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ON THE CRITICAL IDEALS OF COMPLETE MULTIPARTITE GRAPHS*

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Abstract. The notions of critical ideals and characteristic ideals of graphs are introduced by Corrales and Valencia to study properties of graphs, including clique number, zero forcing number, minimum rank and critical group. In this paper, methods are provided to compute critical ideals of complete multipartite graphs and obtain complete answers for the characteristic ideals of complete multipartite graphs.

Key words. Critical ideal, Characteristic ideal, Laplacian matrix, Complete multipartite graph.

AMS subject classifications. 13F20, 05C50.

1. Introduction. Let G be an undirected simple graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and let R be any commutative ring. Define the *generalized Laplacian matrix* of G to be the matrix, denoted as $L_G(X_G)$, with entries given by

$$L_G(X_G)_{ij} = \begin{cases} x_i & \text{if } i = j, \\ -1 & \text{if } i \sim j, \\ 0 & \text{otherwise,} \end{cases}$$

where x_i is an indeterminant associated to vertex i and $i \sim j$ means that there is an edge between vertex v_i and vertex v_j. Alternatively, $L_G(X_G) = \operatorname{diag}(x_1, \ldots, x_n) - A_G$, where $\operatorname{diag}(x_1, \ldots, x_n)$ is the diagonal matrix with diagonal entries $x_1, \ldots, x_n$ and A_G is the adjacency matrix of G. In terms of notation, we view X_G as a set of variables labeled by vertices of G. Similarly, define the *characteristic matrix* of G, denoted as $L_G(t)$, to be $tI - A_G$. Notice that the characteristic matrix can be viewed as a specialization of the generalized Laplacian matrix.

Definition 1.1. For $j = 1, \ldots, n$, the j-th *critical ideal* of G over R, denoted as $I_j^R(G, X_G)$, is defined to be the ideal of $R[x_1, \ldots, x_n]$ generated by all $j \times j$ minors of the generalized Laplacian matrix $L_G(X_G)$ of G. In other words,

$$I_j^R(G, X_G) := \langle \{\det M : M \text{ is a } j \times j \text{ submatrix of } L_G(X_G)\} \rangle \subset R[X_G].$$

And the j-th *characteristic ideal* of G over R, denoted as $I_j^R(G, t)$, is defined to be the ideal of $R[t]$ after specializing all variables $x_1, \ldots, x_n$ to the same variable t in the j-th critical ideal.

Example 1.2. Let G be the graph shown in Figure 1. Its generalized Laplacian matrix and its generalized characteristic matrix are respectively

$$L_G(X_G) = \begin{bmatrix} x_1 & -1 & -1 & 0 \\ -1 & x_2 & -1 & 0 \\ -1 & -1 & x_3 & -1 \\ 0 & 0 & -1 & x_4 \end{bmatrix} \quad \text{and} \quad L_G(t) = \begin{bmatrix} t & -1 & -1 & 0 \\ -1 & t & -1 & 0 \\ -1 & -1 & t & -1 \\ 0 & 0 & -1 & t \end{bmatrix}.$$

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It's then easy to calculate that

$$I_j^R(G, X_G) = \begin{cases} \langle 1 \rangle & k = 1, 2, \\ \langle x_1 + 1, x_2 + 1, x_3x_4 + x_4 - 1 \rangle & k = 3, \text{ and} \\ \langle x_1x_2x_3x_4 - x_1x_2 - x_1x_4 - x_2x_4 - x_3x_4 - 2x_4 + 1 \rangle & k = 4, \end{cases}$$

$$I_j^R(G, t) = \begin{cases} \langle 1 \rangle & k = 1, 2, 3, \\ \langle t_4 - 4t^2 - 2t + 1 \rangle & k = 4. \end{cases}$$

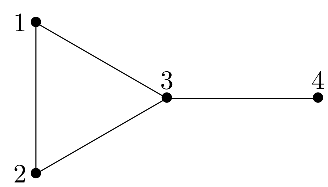

FIGURE 1. *A graph G as an example.*

The notions of critical ideals and characteristic ideals were first introduced by Corrales and Valencia [3], where many properties of these ideals are discussed and computed for some classes of graphs. The study of critical ideals and characteristic ideals has many applications and strong connections with other properties of graphs. For example, we can define the *algebraic co-rank* of G, denoted as γ_G^R, to be the maximum j such that $I_j^R(G, X_G)$ is trivial. Let $Z(G)$ be the zero forcing number of G and let $\mathrm{mz}(G) = |V(G)| - Z(G)$. If R is a field, let $\mathrm{mr}_R(G)$ be the minimum rank of G. It is shown in [2] that $\mathrm{mz}(G) \leq \gamma_G^R$ for any commutative ring R and it is proved in [4] that $\mathrm{mz}(G) \leq \mathrm{mr}_R(G)$ when R is a field. But in general, the relation between $\mathrm{mr}_R(G)$ and γ_G^R still remains an interesting open problem.

Another important connection of critical ideals to other properties of graphs concerns critical groups. As the name suggests, we can get the Laplacian matrix of G by specializing each variable x_i to the degree of i in the generalized Laplacian matrix $L_G(X_G)$. Therefore, if we are able to completely understand the structure of critical ideals of G, we can obtain the critical group of G, or equivalently the Smith normal form of its Laplacian matrix, for free via such specialization. In particular, we will be discussing critical ideals and characteristic ideals of complete multipartite graphs in this paper, whose critical groups are computed by Jacobson, Niedermaier and Reiner [5].

In this paper, we focus on computing the critical ideals and characteristic ideals of complete multipartite graphs, aiming to generalize results on their critical groups. Throughout the paper, let G be a complete multipartite graph with m parts with size $r_1, \dots, r_m$. For simplicity, let $r_1, \dots, r_m \geq 2$. Denote the i-th part as V_i with $|V_i| = r_i$. In Section 2, we will compute characteristic ideals of G explicitly for $m = 2, 3$, i.e., complete bipartite graphs and complete tripartite graphs. In Section 3, we will specialize to characteristic ideals and obtain formulas for general m and in Section 4, we make some further specializations of interests to gain nicer expressions for the characteristic ideal when the graph is balanced and when R is a field.

