Applications of Hall–Littlewood polynomials (I)

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-1 Introduction

"Macdonald theory"

combinatorial representation theory

Macdonald. Symmetric functions

rep theory of S_n, U_n, GL_n, sl_n $\longleftrightarrow$ Schur functions

semi-simple (character theory)

affine Hecke algebras $\longleftrightarrow$ Hall-Littlewood function

Kazhdan-Lusztig theory

Haiman theory:

Double affine Hecke algebra $\longleftrightarrow$ Hilbert schemes (Macdonald sym)

algebra variety

0 Recall

partition

$$P_\lambda = \frac{1}{W_\lambda(t)} \sum_{w\in \cancel{W} S_n} w\left(x^\lambda \prod_{1\le i<j\le n}\frac{x_i - t x_j}{x_i - x_j}\right)$$

Kostka--Foulkes polynomials

For λ a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$,

$$W_\lambda = S_\alpha := S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_m}$$

e.g. $\lambda = (5,5,3,2,2,2,0)$

$\alpha = (2,1,3,1)$

$W_\lambda = S_2 \times S_1 \times S_3 \times S_1$

$W_\lambda = (1+t)\cdot 1 \cdot (1+t+t^2)\cdot 1$

Last time:

$$P_\lambda(x_1,\cdots,x_n;t) \in S_\lambda(x_1,\cdots,x_n) + \sum_{\substack{\mu<\lambda \\ \text{dom}}} t\mathbb{Z}[t]\, S_\mu(x_1,\cdots,x_n)$$

Kostka-Foulkes polynomials

does not depend on n.

$$S_\lambda(x_1,\cdots,x_n) = \sum_\mu K_{\lambda\mu}(t)\, P_\mu(x_1,\cdots,x_n,t).$$

Fact: $K_{\lambda\mu}(t) \in \mathbb{N}[t]$ (KL theory)

$t=0$, $P_\lambda = S_\lambda$

$t=1$, $P_\lambda = m_\lambda$

3.12. Example. One can compute Hall–Littlewood function in SageMath

```
Sym = SymmetricFunctions(FractionField(QQ["t"]))
HLP = Sym.hall_littlewood().P();
HLP([3,1,1]).expand(3)
```

The expansion to Schur functions (similar to other basis)

```
Sym = SymmetricFunctions(FractionField(QQ["t"]))
HLP = Sym.hall_littlewood().P();
s = Sym.Schur();
s(HLP([3,1,1]))
```

See the documentation.

modified version,

$$\tilde{P}_\lambda = t^{\langle \rho,\lambda\rangle} P_\lambda\big|_{t\leftrightarrow t^{-1}} \qquad \text{(all types)}$$

$$\tilde{P}_\lambda = t^{n(\lambda)} P_\lambda\big|_{t\leftrightarrow t^{-1}} \qquad \text{(Type A)}$$

(still a polynomial in t).

$$n(\lambda) = \sum_{\square\in\lambda}(\mathrm{row}(\square)-1) \qquad \text{e.g. } n\left(\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\ \hline 1&1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}\right) = 4$$

1 The question

finite p-groups

Fix prime # p. Assume we have 3 finite abelian p-groups A, B, C.

$$c^{C}_{A,B} = \#\left\{ M \le C : \begin{matrix} M \cong B \\ C/M \cong A \end{matrix} \right\}$$

subgroups $\cong B$ with quotient $\cong A$

Recall a finite abelian p-group

$$A_\lambda = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\lambda_n}\mathbb{Z}$$

for a partition λ. We say

M of type $\lambda \iff M \cong A_\lambda$.

$$\{\text{finite abelian p-groups}\}/\cong \; = \{\text{partitions}\}$$

let $A = A_\lambda \quad B = A_\mu, \quad C = A_\nu$

$$C^{\nu}_{\lambda\mu}(p) = c^{C}_{AB}. \qquad C^{\nu'}_{\lambda'\mu'}(p)$$

$\#$

Question: How to compute $C^{\nu}_{\lambda\mu}(p)$?

$= 0$ unless $|\lambda| + |\mu| = |\nu|$.

Example

$\lambda = (2,1)$

$\nu = (2,1) = \square\!\square\!\!\square \qquad A_\nu = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}.$

Let us classify $M \le A_{(2,1)}$ s.t.

$\mu = (1) = \square \qquad M \cong A_{(1)} \cong \mathbb{Z}/p\mathbb{Z}.$

Exercise:

$(p\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) \setminus \{0\}.$

$$\{\text{order } p \text{ elements}\} = \{(ap, b) : a \ne 0 \text{ or } b \ne 0\}.$$

$M =$ the subgroup generated by (ap, b)

Case A. If $b \ne 0$.

$A_{(2,1)}/M$ is generated by $(1,0)$.

