Hall–Littlewood polynomials

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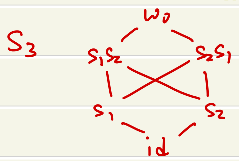
1 Definition

Poincaré polynomial of a Weyl group W (S_n)

$$W(t) = \sum_{w\in W} t^{\ell(w)} \qquad W(1)=|W| \qquad W(0)=1$$

S_3



$$W(t) = t^0+2t^1+2t^2+t^3$$

$$\boxed{S_n:\ W(t)=[n]!}$$

Symmetrizer $\curvearrowright$ $Q_t[\Lambda]$ ← weight lattice of W

$$\mathrm{Symm}(f) = \sum_{w\in W} w\left(f\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right)$$

positive roots

$e^{\alpha}\in Q_t[\Lambda]$

$$\mathrm{Symm}(f)|_{t=1} = \sum_{w\in W} wf$$

$$\mathrm{Symm}(f)|_{t=0} = \sum_{w\in W} w\left(f\prod_{\alpha>0}\frac{1}{1-e^{-\alpha}}\right)$$

(Weyl symmetrizer)

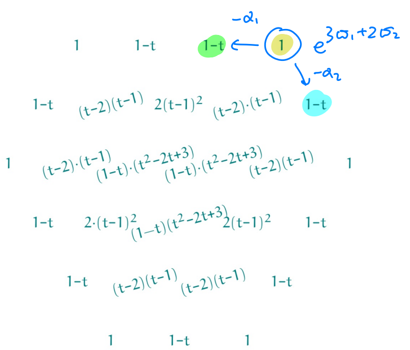
HL polynomial of a dominant weight λ

$$P_\lambda = \frac{1}{W_\lambda(t)}\mathrm{Symm}(e^{\lambda})$$

stabilizer of λ

$$P_\lambda|_{t=1} = \sum_{\mu\in W\lambda} e^{\mu} = \frac{1}{|W_\lambda|}\sum_{w\in W} e^{w\lambda}$$

$$P_\lambda|_{t=0} = \chi_\lambda = \text{Schur polynomial}$$



$$P_{3\varpi_1+2\varpi_2} = e^{3\varpi_1+2\varpi_2}\left(1+(1-t)e^{-\alpha_1}+(1-t)e^{-\alpha_2}+\cdots\right)$$

Type A

$$[n] = 1+t+\cdots+t^{n-1} = \frac{1-t^n}{1-t} \qquad \underbrace{[n]!}_{S_n(t)} = [1][2]\cdots[n]$$

For a partition $\lambda = (\lambda_1\geq\cdots\geq\lambda_n)$

$$W_\lambda = S_\alpha = S_{\alpha_1}\times\cdots\times S_{\alpha_m} \qquad (\alpha_1+\cdots+\alpha_m=n)$$

e.g. $\lambda = (5,5,3,2,2,2,0)$ $\quad \alpha = (2,1,3,1)$

$$S_\alpha = S_2\times S_1\times S_3\times S_1 \subseteq S_7$$

$$W_\lambda(t) = S_2(t)\times S_1(t)\times S_3(t)\times S_1(t)$$

$$= [2][1][3][1] = (1+t)(1+t+t^2)$$

$$\mathrm{Symm}(f) = \sum_{w\in S_n} w\left(f\prod_{i<j}\frac{x_i-tx_j}{x_i-x_j}\right)$$

$f=f(x_1,\dots,x_n)$ $\quad \left(e^{-\alpha} := x_j/x_i \text{ for } \alpha=(\cdots 0,1,0,\cdots,0,-1,0,\cdots),\ i<j\right)$

$$P_\lambda = \frac{1}{[\alpha]!}\sum_{w\in S_n} w\left(x^{\lambda}\prod_{i<j}\frac{x_i-tx_j}{x_i-x_j}\right)$$

$[\alpha_1]!\cdots[\alpha_m]!$ $\qquad x_1^{\lambda_1}\cdots x_n^{\lambda_n}$

$$t=1 \quad P_\lambda|_{t=1} = \frac{1}{\alpha!}\sum_{w\in S_n} x^{w\lambda} = \sum_{\mu\in S_n\lambda} x^{\mu} = \text{monomial symmetric polynomial}$$

$$t=0 \quad P_\lambda|_{t=0} = 1\sum_{w\in S_n} w\left(x^{\lambda}\prod_{i<j}\frac{x_i}{x_i-x_j}\right)$$

$$= \sum_{w\in S_n} w\left(x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\cdots x_n^{\lambda_n}\prod_{i<j}\frac{1}{x_i-x_j}\right)$$

$=$ Schur polynomial.