2. Critical ideals of complete multipartite graphs. Let's further fix some notations. Choosing a $j \times j$ submatrix of $L_G(X_G)$ corresponds to choosing j "row vertices" and j "column vertices". By definition,

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its determinant is a weighted sum of matchings between these row vertices and column vertices, where the weight is given by the sign of the corresponding permutation and the matrix entries. But the matrix entry is zero between distinct vertices with no edge in between. So we can view this determinant as a sum of perfect matchings of those row vertices and column vertices, where a vertex is allowed to be paired with itself. For a $j \times j$ submatrix M of $L_G(X_G)$, let R_i be the set of row vertices chosen in part i and let C_i be the set of column vertices chosen in part j. And let $\mathcal{R} = \bigcup_{i=1}^m R_i$ and $\mathcal{C} = \bigcup_{i=1}^m C_i$. If for some i, $|R_i \setminus C_i| \geq 2$, let $v, w \in R_i \setminus C_i$. We observe that the rows in M corresponding to v and w are identical, meaning $\det M = 0$. Therefore, if $\det M \neq 0$, we necessarily have $|R_i \setminus C_i|, |C_i \setminus R_i| \leq 1$. Accordingly, there are 4 possibilities explained in Definition 2.1 below.

Definition 2.1. We say that block i is of

- type b if $R_i = C_i$;
- type r if $C_i \subset R_i$, $|R_i \setminus C_i| = 1$;
- type c if $R_i \subset C_i$, $|C_i \setminus R_i| = 1$;
- type u if $|R_i \setminus C_i| = |C_i \setminus R_i| = 1$.

We now simplify some computation of minors of $L_G(X_G)$. A *pairing* of $\mathcal{R}$ and $\mathcal{C}$ is a bijective map between these two sets of vertices.

Fix a total ordering of all vertices. For a pairing σ between $\mathcal{R}$ and $\mathcal{C}$, arrange $\mathcal{R}$ in order so that the relative order of their corresponding vertices in $\mathcal{C}$ gives rise to a permutation w_σ. Let the *sign* of σ, denoted $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ to be the sign of the permutation w_σ. Notice that changing a total ordering results in a possibly change of signs for all determinants that we are going to compute, but this does not matter for the computation of the critical ideals.

Definition 2.2. We say that a pairing σ of $\mathcal{R}$ and $\mathcal{C}$ is *valid* if

- in a block V_i of type r, c or u, every vertex in $R_i \cap C_i$ is paired with itself;
- in a type b block v_i, at most one vertex in $R_i = C_i$ is not paired with itself.

Definition 2.3. The *weight* of a pairing σ is the product of its corresponding entries in $L_G(X_G)$. More formally,

$$\mathrm{wt}(\sigma) := \prod_{v \in \mathcal{R}} L_G(X_G)_{v,\sigma(v)},$$

where $\sigma(v)$ means the column vertex paired up with v and $L_G(X_G)_{i,j}$ denotes the entry of the matrix $L_G(X_G)$ in row i and column j.

Lemma 2.4. *With notations as above,*

$$\det M = \pm \sum_{\sigma \text{ valid}} \mathrm{sgn}(\sigma) \cdot \mathrm{wt}(\sigma).$$

Notice that we don't care about the overall sign of $\det M$, which depends on the ordering of vertices and doesn't make a difference for generators of the critical ideals.

Proof. Let's apply some row and column operations on M, which preserve determinant. If part i is of type r, choose the unique $v \in R_i \setminus C_i$. Notice that for any $w \in R_i \cap C_i = C_i$, the row of w and v in M differ by only one entry in the column of w, where $L_G(X_G)_{w,w} = x_w$ and $L_G(X_G)_{v,w} = 0$. Thus, after subtracting row v from row w for every $w \in R_i \cap C_i$, each column corresponding to $w \in C_i$ has only 1 nonzero entry left, which is x_w. Therefore, when computing the determinant, we must select these entries. In the original

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matrix M, this is saying that we are summing over pairings σ that pair $w \in R_i$ with $w \in C_i$, for $w \in R_i \cap C_i$. Similarly, if part i is of type c, apply the above arguments with the roles of rows and columns reversed. If part i is of type u, the same argument works as well, by considering $v \in R_i \setminus C_i$ and concluding that every $w \in R_i \cap C_i$ needs to be paired with itself.

Now consider a part of type b, meaning $R_i = C_i$. Pick any $v \in R_i = C_i$. For any $w \neq v \in R_i = C_i$, subtract row v from row w to obtain M' and observe that all entries become zero except that $M'_{w,v} = -x_v$ and $M'_{w,w} = x_w$. In such a row, when determinant of M' is calculated via sum over permutations, we either choose $-x_v$ or x_w. If for some w, we choose $-x_v$, meaning that column v has been chosen, then for all other $u \in R_i \setminus \{w, v\}$, we need to choose $M'_{u,u} = x_u$. Suppose that in row v, we choose column s, then this choice in matrix M corresponds to choosing $M_{v,v} = x_v$ and $M_{w,s}$. Notice that the sign corresponds as well. If we are not choosing any $-x_v$'s in M', then we choose $M'_{w,w} = M_{w,w} = x_w$ for all $w \in R_i \setminus \{v\}$. No matter which entry we choose in row v, this also corresponds to a valid pairing by definition.

Notice that all row and column operations described above commute with each other since we are only using these operations inside certain parts and row and column operations commute. ∎

Example 2.5. Let's use an example to see how the proof of Lemma 2.4 works.

