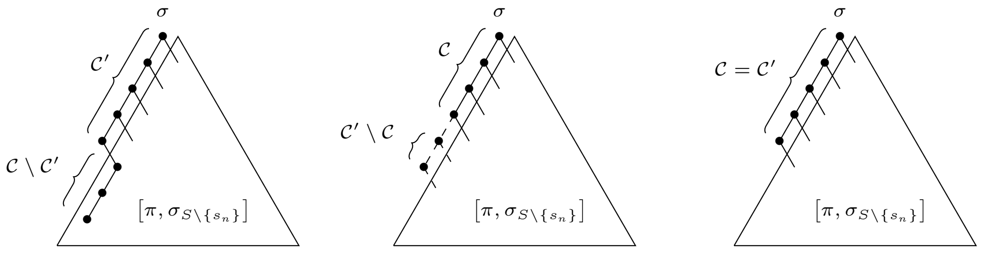

Fig. 2. Sketches of the Hasse diagrams for the interval $[\pi, \sigma]$ with the interval $[\pi, \sigma_{S\setminus\{s_n\}}]$ omitted for clarity. The pertinent parts of $\mathcal{C} \cup \mathcal{C}'$ are labeled.

In the first case, we have

$$\sum_{\tau \in \mathcal{C}'} \mu(\pi, \tau) = \sum_{\tau \in \mathcal{C}} \mu(\pi, \tau) - \sum_{\tau \in \mathcal{C}'\setminus\mathcal{C}} \mu(\pi, \tau) = - \sum_{\tau \in \mathcal{C}'\setminus\mathcal{C}} \mu(\pi, \tau).$$

But $\mathcal{C}' \setminus \mathcal{C}$ is another s_1-anchored saturated chain with a strictly smaller rank of its maximum element, so

$$\sum_{\tau \in \mathcal{C}'} \mu(\pi, \tau) = - \sum_{\tau \in \mathcal{C}'\setminus\mathcal{C}} \mu(\pi, \tau) \in \{-1, 0, 1\},$$

by the induction hypothesis.

Similarly, if $\mathcal{C}' \supseteq \mathcal{C}$, then $\mathcal{C} \setminus \mathcal{C}'$ is another s_1-anchored saturated chain with a strictly smaller rank of its maximum element, so

$$\sum_{\tau \in \mathcal{C}'} \mu(\pi, \tau) = \sum_{\tau \in \mathcal{C}} \mu(\pi, \tau) + \sum_{\tau \in \mathcal{C}\setminus\mathcal{C}'} \mu(\pi, \tau) = \sum_{\tau \in \mathcal{C}\setminus\mathcal{C}'} \mu(\pi, \tau) \in \{-1, 0, 1\},$$

by the induction hypothesis.

Finally, if $\mathcal{C}' = \mathcal{C}$, then

$$\sum_{\tau \in \mathcal{C}'} \mu(\pi, \tau) = \sum_{\tau \in \mathcal{C}} \mu(\pi, \tau) = 0. \quad \square$$

Corollary 4.4. *Consider two finite irreducible Coxeter systems (W, S) and (W', S'), and let $\pi \in W$ and $\sigma \in W'$ such that $\pi \leq \sigma$. If (W', S') is of type A, B, F, H, or I, then $|\mu(\pi, \sigma)| \leq 1$.*

Proof. These types have path graph like Dynkin diagrams, and the set $\{\sigma\}$ is an s_1-anchored saturated chain (for any generator s_1 corresponding to a degree 1 vertex on the Dynkin diagram), so we conclude by Lemma 4.3. $\square$

Proposition 4.5. *Consider two finite irreducible Coxeter systems (W, S) and (W', S'), and let $\pi \in W$ and $\sigma \in W'$ such that $\pi \leq \sigma$. If (W', S') is of type D, then $|\mu(\pi, \sigma)| \leq 2$.*

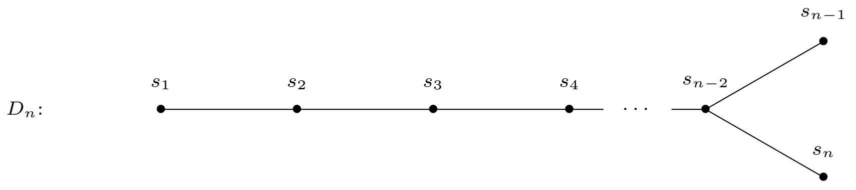

Fig. 3. Dynkin diagram for a Coxeter system of type D_n.

