Diameter and Automorphisms of Gelfand-Tsetlin Polytopes

Yibo Gao[1], Benjamin Krakoff[2] and Lisa Yang[1]

[1]Massachusetts Institute of Technology, [2]Yale University
University of Minnesota Twin Cities REU

Abstract

We determine the diameter of the 1-skeleton and the combinatorial automorphism group of any Gelfand-Tsetlin polytope GT_λ associated to an integer partition λ.

Introduction

Gelfand-Tsetlin (GT) polytopes are compact convex polytopes defined by a set of linear inequalities depending on a partition λ as shown in Figure 1. The polytope GT_λ corresponds to the set of points $\vec{x} = (x_{i,j})_{1\le j\le i\le n} \in \mathbb{R}^{n(n+1)/2}$ where $(x_{i,j})_{1\le j\le i\le n}$ is a filling of this triangular array such that all rows and columns are weakly increasing.

$$\begin{array}{lllll}
\lambda_1 & & & & \\
\rotatebox{90}{$\le$} & & & & \\
x_{2,1} \le & \lambda_2 & & & \\
\rotatebox{90}{$\le$} & \rotatebox{90}{$\le$} & & & \\
x_{3,1} \le & x_{3,2} \le & \lambda_3 & & \\
\rotatebox{90}{$\le$} & \rotatebox{90}{$\le$} & \rotatebox{90}{$\le$} & & \\
x_{4,1} \le & x_{4,2} \le & x_{4,3} \le & \lambda_4 & \\
\vdots & \vdots & \vdots & \ddots & \\
x_{n,1} \le & x_{n,2} \le & \ldots \le & x_{n,n-1} \le & \lambda_n
\end{array}$$

Figure 1: Inequality constraints of GT polytopes.

Background

GT polytopes arise from the study of representations of $GL_n(\mathbb{C})$ and have connections to areas of representation theory and algebraic geometry. For any integer partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, let n be the length of λ and let GT_λ denote the associated GT polytope. The integral points in GT_λ are in bijection with semistandard Young tableaux of shape λ with tableaux entries bounded by n. Furthermore, the integral points of GT_λ parametrize a *Gelfand-Tsetlin basis* of the GL_n-module with highest weight λ, so the number of integral points equals the dimension of this module. GT polytopes can also be viewed as the marked order polytope of a poset as discussed in [1].

Theorem 1 (Diameter of 1-skeleton)

It suffices to consider $\lambda = (1^{a_1}, \ldots, m^{a_m})$ for $a_i \in \mathbb{Z}_{>0}$. For any GT_λ, the diameter of the 1-skeleton is $\mathrm{diam}(\mathrm{GT}_\lambda) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$.

Theorem 2 (Automorphism Group)

It suffices to consider $\lambda = (1^{a_1}, \ldots, m^{a_m})$ for $a_i \in \mathbb{Z}_{>0}$.
$\mathbf{m = 2}$: Suppose $\lambda = (1^{a_1}, 2^{a_2})$ and $a_1, a_2 \ge 2$.
If $a_1 = a_2 = 2$, then

$$\mathrm{Aut}(\mathrm{GT}_\lambda) \cong D_4 \times \mathbb{Z}_2.$$

Otherwise,

$$\mathrm{Aut}(\mathrm{GT}_\lambda) \cong D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\delta_{a_1,a_2}},$$

where D_4 is the dihedral group of order 8 and $\mathbb{Z}_2$ is the cyclic group of order 2.

$\mathbf{m \ge 3}$: Suppose $\lambda = (1^{a_1}, \ldots, m^{a_m})$ and $m \ge 3$. Let r_1 be the number of k such that $a_k, a_{k+1} \ge 2$. Let $r_2 = 1$ if λ is reverse-symmetric and let $r_2 = 0$ otherwise. Then

$$\mathrm{Aut}(\mathrm{GT}_\lambda) \cong (S_{a_2}^{\delta_{1,a_1}} \times S_{a_{m-1}}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{r_1+1}) \rtimes_\varphi \mathbb{Z}_2^{r_2}$$

where $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(S_{a_2}^{\delta_{1,a_1}} \times S_{a_{m-1}}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{r_1+1})$ sends the nonidentity element of $\mathbb{Z}_2$ to the map sending $(\sigma_1, \sigma_2, z_1, \ldots, z_{r_1}, z_{r_1+1}) \mapsto (\sigma_2, \sigma_1, z_{r_1}, \ldots, z_1, z_{r_1+1})$.