Suppose that G has 3 parts and $V_1 = \{1, 2, 3\}$, $V_2 = \{4, 5, 6\}$, $V_3 = \{7, 8, 9\}$ and the submatrix M we are considering is formed by $\mathcal{R} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathcal{C} = \{1, 2, 3, 4, 5, 7, 8, 9\}$. This means part 1 is of type b, part 2 is of type r and part 3 is of type c. By Definition 2.2, a valid pairing σ must map 4,5,7,8 to themselves, and at least two of 1,2,3 to themselves. If σ maps 1,2,3 to themselves, it must map 6 to 9, giving a weight of $-x_1x_2x_3x_4x_5x_7x_8$ and sign of 1. If it maps only 1,2 to themselves, then it must map 3 to 9 and 6 to 3, giving a weight of $x_1x_2x_4x_5x_7x_8$ and sign of -1, and similarly when it maps only 2,3 to themselves or 1,3 to themselves. As a result, the right hand side of Lemma 2.4 equals

$$-(x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3)x_4x_5x_7x_8.$$

The proof of Lemma 2.4 illustrates the following process of row and column operations. We start with

$$M = \begin{bmatrix} x_1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & x_2 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & x_3 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & x_4 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & x_5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & x_7 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & x_8 & 0 \end{bmatrix}.$$

Then subtract row 2,3 from row 1, row 4,5 from row 6, column 7,8 from column 9 to obtain the following matrix that's way sparser:

$$\begin{bmatrix} x_1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ -x_1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & x_7 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & x_8 & 0 \end{bmatrix}.$$

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To calculate determinant, it is clear that row 4 and row 5 must choose x_4 and x_5 respectively, and column 6 and column 7 must choose x_7 and x_8 respectively. The determinant of the remaining 4×4 is easily calculated to be $-(x_1x_2x_3+x_1x_2+x_1x_3+x_2x_3)$ as desired by considering choices in row 2 and 3.

Lemma 2.6. *With notations as above, if* $\det M \neq 0$, *the number of parts of type* r, *which equals the number of parts of type* c, *is at most 1.*

Proof. Since we need the same number of row vertices and column vertices to calculate determinant, the number of parts of type r equals the number of parts of type c. If V_i and V_j are two parts of type r, let $v_i \in R_i \setminus C_i$ and $v_j \in R_j \setminus C_j$. Observing that if σ is a valid paring with nonzero weight, exchanging the image of v_i and v_j results in another pairing σ' with the same weight but different sign. By this involution and by Lemma 2.4, we obtain $\det M = 0$. $\square$

We further fix some notations. Recall that there are m parts, each denoted as V_i with size r_i for $i=1,\dots,m$. Denote vertices in V_i as v_i^k for $1\le k\le r_i$ and its corresponding variable as $x_i^{(k)}$. Define

$$P_{i,j} := \left\{ x_i^{(k_1)}\cdots x_i^{(k_j)} : 1\le k_1<\cdots<k_j\le r_i \right\},$$
$$Q_{i,j} := \left\{ x_i^{(k_1)}\cdots x_i^{(k_j)} \left(\frac{1}{x_i^{(k_1)}}+\cdots+\frac{1}{x_i^{(k_j)}} \right) : 1\le k_1<\cdots<k_j\le r_i \right\}.$$

Note that in this definition, both $P_{i,j}$ and $Q_{i,j}$ are sets of algebraic expressions, indexed by j-element subset of $\{1,2,\dots,r_i\}$. Expressions in $P_{i,j}$ are of degree j while expressions in $Q_{i,j}$ are of degree $j-1$. As convention, let $P_{i,0}=\{1\}$ and $Q_{i,0}=\{0\}$ for all i, which are used in minors that contain empty parts ($R_i=C_i=\emptyset$ which are of type b). In the following discussions, as for notations, we will use some algebraic expressions of such P's and Q's: multiplication uses Cartesian product of index sets while addition is component-wise with the same index set. For example,

$$(P_{1,2}+Q_{1,2})P_{2,1} = \{(x_1^{(a)}x_1^{(b)}+x_1^{(a)}+x_1^{(b)})x_2^{(c)} : 1\le a<b\le r_1, 1\le c\le r_2\}.$$

We are now ready to give full descriptions of the critical ideals of complete multipartite graphs when the number of parts equal 2 or 3, where the case $m=2$ is given in Section 3.3 of [1], for which we formulate a different proof. Our strategy is straightforward: separate cases by types of parts (Definition 2.1) and write down generators via the above notations.

Theorem 2.7. *Let G be a complete bipartite graph with part size* $r_1,r_2\ge 2$. *Then*

$$I_j^R(G,X_G)=\begin{cases}\langle 1\rangle & j=1,\\ \langle P_{1,s}P_{2,t},\ s{+}t{=}j{-}2,\ s{\le}r_1{-}2,\ t{\le}r_2{-}2,\ P_{1,j-r_2}Q_{2,r_2},\ Q_{1,r_1}P_{2,j-r_1}\rangle & 2{\le}j{\le}r_1{+}r_2{-}2,\\ \langle P_{1,r_1-1}P_{2,r_2-1},\ P_{1,r_1-2}Q_{2,r_2},\ Q_{1,r_1}P_{2,r_2-2}\rangle & j{=}r_1{+}r_2{-}1,\\ \langle P_{1,r_1}P_{2,r_2}-Q_{1,r_1}Q_{2,r_2}\rangle & j{=}r_1{+}r_2.\end{cases}$$

Proof. It's clear that $I_1^R(G,X_G)=\langle 1\rangle$.