Proof. Write $S' = \{s_1, s_2, \dots, s_n\}$ where $n = |S'|$ such that s_i, s_{i+1} do not commute for $1 \le i \le n-1$, s_{n-2}, s_n do not commute, and any other pair commute.

For clarity, the Dynkin diagram is shown in Fig. 3

Observe that if $|\sigma| - |\pi| \le 2$, a check of every possible poset interval gives the desired (since σ covers at most 3 elements). Thus, assume $|\sigma| - |\pi| > 2$.

Consider the (possibly empty) set $[\pi, \sigma) \setminus \left[\pi, \sigma_{S' \setminus \{s_1\}}\right]$ where we say $\left[\pi, \sigma_{S' \setminus \{s_1\}}\right] = \emptyset$ if $\pi \not\le \sigma_{S' \setminus \{s_1\}}$. Note that

$$\mu(\pi, \sigma) = -\sum_{\tau \in [\pi, \sigma)} \mu(\pi, \tau) = -\sum_{\tau \in [\pi, \sigma) \setminus \left[\pi, \sigma_{S' \setminus \{s_1\}}\right]} \mu(\pi, \tau),$$

since $\sum_{\tau \in \left[\pi, \sigma_{S' \setminus \{s_1\}}\right]} \mu(\pi, \tau) = 0$. But $[\pi, \sigma) \setminus \left[\pi, \sigma_{S' \setminus \{s_1\}}\right]$ contains $\sigma_{S' \setminus \{s_n\}}$, the saturated chain $\mathcal{C}$ containing the elements

$$\sigma_{S' \setminus \{s_{n-1}\}} > \sigma_{S' \setminus \{s_{n-2}, s_{n-1}\}} > \sigma_{S' \setminus \{s_{n-3}, s_{n-2}, s_{n-1}\}} > \cdots,$$

and nothing else (if $\sigma_{S' \setminus \{s_n\}} = \sigma_{S' \setminus \{s_{n-1}\}}$, then we can ignore $\sigma_{S' \setminus \{s_n\}}$ entirely). But by Corollary 4.4, $\left|\mu(\pi, \sigma_{S' \setminus \{s_n\}})\right| \le 1$. Furthermore, $\mathcal{C}$ is an s_1-anchored saturated chain with all elements consecutively contained in a group element of the Coxeter system $(W'(S' \setminus \{s_n\}), S' \setminus \{s_n\})$, which is of type A, so by Lemma 4.3,

$$\left|\sum_{\tau \in \mathcal{C}} \mu(\pi, \tau)\right| \le 1.$$

It follows that

$$|\mu(\pi, \sigma)| \le \left|\mu(\pi, \sigma_{S' \setminus \{s_n\}})\right| + \left|\sum_{\tau \in \mathcal{C}} \mu(\pi, \tau)\right| \le 2,$$

as desired. $\square$

Proposition 4.6. *Consider two finite irreducible Coxeter systems (W, S) and (W', S'), and let $\pi \in W$ and $\sigma \in W'$ such that $\pi \le \sigma$. If (W', S') is of type E, then $|\mu(\pi, \sigma)| \le 2$.*

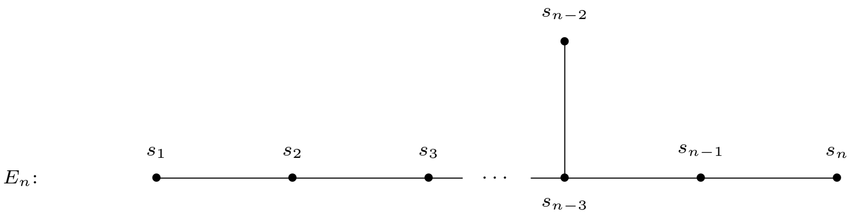

Fig. 4. Dynkin diagram for a Coxeter system of type E_n.