Example $\quad (n=2)\quad S_2=\{id,s\}\quad (x_1,x_2)=(x,y)$

$\lambda=(a\geq b)$ $\quad a=b \quad \overset{s}{(a,b)} = (a,b) \quad W_\lambda = S_2$

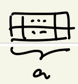
$\qquad\qquad a>b \quad (a,b)\neq(a,b) \quad W_\lambda = S_1\times S_1$

① $a=b$

$$P_\lambda = \frac{1}{1+t}\left(x^ay^b\frac{x-ty}{x-y}+y^ax^b\frac{y-tx}{y-x}\right)$$

$$= \frac{x^ay^a}{1+t}\,\frac{(x-ty)-(y-tx)}{x-y} = \frac{x^ay^a}{1+t}(1+t) = x^ay^a.$$

$$= s_{(a,a)}(x,y)$$



② $a>b$

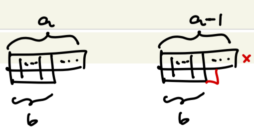
$$P_\lambda = \frac{1}{1}\left(x^ay^b\frac{x-ty}{x-y}+y^ax^b\frac{y-tx}{y-x}\right)$$

$$= x^by^b\left(x^{a-b}\frac{x-ty}{x-y}+y^{a-b}\frac{y-tx}{y-x}\right)$$

$$= x^by^b\left(\frac{x^{a-b+1}-y^{a-b+1}}{x-y} - txy\,\frac{x^{a-b-1}-y^{a-b-1}}{x-y}\right)$$

$$= x^by^b\left(h_{a-b}(x,y) - \begin{cases} txy\,h_{a-b-2}(x,y) & \text{if } a>b+1\\ 0 & \text{if } a=b+1\end{cases}\right)$$

$$= s_{(a,b)}(x,y) - \begin{cases} t\,s_{(a-1,b+1)}(x,y), & \text{if } a>b+1,\\ 0, & \text{if } a=b+1.\end{cases}$$



(Macdonald identity)

(for finite Weyl group)

No known analogy for Coxeter group

Lemma We have $P_0 = 1$, i.e.

$$\mathrm{Symm}(1) = \sum_{w\in W} w\left(\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right) = W(t).$$

We will give three different proofs of this formula.

$\frac{1}{2}\sum_{\alpha>0}\alpha$ Type A $(n-1, n-2, \cdots, 0)$

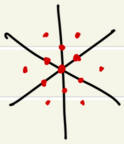
prove via Weyl denominator

For a weight $\lambda \in \mathrm{Conv}(w\rho : w\in W)$

$$(*)\quad \sum_{w\in W} w\left(\prod_{\alpha>0}\frac{e^{\lambda-\rho}}{1-e^{-\alpha}}\right) = \begin{cases} (-1)^{\ell(w)}, & \lambda = w\rho \\ 0, & \text{otherwise.}\end{cases}$$

$\Uparrow$

(Weyl character formula)



$\prod_{\alpha>0}(1-te^{-\alpha})$ supported over $\rho + \mathrm{Conv}(w\rho : w\in W)$.

$$= \sum_{w\in W}(-t)^{\ell(w)} e^{w\rho-\rho} + (\text{other terms})$$

By $(*)$

$$\sum_{w\in W} w\left(\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right) = \sum_{w\in W}(-t)^{\ell(w)}(-1)^{\ell(w)}$$

$$= \sum_{w\in W} t^{\ell(w)} = W(t).$$

prove via rigidity

Trick "rigidity"

$f(x)\in\mathbb{C}[x^{\pm1}]$ $\left.\begin{array}{l}\lim_{x\to0} f(x)\ \exists\\ \lim_{x\to\infty} f(x)\ \exists\end{array}\right\} \Rightarrow f$ is a constant

$\underset{\neq 0}{c_a}x^a + \cdots + \underset{\neq 0}{c_b}x^b$

① Symm(1) is a Laurent polynomial in e^{α}

$$\lim_{x\to0}\frac{1-tx}{1-x} = 1 \qquad \lim_{x\to\infty}\frac{1-tx}{1-x} = t$$

Each term $\lim \frac{1-te^{w\alpha}}{1-e^{w\alpha}}\ \exists$.

By "rigidity" $\Rightarrow$ Symm(1) only depends on t.