First, consider $2\le j\le r_1+r_2-2$. If the types of two parts are u and u, there is a unique valid pairing with nonzero weight, which pairs the vertex in $R_1\setminus C_2$ with the vertex in $C_2\setminus R_1$ and vice versa. This gives a determinant of $P_{1,s}P_{2,t}$ for $s+t=j-2$, $s\le r_1-2$, $t\le r_2-2$. If the types are r and c, there is also a unique valid pairing which gives a determinant of $P_{1,s}P_{2,t}$ for $s+t=j-1$, $s\le r_1-1$, $t\le r_2-1$. In this case of (r,c), if $s=r_1-1$ and $t=r_2-1$, we would have $j=r_1+r_2-1$ which is a contradiction; but

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if $s < r_1 - 1$, the expression $P_{1,s}P_{2,t}$ is already generated by $P_{1,s}P_{2,t-1}$ in the case of (u,u), and similarly with $t < r_2 - 1$. Thus, the case (r,c) does not provide us with more generators for the ideal $I_j^R(G, X_G)$. If the types are u (say part 1) and b, a valid pairing pairs $R_1 \setminus C_1$ to some vertex v_2^k and pairs the same v_2^k to $C_1 \setminus R_1$. We obtain generators $P_{1,s}Q_{2,t}$ with $s + t = j - 1$, $s \leq r_1 - 2$, $t \leq r_2$. The only ones not generated by case (u,u) are $s = j - r_2$ and $t = r_2$, which are $P_{1,j-r_2}Q_{2,r_2}$ and $Q_{1,r_1}P_{2,j-r_1}$ (if such expressions exist). If the types are b and b, the determinant is $P_{1,s}P_{2,t} - Q_{1,s}Q_{2,t}$ with $s + t = j$. This expression is already generated by the case (u,u) and (b,u): if $s = r_1 - 1$ and $t = r_2 - 1$, $P_{1,s}P_{2,t}$ is generated by $P_{1,s-1}P_{2,t-1}$ in the case (u,u) and if $s \leq r_1 - 2$, $P_{1,s}P_{2,t}$ is generated by $P_{1,s}P_{2,t-2}$ in the case (u,u) as well; assume $s \leq r_1 - 1$ without loss of generality, then $Q_{1,s}Q_{2,t}$ can be grouped into $P_{1,s-1}P_{2,t}$, which are generated in the case (u,b). Thus, type (b,b) doesn't provide new generators.

Next, consider $j = r_1 + r_2 - 1$. The significant difference is that type (u,u) is no longer possible. If types are (r,c), we get $P_{1,r_1-1}P_{2,r_2-1}$. If types are (u,b), we get $P_{1,r_1-2}Q_{2,r_2}$, $Q_{1,r_1}P_{2,r_2-2}$. If types are (b,b), we get $P_{1,s}P_{2,t} - Q_{1,s}Q_{2,t}$ with $s + t = r_1 + r_2 - 1$, which can be generated by the previous two.

Finally, if $j = r_1 + r_2$, only type (b,b) is possible, and we obtain $P_{1,r_1}P_{2,r_2} - Q_{1,r_1}Q_{2,r_2}$. ∎

With a similar strategy, we can proceed to complete tripartite graphs. Note that $I_j^R(G, X_G) = \langle 1 \rangle$ for $j = 1, 2$. We will then deal with the most general cases where $3 \leq j \leq n - 3$, where $n = r_1 + r_2 + r_3$ is the total number of vertices.

THEOREM 2.8. *Let G be a complete tripartite graph with part size $r_1, r_2, r_3 \geq 2$. Fix j with $3 \leq j \leq n-3$. Then the critical ideal $I_j^R(G, X_G)$ is generated by the following generators:*

- $2P_{1,s_1}P_{2,s_2}P_{3,s_3}$, $\sum_{i=1}^{3} s_i = j - 3$, $s_i \leq r_i - 2$, $i = 1, 2, 3$;
- $P_{1,s_1}P_{2,s_2}P_{3,s_3}$, $\sum_{i=1}^{3} s_i = j - 2$, $s_i \leq r_i - 1$, $i = 1, 2, 3$;
- $P_{i_1,s_{i_1}}P_{i_2,s_{i_2}}\left(P_{i_3,s_{i_3}} + 2Q_{i_3,s_{i_3}}\right)$, $\{i_1, i_2, i_3\} = \{1, 2, 3\}$, $s_{i_1} \leq r_{i_1} - 2$, $s_{i_2} \leq r_{i_2} - 2$, $s_{i_3} = r_{i_3}$, $s_{i_1} + s_{i_2} = j - r_{i_3} - 2$;
- $P_{i_1,s_{i_1}}\left(P_{i_2,s_{i_2}}Q_{i_3,s_{i_3}} + Q_{i_2,s_{i_2}}P_{i_3,s_{i_3}} + 2Q_{i_2,s_{i_2}}Q_{i_3,s_{i_3}}\right)$, $\{i_1, i_2, i_3\} = \{1, 2, 3\}$, $s_{i_1} = j - r_{i_2} - r_{i_3} - 1$, $s_{i_2} = r_{i_2}$, $s_{i_3} = r_{i_3}$.

Proof. We perform a similar analysis as above, by considering possible types of selection of row and column vertices. As we see from the above arguments, all of our generators will be linear combinations of "monomials" of the form $A_{1,s_1}B_{2,s_2}C_{3,s_3}$ where $A, B, C \in \{P, Q\}$, we will then shorten the notation by writing ABC instead, with certain conditions on s_i's.

Case (u,u,u). We obtain $2P_{1,s_1}P_{2,s_2}P_{3,s_3}$, where $s_1 + s_2 + s_3 = j - 3$ and $s_i \leq r_i - 2$, $i = 1, 2, 3$. The coefficient 2 here can be seen from Lemma 2.4 as there are two nonzero-weight valid pairing with the same weight and sign, or by Lemma 3.1 which is more general.