$(ap, b) \in M \Rightarrow (a'p, 1) \in M$ for some a'

$\Rightarrow (0,1)$ can be generated by $(1,0)$ mod M.

$(2)\qquad A_{(2,1)}/M \cong A_{(2)} \cong \mathbb{Z}/p^2\mathbb{Z}.$

Case B If $b = 0$, $M = p\mathbb{Z}/p^2\mathbb{Z} \oplus \{0\}$.

$(1,1)\qquad A_{(2,1)}/M \cong A_{(1,1)} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$

Summary:

$$C^{(2,1)}_{(2),(1)}(p) = p, \qquad C^{(2,1)}_{(1,1),(1)}(p) = 1.$$

Examples

$\lambda = (3)$ and $\lambda = (1,1,1)$

① $(3)\qquad A_{(3)} = \mathbb{Z}/p^3\mathbb{Z}.$

Classify $M \le A_{(3)}, \quad M \cong A_{(1)} = \mathbb{Z}/p\mathbb{Z}$

$$C^{(3)}_{(2),(1)}(p) = 1.$$

$$C^{(3)}_{(1,1),(1)}(p) = 0$$

② $(1,1,1)\qquad A_{(1,1,1)} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$

Classify $M \le A_{(1,1,1)} \quad M \cong A_{(1)} = \mathbb{Z}/p\mathbb{Z}$

$A_{(1,1,1)}$ is a $\mathbb{F}_p$-vector vector space.

$M \cong \mathbb{Z}/p\mathbb{Z} \iff M$ is a one-dim subspace.

Choice of basis of $M \longrightarrow p^3 - 1$

different choice for one $M \rightarrow p - 1$

$$\frac{p^3-1}{p-1} = 1 + p + p^2.$$

$$C^{(1,1,1)}_{(1,1),(1)}(p) = \#(\mathbb{P}^2(\mathbb{F}_p)) = 1 + p + p^2.$$

$$C^{(1,1,1)}_{(2),(1)}(p) = 0.$$

2 Hall algebras

Consider

$$H = \left\{ \{\text{finite abelian p-groups}\} \xrightarrow{f} \mathbb{Q} : \begin{matrix} A \cong B \\ f(A) = f(B) \end{matrix} \right\}.$$

We define Hall product

$$(f * g)(A) = \sum_{M \le A} f(A/M)\, g(M)$$

Exercise: $(H, *)$ is a ring.

$$((f*g)*h)(A) \overset{?}{=} (f*(g*h))(A)$$

$$\| \qquad\qquad\qquad \|$$

$$\sum_{M \le A} (f*g)(A/M)\, h(M) \qquad \sum_{N \le A} f(A/N)\, (g*h)(N)$$

$$\| \qquad\qquad\qquad \|$$

$$\sum_{\substack{M \le A \\ K \le A/M}} f\!\left(\tfrac{A/M}{K}\right) g(K)\, h(M) = \sum_{M \le N \le A} f(A/N)\, g(N/M)\, h(M)$$

N/M

Let $\mathbb{P}_\lambda$ = characteristic function of A_λ up to $\cong$.

By definition:

$$\mathbb{P}_\lambda * \mathbb{P}_\mu = \sum_\nu C^{\nu}_{\lambda\mu}(p)\, \mathbb{P}_\nu.$$

Formally, we are using

$$\mathcal{H} = \bigoplus_\lambda \mathbb{Q} \cdot \mathbb{P}_\lambda \subseteq H$$

the subalgebra of finite support.

Theorem The linear map

$$\mathcal{H} \longrightarrow \Lambda, \qquad \mathbb{P}_\lambda \longmapsto \tilde{P}_\lambda|_{t\mapsto q}$$

is a ring isomorphism.

$t^{n(\lambda)} P_\lambda|_{t\leftrightarrow t^{-1}}$

$= q^{-n(\lambda)} P_\lambda(t=q^{-1})$

Example

① From 3 examples above

$$\mathbb{P}_{(1,1)}\mathbb{P}_{(1)} = (1+p+p^2)\,\mathbb{P}_{(1,1,1)} + \mathbb{P}_{(2,1)} + 0$$

$$\mathbb{P}_{(2)}\mathbb{P}_{(1)} = 0 + p\,\mathbb{P}_{(2,1)} + \mathbb{P}_{(3)}$$

② Compare:

$$P_{(1,1)}P_{(1)} = (t^2+t+1)P_{(1,1,1)} + P_{(2,1)}$$

$$P_{(2)}P_{(1)} = P_{(2,1)} + P_{(3)}$$

λ	(1)	$(1,1)$	(2)	$(1,1,1)$	$(2,1)$	(3)
$n(\lambda)$	0	1	0	3	1	0

$$\tilde{P}_{(1,1)}\tilde{P}_{(1)} = t^2(t^{-2}+t^{-1}+1)\tilde{P}_{(1,1,1)} + 1\,\tilde{P}_{(2,1)}$$

$$\tilde{P}_{(2)}\tilde{P}_{(1)} = t\,\tilde{P}_{(2,1)} + \tilde{P}_{(3)}$$

$$P_\lambda = s_\lambda + (\quad) + t^{n(\lambda)} h_{|\lambda|}.$$

Remark

Why it is commutative?