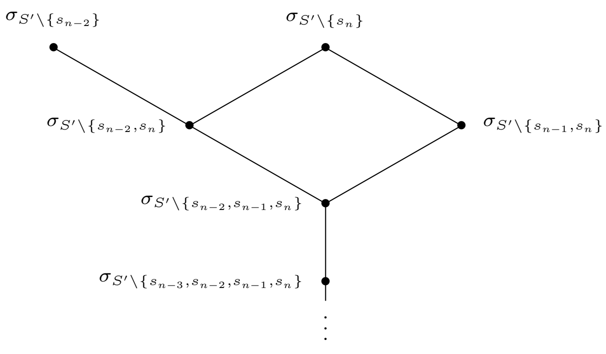

Fig. 5. Hasse Diagram for the set $[\pi,\sigma)\setminus\left[\pi,\sigma_{S'\setminus\{s_1\}}\right]$, assuming no duplicates.

Proof. As with Proposition 4.5, we label S' such that the Dynkin diagram is as shown in Fig. 4.

By the classification of finite irreducible Coxeter groups, $n \le 8$, but we do not use that fact here.

Similarly to Proposition 4.5, if $|\sigma| - |\pi| \le 2$, a check of every possible poset interval gives the desired (since σ covers at most 3 elements). Thus, assume $|\sigma| - |\pi| > 2$.

Consider the (possibly empty) set $[\pi,\sigma)\setminus\left[\pi,\sigma_{S'\setminus\{s_1\}}\right]$ where we say $\left[\pi,\sigma_{S'\setminus\{s_1\}}\right] = \emptyset$ if $\pi \not\le \sigma_{S'\setminus\{s_1\}}$. Note that

$$\mu(\pi,\sigma) = -\sum_{\tau\in[\pi,\sigma)} \mu(\pi,\tau) = -\sum_{\tau\in[\pi,\sigma)\setminus\left[\pi,\sigma_{S'\setminus\{s_1\}}\right]} \mu(\pi,\tau).$$

The set $[\pi,\sigma)\setminus\left[\pi,\sigma_{S'\setminus\{s_1\}}\right]$ contains only those σ_J such that $s_1 \in J$ (ignoring duplicates). Its Hasse diagram is shown in Fig. 5.

Observe that there is an s_1-anchored saturated chain $\mathcal{C}_1$ with elements,

$$\sigma_{S'\setminus\{s_{n-2}\}} > \sigma_{S'\setminus\{s_{n-2},s_n\}} > \sigma_{S'\setminus\{s_{n-2},s_{n-1},s_n\}} >> \sigma_{S'\setminus\{s_{n-3},s_{n-2},s_{n-1},s_n\}} > \cdots .$$

The rest of the set $[\pi,\sigma)\setminus\left[\pi,\sigma_{S'\setminus\{s_1\}}\right]$ consists of the two elements $\sigma_{S'\setminus\{s_n\}}$ and $\sigma_{S'\setminus\{s_{n-1},s_n\}}$. But $\sigma_{S'\setminus\{s_n\}}$ is a group element of the Coxeter system $(W'(S'\setminus\{s_n\}), S'\setminus\{s_n\})$, which is of type D, so as we did for the proof of Proposition 4.5, consider the set

$\left[\pi, \sigma_{S'\setminus\{s_n\}}\right] \setminus \left[\pi, \sigma_{S'\setminus\{s_1,s_n\}}\right]$. This contains $\sigma_{S'\setminus\{s_{n-1},s_n\}}$ and an s_1-anchored saturated chain $\mathcal{C}_2$ with elements,

$$\sigma_{S'\setminus\{s_{n-2},s_n\}} > \sigma_{S'\setminus\{s_{n-2},s_{n-1},s_n\}} >> \sigma_{S'\setminus\{s_{n-3},s_{n-2},s_{n-1},s_n\}} > \cdots .$$

Note that although they look very similar, the chain $\mathcal{C}_2$ does not necessarily end at the same minimal element as the chain $\mathcal{C}_1$. Now we have,

$$\begin{aligned} \mu\left(\pi, \sigma_{S'\setminus\{s_n\}}\right) + \mu\left(\pi, \sigma_{S'\setminus\{s_{n-1},s_n\}}\right) &= \mu\left(\pi, \sigma_{S'\setminus\{s_{n-1},s_n\}}\right) \\ &\quad - \sum_{\tau \in \left[\pi,\sigma_{S'\setminus\{s_n\}}\right]\setminus\left[\pi,\sigma_{S'\setminus\{s_1,s_n\}}\right]} \mu(\pi,\tau) \\ &= -\sum_{\tau \in \mathcal{C}_2} \mu(\pi,\tau). \end{aligned}$$