Case (r,c,u). We obtain PPP with $s_1 + s_2 + s_3 = j - 2$ and $s_1 \leq r_1 - 1$, $s_2 \leq r_2 - 1$, $s_3 \leq r_3 - 2$ (and two more cases by symmetry indices 1,2,3). However, as $j - 2 < j \leq (r_1 - 1) + (r_2 - 1) + (r_3 - 1)$, we can simplify the condition on indices to be $s_i \leq r_i - 1$, $i = 1, 2, 3$ to unify all those three cases.

Case (r,c,b). We obtain $PP(P + Q)$ with $s_1 + s_2 + s_3 = j - 1$, $s_1 \leq r_1 - 1$, $s_2 \leq r_2 - 1$ and two more cases by symmetry of indices 1,2,3. As each element in the set Q_{3,s_3} is a linear combination of degree $s_3 - 1$ monomials and each element in the set P_{3,s_3} is a multiple of a degree $s_3 - 1$ monomial, we see that the generators produced here can already be generated by PPP with $s_1 + s_2 + s_3 = j - 2$, $s_i \leq r_i - 1$ described above in the case (r,c,u).

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Case (u,u,b). We obtain $PP(P+2Q)$ with $s_1+s_2+s_3=j-2$, $s_1\le r_1-2$, $s_2\le r_2-2$ and two more cases by symmetry of indices 1,2,3. If $s_3\le r_3-1$, then PPP can be generated by case (r,c,u) and $2PPQ$ can be generated by case (u,u,u). But when $s_3=r_3$, we obtain more generators (along with two more cases).

Case (u,b,b). We obtain $P(PQ+QP+2QQ)$ with $s_1+s_2+s_3=j-1$, $s_1\le r_1-2$. Notice that if $s_2\le r_2-1$, then PPQ can be generated by case (r,c,u) by splitting up monomials in Q_{3,s_3} and the rest $PQ(P+2Q)$ can be generated by case (u,u,b) by splitting up monomials in Q_{2,s_2}. This means that we do not have more generators when $s_2<r_2$ or $s_3<r_3$. So the only new generators we get are $P(PQ+QP+2QQ)$ with $s_1=j-1-r_2-r_3$, $s_2=r_2$, $s_3=r_3$ and two more cases by symmetry.

Case (b,b,b). We obtain $PPP-PQQ-QPQ-QQP-2QQQ$ with $s_1+s_2+s_3=j$. Since $s_1+s_2+s_3=j<r_1+r_2+r_3$, let's assume without loss of generality that $s_1\le r_1-1$. Then every element in the first two terms $P_{1,s_1}P_{2,s_2}P_{3,s_3}$ and $-P_{1,s_1}Q_{2,s_2}Q_{3,s_3}$ is a multiple of some element in $P_{1,s_1}P_{2,s_2-1}P_{3,s_3-1}$, which belongs to case (r,c,u). For the rest $Q(PQ+QP+2QQ)$, if we split up monomials in Q_{1,s_1}, we obtain a sum of $P_{1,s_1-1}(PQ+QP+2QQ)$'s, which belong to case (u,b,b). Thus, no new generators are created here. ∎

Note that in the case of complete tripartite graphs when $j\ge r_1+r_2+r_3-2$, we have fewer cases but messier expressions with the same method. Since there are at most 2 vertices that are not selected as row vertices and at most 2 vertices that are not selected as column vertices, the overall possibilities are very limited so we won't enumerate them here.

3. Characteristic ideals of complete multipartite graphs. As a major simplification, we then consider characteristic ideals by specializing all variables to t. This makes computation a lot easier and allows us to deduce nice formula for the general case.

We note that the known spectrum of the complete multipartite graphs (either adjacency matrix or Laplacian matrix) will not be sufficient to provide us formula for the characteristic ideals of these graphs, when R is a general ring.

Lemma 2.4 and Lemma 2.6 still remain valid after the specialization. Recall that our complete multipartite graph contains m parts with size $r_1,\dots,r_m\ge 2$ and $n=\sum_{i=1}^m r_i$ is the total size of the graph. As our usual convention, let j be the size of a chosen minor. Here is a simple lemma that helps the computation.

Lemma 3.1. *Let $\mathbf{J}_n$ denote the all 1 matrix of size n and $\mathbf{I}_n$ denote the identity matrix of size n. Then* $\det(\mathbf{I}_n-\mathbf{J}_n)=-n+1$ *and* $\det([0]\oplus\mathbf{I}_{n-1}-\mathbf{J}_n)=-1$*, where $[0]\oplus\mathbf{I}_{n-1}$ refers to the diagonal matrix with diagonal entries* $0,1,1,\dots,1$.

Proof. It is clear that $\mathbf{J}_n$ has eigenvalue n with multiplicity 1 and eigenvalue 0 with multiplicity $n-1$. Thus, $\mathbf{J}_n$ has characteristic polynomial $(x-n)x^{n-1}$. Assign $x=1$ gives the determinant of $\mathbf{I}_n-\mathbf{J}_n$ by definition so $\det(\mathbf{I}_n-\mathbf{J}_n)=-n+1$. By a subtraction of determinant of $\mathbf{I}_n-\mathbf{J}_n$ from $\mathbf{I}_{n-1}-\mathbf{J}_{n-1}$, we obtain $\det([0]\oplus\mathbf{I}_{n-1}-\mathbf{J}_n)=-1$. ∎

Recall that the *elementary symmetric function* of degree d in ℓ variables is defined to be

$$e_d(y_1,\dots,y_\ell):=\sum_{1\le i_1<\cdots<i_d\le\ell} y_{i_1}\cdots y_{i_d}.$$

The following is the main theorem of this section.