Taking $e^{\alpha_i}\to\infty$

$$\lim\prod_{\alpha>0}\frac{1-te^{-w\alpha}}{1-e^{-w\alpha}} = t^{\ell(w)}$$

Thus

$$\sum_{w\in W}\prod_{\alpha>0}\frac{1-te^{-w\alpha}}{1-e^{-w\alpha}} = \sum_{w\in W} t^{\ell(w)}$$

KL basis characterization $\chi_\lambda \in P_\lambda + \sum_{\mu<_{dom}\lambda} t\mathbb{Z}[t]P_\mu$

Proposition For any dominant weight $\lambda\in\Lambda$, we have

$$P_\lambda = \chi_\lambda + \sum_{\mu<_{\mathrm{dom}}\lambda} t\mathbb{Z}[t]\cdot\chi_\mu.$$

Kostka-Foulkes polynomial

$$P_\lambda = \frac{1}{W_\lambda(t)}\sum_{w\in W} w\left(e^\lambda\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right)$$

e.g. λ strictly dominant s.t. $W_\lambda = \{id\}$

$$P_\lambda = \sum_{w\in W} w\left(e^\lambda\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right)$$

$$= \sum_{w\in W} w\left(e^\lambda\prod_{\alpha>0}(1-te^{-\alpha})\prod\frac{1}{1-e^{-\alpha}}\right)$$

(each e^μ contributes $\pm\chi_{\mu^+}$, μ^+ dom, $\mu^+\in W\mu$)

When λ strictly dom $\mu\in\mathrm{Conv}(w\lambda : w\in W)$.

overall contribution $\mu\le_{dom}\lambda$.

In general, W_λ-invariant

$$\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}} = \prod_{\langle\alpha,\lambda\rangle>0}\ \prod_{\langle\alpha,\lambda\rangle=0}$$

W_λ positive roots $\quad W_\lambda(t)$

$$\sum_{w\in W} w(\cdots) = \sum_{u\in W^\lambda} u\left(\sum_{v\in W_\lambda} v(\cdots)\right)$$

We proved $P_\lambda = \chi_\lambda + \sum_{\mu<_{dom}\lambda}\mathbb{Z}[t]\chi_\mu$

2 Demazure–Lusztig operators

S_n $\quad s_i = i\leftrightarrow i+1\quad (1\le i\le n-1)$

For a simple reflection $s_i\in W$, $i\in I$

$$T_i = ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - id)$$

Demazure-Lusztig operator

$$f\mapsto t(s_if) + \frac{t-1}{e^{\alpha_i}-1}((s_if)-f)$$

$T_i^2 = (t-1)T_i + t\cdot id$ (quadratic relations)

$(i\neq j)$ $\underbrace{T_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_i\cdots}_{m_{ij}}$ (braid relations)

$\left(\underbrace{s_is_j\cdots}_{m_{ij}} = \underbrace{s_js_i\cdots}_{m_{ij}}\right)$

reduced word

For $w = s_{i_1}\cdots s_{i_\ell}$ $\quad\ell(w) = \ell$.

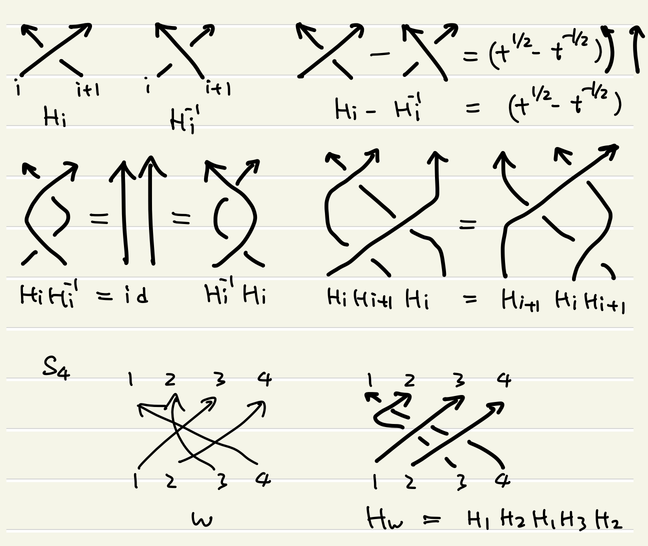
$T_w = T_{i_1}\cdots T_{i_\ell}$

(NOT depends on the choice of $s_{i_1},\cdots,s_{i_\ell}$)