Example: $\mathrm{GT}_\lambda, \lambda = (1, 2, 3)$

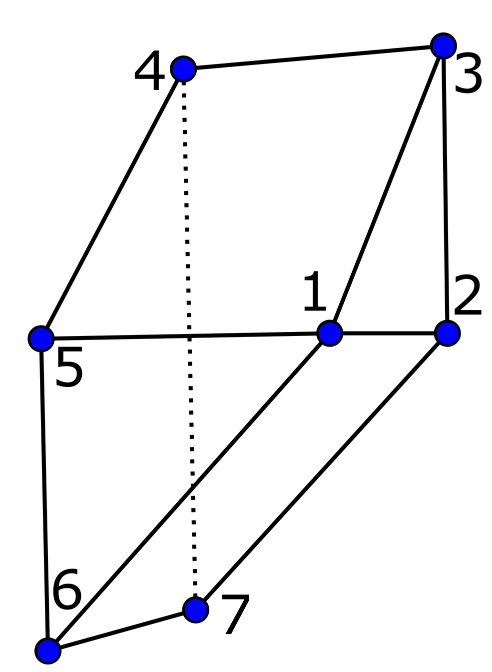


Figure 2: Polytope $\mathrm{GT}_{(1,2,3)}$

One can see from this figure that the diameter of the 1-skeleton is 2 and there are 4 automorphisms.

Ladder Diagrams and Face Poset

Definition (Γ_λ and Ladder Diagrams)
Given a partition $\lambda = (1^{a_1}, \ldots, m^{a_m})$, define the *initial vertex* to be $(0, 0)$ and the *terminal vertices* to be $t_j = (s_j, n - s_j)$ for $0 \le j \le m$ where $s_j := \Sigma_{i=1}^{j} a_i$. A ladder diagram is a graph that is a union of northeast lattice paths from $(0, 0)$ to t_j such that there is at least 1 path to each t_j.

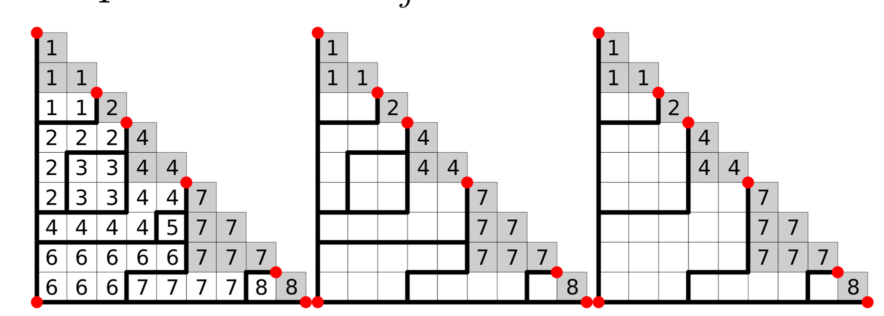


Figure 3: Ladder Diagrams for $\lambda = (1^2, 2, 4^2, 7^3, 8)$. A vertex is a set of $m - 1$ noncrossing paths.

Theorem [2, Theorem 1.9] Let $\mathcal{F}(\mathrm{GT}_\lambda)$ be the face poset of GT_λ, and let $\mathcal{F}(\Gamma_\lambda)$ be the poset of ladder diagrams ordered by inclusion. Then $\mathcal{F}(\mathrm{GT}_\lambda) \cong \mathcal{F}(\Gamma_\lambda)$.

Proof Idea for Theorem 1

We give an algorithm to construct a path between any two vertices that has length $2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$. We construct two vertices z_h and z_v and prove that any path between them has length at least $2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$ by associating a sequence of sets to any such path (these sets are subject to certain combinatorial constraints).