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THEOREM 3.2. *Let G be a complete multipartite graph with $m(\geq 2)$ parts of size $r_1, \ldots, r_m \geq 2$. Let $n = \sum_{i=1}^{m} r_i$. Then*

$$I_j^R(G,t) = \begin{cases} \langle 1 \rangle & j \leq m-1, \\ \langle (m-1)t^{j-m}, t^{j-m+1} \rangle & m \leq j \leq n-m, \\ \langle t^{j-m+1} \prod_{a=1}^{m-k-1}(t+r_{i_a}), \sum_{a=0}^{m-k}(k-1+a)e_a(r_{i_1}, \ldots, r_{i_{m-k}})t^{j-k-a}, & \\ \quad k = n-j \geq 1,\ 1 \leq i_1 < \cdots < i_{m-k} \leq m \rangle & n-m < j < n, \\ \langle \sum_{a=0}^{m}(a-1)e_a(r_1, \ldots, r_m)t^{n-a} \rangle & j = n, \end{cases}$$

where e_a is the elementary symmetric function of degree a.

Proof. If $j < m$, let's choose a submatrix M where parts i, for $i = 1, \ldots, j-1$, is of type u with $|R_i| = |C_i| = 1$, part j is of type r with $|R_j| = 1, |C_j| = 0$ and part $j+1$ is of type c with $|R_{j+1}| = 0, |C_{j+1}| = 1$. By the second determinant in Lemma 3.1, $\det M = \pm 1$, giving us $I_j^R(G,t) = \langle 1 \rangle$. As a remark, this is true for critical ideals as well.

Then consider the case $m \leq j \leq n-m$. As we see from Section 2, this is the case where the most number of types are possible. Type $u, u, \ldots, u$ (all parts are of type u) gives a determinant of $(m-1)t^{j-m}$ by Lemma 2.4 and the first determinant in Lemma 3.1. Type $r, c, u, u, \ldots, u$ gives a determinant of t^{j-m+1} by Lemma 2.4 and the second determinant in Lemma 3.1. We claim that in fact

$$I_j^R(G,t) = \langle (m-1)t^{j-m}, t^{j-m+1} \rangle, \quad \text{for } m \leq j \leq n-m.$$

For an arbitrary submatrix M, if it contains a part of type r/c, then each valid pairing (Definition 2.2) gives a monomial of degree at least $j-m+1$ so the determinant is divisible by t^{j-m+1}. So we can next assume M contains k parts of type b and $m-k$ parts of type u. We see that the pairings whose weights are not divisible by t^{j-m+1} must have precisely 1 vertex not paired with itself in each of the parts of type b. Grouping together such pairings via the vertices not paired with themselves, by Lemma 3.1, such weights left are divisible by $(m-1)t^{j-m}$.

We are left with the case $j > n-m$. Fix j and write $k = n-j \leq m-1$ for notation. Intuitively, the minor we are considering is very big comparing to the overall generalized characteristic matrix $L_G(t)$. Thus, we are expecting a lot of parts V_i with $R_i = C_i = V_i$. We call such part *full*. We divide our discussion into two cases by whether there is a part of type r/c.

Type $rcu^{\ell}b^{m-\ell-2}$. This can only happen when $j < n$ or equivalently, $k \geq 1$. Since $|\mathcal{R}| = j$ and $|V_i \setminus R_i| \geq 1$ for a type c, u part, we have at most $(n-j)-(1+\ell) = k-\ell-1$ parts of type b that are not full. This means we have at least $(m-\ell-2)-(k-\ell-1) = m-k-1$ parts that are full. Let $s_1, \ldots, s_{m-\ell-2}$ be the size of selected vertices in these type b parts. We obtain

$$\det M = t^{j-m+1}(t+s_1)\cdots(t+s_{m-\ell-2}).$$

To see this, consider Lemma 2.4 and group together valid pairings by the number of vertices paired with themselves in each part. In a type b part with s_i selected vertices, either all of them are paired with themselves or $s_i - 1$ of them are. In the latter case, there are s_i ways to choose these $s_i - 1$ vertices. Thus, if type b parts, with selected vertices of sizes $s_{i_1}, \ldots, s_{i_a}$, contain vertices not paired with themselves while the rest of type b parts do not, we obtain (by Lemma 3.1 as well) a monomial $s_{i_1}\cdots s_{i_a}t^{j-\ell-a-1}$. Summing over all $i_1 < \cdots < i_a \in \{1, \ldots, m-\ell-2\}$, we obtain $t^{j-m+1}(t+s_1)\cdots(t+s_{m-\ell-2})$. As we have at least

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$m-k-1$ full parts, this expression is a multiple of $t^{j-m+1}(t+r_{i_1})\cdots(t+r_{i_{m-k-1}})$, which is the determinant of a special case $rcu^{k-1}b^{m-k-1}$. This means that we can eliminate variable ℓ and only need to keep the generators $t^{j-m+1}(t+r_{i_1})\cdots(t+r_{i_{m-k-1}})$ where $i_1,\ldots,i_{m-k-1}$ range through all $(m-k-1)$-element subsets of $\{r_1,\ldots,r_m\}$.

Type $u^\ell b^{m-\ell}$. To use some generators from above, let's first assume $k\geq 1$ as the case $k=0$ is just a corner case that is easily computed with the same argument. Similarly, we have at most $(n-j)-\ell=k-\ell$ non-full parts of type b so at least $m-k$ full parts. This also shows $\ell\leq k$. Let $s_1,\ldots,s_{m-\ell}$ be $|R_i|=|C_i|$ for all parts of type b and let $e_a(s_1,\ldots,s_{m-\ell})$ be the a-th (degree a) elementary symmetric polynomial, i.e.,

$$e_a(s_1,\ldots,s_{m-\ell})=\sum_{1\leq i_1<\cdots<i_a\leq m-\ell} s_{i_1}\cdots s_{i_a}.$$