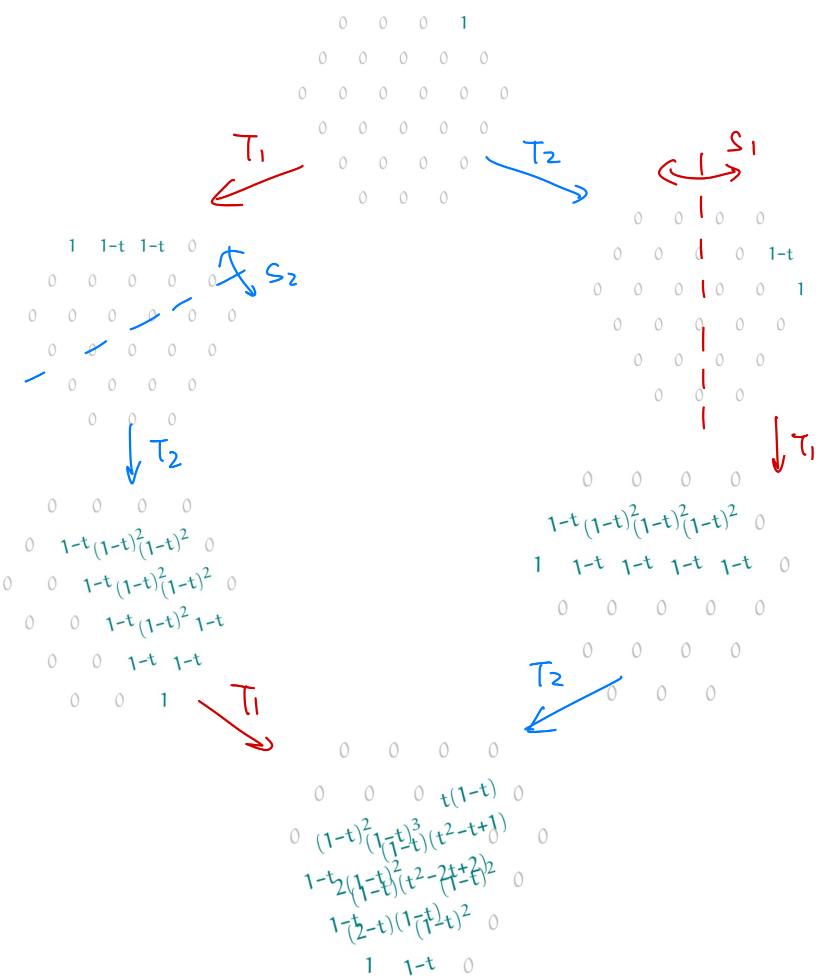
Rmk. The $\mathbb{Q}(t)$ algebra generated by T_i is called Hecke algebra.

diagram in type A

$W = S_n, \quad H_i = t^{-1/2} T_i.$



Example



Lemma We have

$$\mathrm{Symm}(f) = \sum_{w\in W} w\left(f \prod_{\alpha>0} \frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right) = \sum_{w\in W} T_w(f).$$

Sketch

RHS, as an operator on f

$$= \sum_{w\in W} w \circ \text{multiplication by } a_w \in \mathbb{Q}_t[\Lambda]_{loc}.$$

So it reduces to show

$$a_w = \prod_{\alpha>0} \frac{1-te^{-\alpha}}{1-e^{-\alpha}} \quad \forall w \in W.$$

① $w = w_0$,

e.g. S_3 $\quad w_0 = s_1 s_2 s_1$

id $\quad T_1 \quad T_2 \quad T_1T_2 \quad T_2T_1 \quad T_1T_2T_1$

$$T_1T_2T_1 = (s_1(\ )+(\ ))\circ(s_2(\ )+(\ ))\circ(s_1(\ )+(\ ))$$

$$\Rightarrow a_{w_0} = \prod_{\alpha>0} \frac{1-te^{-\alpha}}{1-e^{-\alpha}}$$

② $(T_i - t)\left(\sum_{w\in W} T_w(f)\right) = 0$

$$T_i T_w = \begin{cases} T_{s_i w}, & s_i w > w, \\ T_{s_i w} + (t-1) T_w, & s_i w < w. \end{cases}$$

$(T_i - t) f = 0 \Rightarrow s_i f = f$ (expansion)

This means

$$\sum_{w\in W} T_w = \left(\sum_{w\in W} w\right) \circ \text{multiplication by some } a \in \mathbb{Q}_t[\Lambda]_{loc}$$

$$\Rightarrow a_w = a_{w_0} = \prod_{\alpha>0} \frac{1-te^{-\alpha}}{1-e^{-\alpha}}$$

$$\mathrm{Symm}(1) = \sum_{w\in W} T_w(1) = \sum_{w\in W} t^{\ell(w)}.$$

$$T_i(1) = t \Rightarrow T_w(1) = t^{\ell(w)}$$

Learning seminar Macdonald

link $\qquad$ notes