It follows that

$$\mu(\pi,\sigma) = -\sum_{\tau \in \mathcal{C}_1} \mu(\pi,\tau) + \sum_{\tau \in \mathcal{C}_2} \mu(\pi,\tau).$$

But both $\mathcal{C}_1$ and $\mathcal{C}_2$ are s_1-anchored saturated chains, so

$$|\mu(\pi,\sigma)| = \left|\sum_{\tau \in \mathcal{C}_1} \mu(\pi,\tau)\right| + \left|\sum_{\tau \in \mathcal{C}_2} \mu(\pi,\tau)\right| \le 2,$$

as desired. $\square$

We summarize the previous results in Theorem 4.7:

Theorem 4.7. *If σ is an element of a finite irreducible Coxeter system, then $|\mu(\pi,\sigma)| \le 2$.*

However, things are different in infinite Coxeter groups.

Theorem 4.8. *If the Coxeter system to which π belongs, i.e. (W', S'), is not necessarily finite, then $|\mu(\pi,\sigma)|$ can be unbounded.*

Proof. We provide an explicit construction.

Consider the Coxeter system (W, S) with $S = \{s_0, s_1, s_2, \ldots, s_{2n}\}$ for some positive integer n such that s_i and s_j commute for all $1 \le i, j \le 2n$ and s_0 has no relation with any s_i for $1 \le i \le 2n$ (i.e. $m_{0,i} = \infty$). Let

$$\sigma = s_2 s_0 s_4 s_0 s_6 s_0 \cdots s_{2n} s_0 s_1 s_3 s_5 \cdots s_{2n-1}.$$

We can check that, using the fact that s_1, s_3, s_5,... s_{2n-1} commute but s_0 has no relation with them, for any $1 \leq i \leq 2n$,

$$\sigma_{S\setminus\{s_i\}} = \begin{cases} s_1 s_3 \dots s_{i-2} s_{i+2} \dots s_{2n-1} & \text{if } i \text{ is odd} \\ s_0 s_{i+2} s_0 \dots s_{2n} s_0 s_1 s_3 \dots s_{2n-1} & \text{if } i \text{ is even} \end{cases}.$$

Similarly, $\sigma_{S\setminus\{s_{2i-1},s_{2i}\}} = s_1 s_3 \dots s_{i-2} s_{i+2} \dots s_{2n-1}$ for any $1 \leq i \leq n$. Notice that these are all isomorphic by permuting the tuple of pairs $((s_1, s_2), (s_3, s_4), \dots, (s_{2n-1}, s_{2n}))$. However, all of the ones of the form $s_0 s_{i+2} s_0 \dots s_{2n} s_0 s_1 s_3 \dots s_{2n-1}$ are distinct.

Thus if we pick $\pi = \sigma_{S\setminus\{s_1,s_2\}} = \sigma_{S\setminus\{s_3,s_4\}} = \dots = \sigma_{S\setminus\{s_{2n-1},s_{2n}\}}$, we have that the interval $[\pi, \sigma]$ is a poset with $n + 1$ other elements, namely $\tau = \sigma_{S\setminus\{s_1\}}$ and $\tau = \sigma_{S\setminus\{s_2\}}, \sigma_{S\setminus\{s_4\}}, \dots, \sigma_{S\setminus\{s_{2n}\}}$, which each cover π and are covered by σ.

For each of these τ, $\mu(\pi, \tau) = -1$. $\mu(\pi, \pi) = 1$, so we have

$$\mu(\pi, \sigma) = -(1 + (n + 1) \cdot (-1)) = n.$$

Since n can be arbitrarily large, we are done. □

Nevertheless, it appears that $\mu(\pi, \sigma)$ cannot grow too quickly in terms of $|\sigma|$. In fact, we conjecture the following:

Conjecture 4.9. *For any π, σ such that $\pi \leq \sigma$, $|\mu(\pi, \sigma)| \leq |\sigma|$.*

Data availability

No data was used for the research described in the article.

Acknowledgments

We would like to thank Sergi Elizalde for many helpful conversations. We would also like to thank the MIT PRIMES program, under which this research was conducted.

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