Then with the same argument as above, up to an overall sign,

$$\det M=\sum_{i=0}^{m-\ell}(\ell-1+i)e_i(s_1,\ldots,s_{m-\ell})t^{j-\ell-i}.$$

We claim that this expression can be generated by the special cases from $\ell=k$ and $t^{j-m+1}(t+r_{i_1})\cdots(t+r_{i_{m-k-1}})$ mentioned in type $rcu^\ell b^{m-\ell-2}$ above. To do this, apply backward induction on ℓ. Recall that $\ell\leq k$ and the base case $\ell=k$ leaves nothing to be proved. Now suppose $\ell\leq k-1$ and that we are done with $\ell+1$. Consider $f=\sum_{i=0}^{m-\ell}(\ell-1+i)e_i(s_1,\ldots,s_{m-\ell})t^{j-\ell-i}$. As $\ell<k$, we must have some part i_0 such that $R_{i_0}\cup C_{i_0}\neq V_{i_0}$ (by simple counting). If part i_0 is of type b, let's assume $|R_{i_0}|=|C_{i_0}|=s_{m-\ell}<r_{i_0}$. Since part i_0 is not full, we can adjust it into type u, resulting in case $\ell+1$ where we can apply induction hypothesis. Here, $g=\sum_{i=0}^{m-\ell-1}(\ell+i)e_i(s_1,\ldots,s_{m-\ell-1})t^{j-\ell-1-i}$ is already generated by induction hypothesis. Then

$$\begin{aligned}
g(t+s_{m-\ell})=&\sum_{i=0}^{m-\ell-1}(\ell+i)e_i(s_1,\ldots,s_{m-\ell-1})t^{j-\ell-i-1}(t+s_{m-\ell})\\
=&\sum_{i=0}^{m-\ell-1}(\ell+i)e_i(s_1,\ldots,s_{m-\ell-1})t^{j-\ell-i}+\sum_{i=1}^{m-\ell}(\ell+i-1)s_{m-\ell}e_{i-1}(s_1,\ldots,s_{m-\ell-1})t^{j-\ell-i}\\
=&\sum_{i=0}^{m-\ell}(\ell-1+i)e_i(s_1,\ldots,s_{m-\ell})t^{j-\ell-i}+\sum_{i=0}^{m-\ell-1}e_i(s_1,\ldots,s_{m-\ell-1})t^{j-\ell-i}\\
=&f+t^{j-m+1}(t+s_1)\cdots(t+s_{m-\ell-1}).
\end{aligned}$$

So $f=g(t+s_{m-\ell})-t^{j-m+1}(t+s_1)\cdots(t+s_{m-\ell-1})$, which is already generated as desired. If part i_0 is of type u, we add one vertex in both R_{i_0} and C_{i_0} and remove one vertex from the type b part corresponding to $s_{m-\ell}$. Here, $s_{m-\ell}$ is chosen among $s_1,\ldots,s_{m-\ell}$ which is nonzero. Notice that we cannot have all type b parts to be empty since otherwise, $j\leq r_1+\cdots+r_m-m$. The determinant corresponding to this adjustment is $f_1=\sum_{i=0}^{m-\ell}(\ell-1+i)e_i(s_1,\ldots,s_{m-\ell}-1)t^{j-\ell-i}$, which is already generated by the argument for i_0 being type b. Then,

$$f-f_1=\sum_{i=1}^{m-\ell}(\ell-1+i)e_{i-1}(s_1,\ldots,s_{m-\ell-1})t^{j-\ell-i}=g.$$

So $f=f_1+g$ is generated as well as desired. Thus, in the type $u^\ell b^{m-\ell}$ case, the new generators we need to add are $\sum_{i=0}^{m-k}(k-1+i)e_i(s_1,\ldots,s_{m-k})t^{j-k-i}$ but by counting, every type b part must be full so $\{s_1,\ldots,s_{m-k}\}\subset$

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$\{r_1, \ldots, r_m\}$ as multi-sets. We can also rewrite them as $\sum_{a=0}^{m-k}(k-1+a)e_a(r_{i_1}, \ldots, r_{i_{m-k}})t^{j-k-a}$, where $1 \le i_1 < \cdots < i_{m-k} \le m$.

This concludes the discussion for $j > n-m$. □

4. Further specializations. One special case of interest is the complete *balanced* multipartite graph where each part has the same size r. In this case, generators from Theorem 3.2 can be greatly simplified and we obtain the following corollary.

COROLLARY 4.1. *Let G be a complete multipartite graph with $m(\ge 2)$ parts of size $r \ge 2$. Then*

$$I_j^R(G,t) = \begin{cases} \langle 1 \rangle & j \le m-1, \\ \langle (m-1)t^{j-m}, t^{j-m+1} \rangle & m \le j \le mr-m, \\ \langle t^{j-m+1}(t+r)^{m+j-mr-1}, & \\ \ \sum_{a=0}^{m+j-mr}(mr-j-1+a)r^a\binom{m+j-mr}{a}t^{2j-mr-a} \rangle & mr-m < j < mr, \\ \langle \sum_{a=0}^{m}(a-1)r^a\binom{m}{a}t^{mr-a} \rangle & j = n. \end{cases}$$

Theorem 3.2 makes the computation of characteristic ideals of complete multipartite graphs a lot easier. However, when $n-m < j < n$, the generators for $I_j^R(G,t)$ are not minimal in any sense. By subtracting, we can reduce these generators to smaller degrees, but possibly larger and more complicated coefficients may appear. For example, consider the ideal $I_j^R(G,t)$ with $j = n-m+1$. In notations in Section 3, $k = m-1$ and by Theorem 3.2, we obtain generators $t^{j-m+1}, (m-2)t^{j-m+1} + (m-1)r_i t^{j-m}$ for each $i = 1, \ldots, m$, which can be reduced to $t^{j-m+1}, (m-1)r_i t^{j-m}$ for $i = 1, \ldots, m$. This gives

$$I_{n-m+1}^R(G,t) = \langle t^{n-2m+2}, (m-1)\gcd(r_1, \ldots, r_m)t^{n-2m+1} \rangle.$$

The answer is satisfactory here but as j becomes larger, it is harder to control the leading coefficients of our generators that come to front.