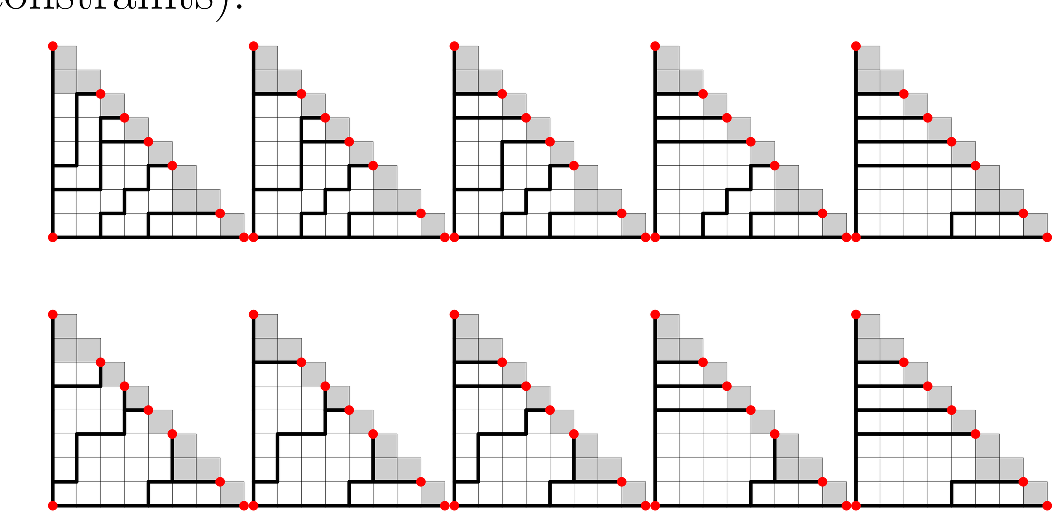


Figure 4: Algorithm showing the upperbound of the diameter.

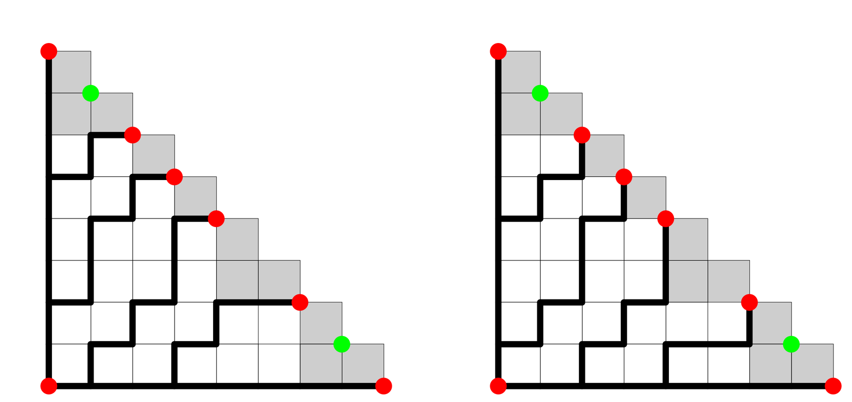


Figure 5: Vertices z_h and z_v of GT_λ.

Proof Idea for Theorem 2

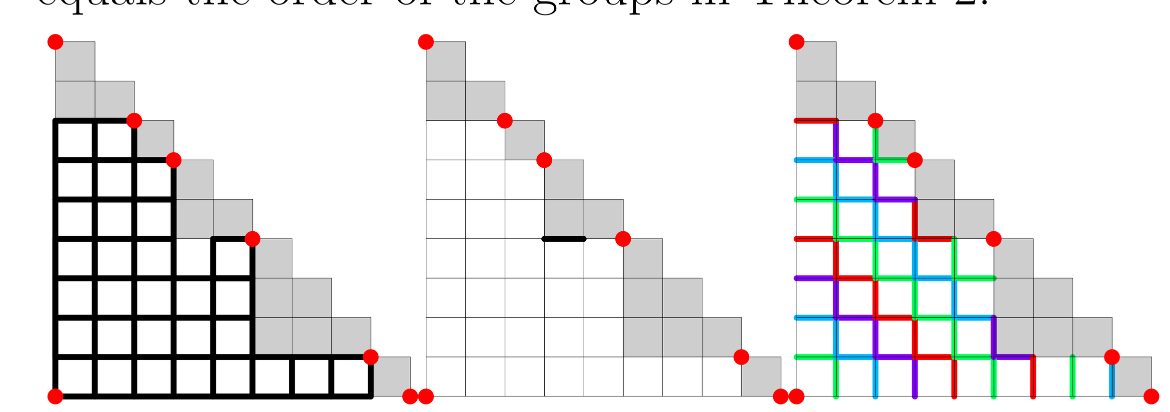
First we exhibit a set of automorphisms and show that they generate the groups in Theorem 2.

Noting that facets in a GT-polytope are in bijection with single edges in Γ_λ, we represent facets by their associated edge. Two facets are called *dependent* if their intersection is $d - 3$ dimensional. We partition the edges of Γ_λ into maximal chains of dependent facets.

For any $\phi \in \mathrm{Aut}(\mathrm{GT}_\lambda)$ and chains C_1, C_2:

- If $\phi(C_1) = C_2$, then C_1 is mapped to C_2 or its flip.
- ϕ preserves the lengths of chains.
- ϕ preserves adjacency of chains.

Starting with the facets in chains of length 1 and length 2, we bound the size of the orbits of these facets and iteratively apply the Orbit-Stabilizer Theorem. We show that the order of the automorphism group equals the order of the groups in Theorem 2.



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