$\mathcal{H}$ is commutative $\quad c^{\nu}_{\lambda\mu}(p) = c^{\nu}_{\mu\lambda}(p)$.

Exercise: prove it directly. (Hint: $\mathrm{Hom}(-,\mathbb{C}^\times)$)

(Gelfand pair)

Example

A proof can be found in [Mac, chapII, chapIII]

The proof on the right is also in Macdonald's book. Chap V.

3 Why?

Affine Grassmannian; convolutions

Affine Hecke algebras

Satake isomorphism

$G = GL_n$

$\mathcal{K}$ = local field $\quad \mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

$\mathcal{O}$ = ring of integers $\quad \mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$

Spherical Hecke algebra $\quad G(K) = GL(\mathbb{Q}_p)$ or $GL(\mathbb{F}_p((t)))$.

$$C_c(G_\mathcal{O}\backslash G_\mathcal{K}/G_\mathcal{O}) = \left\{ \begin{array}{c} G_\mathcal{K} \xrightarrow{f} \mathbb{Q} \text{ of} \\ \text{compact support} \end{array} \middle| \begin{array}{c} g_1, g_2 \in G_\mathcal{O} \\ f(g_1 x g_2) = f(x) \end{array} \right\}$$

under convolution

$$(f*g)(x) = \int_{G_\mathcal{K}} f(xy^{-1})g(y)\,dy \qquad \mathrm{vol}(G_\mathcal{O}) = 1$$

$$G_\mathcal{K} \cong \bigsqcup_{\lambda \text{ dom}} G_\mathcal{O}\,\varpi^\lambda G_\mathcal{O}$$

GL_n: $\varpi^\lambda = \begin{bmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_n}\end{bmatrix}$ or $\begin{bmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n}\end{bmatrix}$

Exercise: $A \in GL_n(\mathbb{F}_p((t)))$ $\exists\, M, N \in GL_n(\mathbb{F}_p[[t]])$ s.t. $MAN = \begin{bmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n}\end{bmatrix}$

$$\mathcal{C}_c(G_\mathcal{O}\backslash G_\mathcal{K}/G_\mathcal{O}) = \bigoplus_{\lambda \text{ dom}} \mathbb{1}_\lambda.$$

For $G = GL_n$, $\quad L \subseteq K^{\oplus n}$ $\mathcal{O}$-submodule free of rank n.

$$G_\mathcal{K}/G_\mathcal{O} = \{\mathcal{O}\text{-lattices } L \text{ in } \mathcal{K}^{\oplus n}\}$$

$$\varpi^\lambda G_\mathcal{O} \longleftrightarrow \underbrace{p^{\lambda_1}\mathcal{O} \oplus \cdots \oplus p^{\lambda_n}\mathcal{O}}_{L_\lambda} \subseteq \mathcal{K}^{\oplus n}.$$

We have $L_0 = \mathcal{O} \oplus \cdots \oplus \mathcal{O} \subseteq \mathcal{K}^{\oplus n}$

Then λ a partition $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$

$G_\mathcal{O}$-orbit of $\varpi^\lambda G_\mathcal{O}$ $\quad \mathcal{O}/p^{\lambda_1}\mathcal{O} \oplus \cdots \oplus \mathcal{O}/p^{\lambda_n}$.

$= \{\text{lattice } L \supseteq L_0 : L/L_0 \cong A_\lambda\}$

$$\mathbb{1}_\lambda * \mathbb{1}_\mu = \sum_\nu \# \left\{ L : \begin{array}{l} L_0 \subseteq L \subseteq L_\nu \\ L/L_0 \cong A_\lambda \\ L_\nu/L \cong A_\mu \end{array} \right\} \mathbb{1}_\nu$$

|| not hard

$c^{\nu}_{\lambda\mu}$

1-para subgroup of a max torus $T \subseteq G$

cocharacter lattic

Classical Satake isomorphism

$$C_c(G_\mathcal{O}\backslash G_\mathcal{K}/G_\mathcal{O}) \cong \mathbb{Q}[\Lambda]^W$$

$$\mathbb{1}_\lambda \longmapsto \tilde{P}_\lambda$$

Due to Macdonald

Macdonald's Spherical function

Learning seminar Macdonald

link

notes