To avoid this problem, let's assume further that our ambient ring R contains a copy of $\mathbb{Q}$, the field of rational numbers. In other words, we assume that there is an injective ring morphism $\mathbb{Q} \hookrightarrow R$. This choice of specializations allows $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, the fields of particular interests. In this way, as all our generators listed in Theorem 3.2 have integer coefficients, we can perform Euclidean's algorithm and deduce that each $I_j^R(G,t)$ is principal. Essentially, this specialization is effectively the same as requiring R to be a field so that $R[t]$ is a PID (and requiring characteristic 0 for simplicity).

Notice that the following corollary can be obtained from the well-known spectrum of the complete multipartite graphs (see for example [5]). However, Corollary 4.1 also provides us a simple way to compute the characteristic ideals.

COROLLARY 4.2. *Let G be a complete multipartite graph with β_i parts of size $p_i \ge 2$, for $i = 1, \ldots, w$. Write $m = \sum_{i=1}^{w}\beta_i$ be the number of parts and $n = \sum_{i=1}^{w}\beta_i p_i$ be the number of vertices. Assume $m \ge 2$ and let R be a commutative ring that contains a copy of $\mathbb{Q}$. Then*

$$I_j^R(G,t) = \begin{cases} \langle 1 \rangle & j \le m-1, \\ \langle t^{j-m} \rangle & m \le j \le n-m, \\ \langle t^{j-m}\prod_{i=1}^{w}(t+p_i)^{(\beta_i+j-n-1)_+} \rangle & n-m < j < n, \\ \langle \sum_{a=0}^{m}(a-1)e_a(r_1, \ldots, r_m)t^{j-a} \rangle & j = n, \end{cases}$$

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where $(s)_+ = (s+|s|)/2$ *is the positive part of* s*,* e_a *is the elementary symmetric function of degree* a *and* $r_1, \dots, r_m$ *contain* β_i *copies of* p_i *for each* $i = 1, \dots, w$*.*

Proof. By observing results from Theorem 3.2, we see that the nontrivial cases are in the range of $n - m < j < n$. Since we know our ideal $I_j^R(G,t)$ is principal, and one of the generators factors into linear factors, it suffices to check the multiplicity of each linear factors.

Suppose that $I_j^R(G,t)$ is generated by the polynomial $g_j \in R[t]$ with leading coefficient 1. The multiplicity of t in g_j is $j-m$ by Theorem 3.2 so now we only need to figure out the multiplicity of $t+p_i$ in g_j for each $i = 1, \dots, w$. Denote such multiplicity by $\delta_i := v_{t+p_i}(g_j)$, i.e., $(t+p_i)^{\delta_i} \mid g_j$ and $(t+p_i)^{\delta_i+1} \nmid g_j$. We claim that $\delta_i = (\beta_i - k - 1)_+$, where $k = n-j$.

To prove this claim, we first show that $(t+p_i)^{(\beta_i-k-1)_+} \mid g_j$. Theorem 3.2 gives us two kinds of generators and we need to show that each one of them is divisible by $(t+p_i)^{(\beta_i-k-1)_+}$. The first kind of generators has the form $f = t^{j-m+1}\prod_{a=1}^{m-k-1}(t+r_{i_a})$, where $1 \le i_1 < \cdots < i_a \le m$. Since there are $m - \beta_i$ parts in G whose sizes are not p_i, among the $m-k-1$ chosen parts, at least $\big((m-k-1)-(m-\beta_i)\big)_+ = (\beta_i-k-1)_+$ parts have size p_i so we obtain $(t+p_i)^{(\beta_i-k-1)_+} \mid f$. For the second kind of generators with form $f = \sum_{a=0}^{m-k}(k-1+a)e_a(r_{i_1}, \dots, r_{i_{m-k}})t^{j-k-a}$, let's do some calculations.

Let $h = (t+r_{i_1})\cdots(t+r_{i_{m-k}})t^{j-m}$. Then

$$\begin{aligned}
jh - (th)' &= \sum_{a=0}^{m-k} je_a(r_{i_1}, \dots, r_{i_{m-k}})t^{j-k-a} - \frac{d}{dt}\sum_{a=0}^{m-k} e_a(r_{i_1}, \dots, r_{i_{m-k}})t^{j-k-a+1} \\
&= \sum_{a=0}^{m-k} je_a(r_{i_1}, \dots, r_{i_{m-k}})t^{j-k-a} - \sum_{a=0}^{m-k}(j-k-a+1)e_a(r_{i_1}, \dots, r_{i_{m-k}})t^{j-k-a} \\
&= \sum_{a=0}^{m-k}(k-1+a)e_a(r_{i_1}, \dots, r_{i_{m-k}})t^{j-k-a} = f.
\end{aligned}$$

The divisibility we desire is satisfied when $\beta_i \le k+1$ so let's assume $\beta_i - k - 1 \ge 1$. Similar as above, among the $m-k$ part sizes chosen in h, at least $(m-k)-(m-\beta_i) = \beta_i - k$ parts have size p_i. Thus, $(t+p_i)^{\beta_i-k} \mid h$, $(t+p_i)^{\beta_i-k} \mid th$, $(t+p_i)^{\beta_i-k-1} \mid (th)'$ so $(t+p_i)^{\beta_i-k-1} \mid f$ as desired.

The next step is to show that $(t+p_i)^{\delta_j+1} \nmid g_j$. To do this, it suffices to find one generator that is not divisible by $(t+p_i)^{\delta_j+1}$. We can select $f = t^{j-m+1}\prod_{a=1}^{m-k-1}(t+r_{i_a})$ by choosing $i_1, \dots, i_{m-k-1}$ to be all parts that are not of size p_i and the leftover $(\beta_i-k-1)_+ = \delta_j$ parts to be of size p_i. In this way, we obtain that $(t+p_i)^{\delta_j+1} \nmid f$. ∎

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