Applications of Hall–Littlewood polynomials (II)

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0 Normalized Hall–Littlewood

H_λ vs P_λ
another basis, Q_λ

$P_\lambda \qquad \tilde{P}_\lambda = t^{n(\lambda)} P_\lambda |_{t \leftrightarrow t^{-1}}$

Define modified HL

$H_\lambda =$ dual basis of P_λ wrt Hall pairing

$\tilde{H}_\lambda = t^{n(\lambda)} H_\lambda |_{t \leftrightarrow t^{-1}}$

Rmk. Q_λ dual to P_λ wrt to t-deformation of Hall pairing

$H_\lambda = Q_\lambda [Z \frac{1}{1-t}]$

1 Springer theory

Representation theory of S_n

bijection with partitions
Frobenius character
example

$S_n = \{ \{1, \dots, n\} \circlearrowleft \text{bijections} \}$ Symmetric group

Eg ($n=3$). S_3 a group of order $3! = 6$.

tri = trivial representation

$S_3 \to \{id\} \to GL(1) = \mathbb{C}^\times$

alt = sgn $\quad w \mapsto (-1)^{\ell(w)}$

$S_3 \to \{\pm 1\} \to GL(1) = \mathbb{C}^\times$

std = 2-dimensional rep $\mathbb{C}^3/\text{tri}$.

$S_3 \subseteq GL(3)$ defines $\mathbb{C}^3 \supseteq \text{tri} = \{(x,x,x) : x \in \mathbb{C}\}$

In general, $\{S_n\text{-irreps}\} \leftrightarrow \{\lambda \vdash n\}$

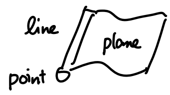
$\text{Specht}_\lambda \longmapsto \lambda$

Frob : $[S_n\text{-reps}] \longmapsto \Lambda$

$\text{Specht}_\lambda \longmapsto s_\lambda$

Grothendieck group

Flag varieties



Flag variety itself
n=2: projective space
n=3: incidences variety
G/B

$n \geq 1$. $Fl_n = \{ 0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n : \dim V_i = i \}$ (V_0 … V_n: subspaces)

e.g. $n=2$, $Fl_2 = \{ 0 \subset V_1 \subset \mathbb{C}^2 : \dim V_1 = 1 \}$

$= (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times = \mathbb{P}^1$.

$n=3$, $Fl_3 = \{ 0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 : \dim V_1 = 1, \dim V_2 = 2 \}$

"incidence variety"

$\{ 0 \subset V_1 \subset \mathbb{C}^3 : \dim V_1 = 1 \} = \mathbb{P}^2$ incidence variety

pts $\Leftrightarrow$ 1-dim subspaces $\mathbb{C}^3$

lines $\Leftrightarrow$ 2-dim subspaces

We can identify $Fl_n = G/B$

$G = GL_n(\mathbb{C})$

$B = \left\{ \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{bmatrix} \right\} \subseteq GL_n(\mathbb{C})$

$G \curvearrowright Fl_n$ by changing basis transitively (ex.)

standard flag $V_\bullet^0 = (0 \subset V_1^0 \subset \cdots \subset V_{n-1}^0 \subset \mathbb{C}^n)$

$V_i^0 = \text{span}(e_1, \dots, e_i) \qquad e_i = (\cdots 0, \overset{i}{1}, 0, \cdots) \in \mathbb{C}^n$

Stabilizer of $V_\bullet^0 = B$ (ex)

Nilpotent matrices

Jordan blocks
Springer fiber
S_n-actions

$A \in M_n(\mathbb{C})$ nilpotent $A^m = 0$ for $m \gg 0$.

A is $(GL_n(\mathbb{C}))$ conjugate to $\begin{bmatrix} J_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_k} \end{bmatrix}$ with $\lambda \vdash n$

each $J_k = \underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{k} \Big\} k$

$\underbrace{\{ \text{nilpotent matrices in } M_n(\mathbb{C}) \}}_{= N} \Big/ GL_n \leftrightarrow \{ \lambda \vdash n \}$ conjugation

Springer theory

$A(V_i) \subseteq V_{i-1}$ (ex) $\Updownarrow$ A is nilpotent

$\tilde{N} = \{ (V_\bullet, A) \in Fl_n \times N : A(V_i) \subseteq V_i \}$

vector bundle $\to pr_1 \swarrow \quad \searrow pr_2 \leftarrow$ Springer resolution

$Fl_n \qquad N$

Springer fibre

For a nilpotent matrix A of type λ.

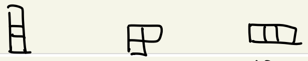
$Fl_\lambda = \{ V_\bullet \in Fl_n : A(V_i) \subseteq V_i \}$

Springer theory : $S_n \curvearrowright H^*(Fl_\lambda)$

(equivalently $H_*(Fl_\lambda)$).

Example $S_3 \curvearrowright H^*(\mathcal{F}\ell_3)$, $H^*(pt)$, $H^*(\mathbb{P}^1 \times_{pt} \mathbb{P}^1)$

$\underline{n=3}$



(1,1,1) (2,1) (3)

$$\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

For $\lambda = (1,1,1)$, $A = 0$ $\quad (A(V_i) \subseteq V_i)$

$\Rightarrow \mathcal{F}\ell_\lambda = \mathcal{F}\ell_3$

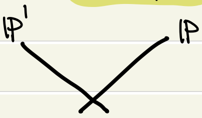
For $\lambda = (3)$, $0 \xleftarrow{A} e_1 \xleftarrow{A} e_2 \xleftarrow{A} e_3$ $\quad (A(V_i) \subseteq V_i)$

$V_1 = \text{span}(e_1)$ $\quad \Rightarrow \mathcal{F}\ell_\lambda = \{\text{std flag}\}$

$V_2 = \text{span}(e_1, e_2)$ $\quad = \text{a point}.$

For $\lambda = (2,1)$,

$\mathcal{F}\ell_\lambda = \mathbb{P}^1 \cup_{pt} \mathbb{P}^1$



$0 \xleftarrow{A} e_1 \xleftarrow{A} e_2$

$0 \xleftarrow{A} e_3$

$\ker(A) \supseteq V_1$ $\quad \mathbb{C}^3$ $\quad \text{im}(A) \subseteq V_2$

$AV_1 \subseteq V_0 = 0$ $\quad AV_2 \subseteq V_1$ $\quad AV_3 \subseteq V_2$

$V_1 \subseteq \text{span}(e_1, e_3)$ $\quad \text{\\\\\\}$ $\quad \text{span}(e_1) \subseteq V_2$

The choice of V_1 is

$\{V_1 \subset \text{span}(e_1, e_3)\} = \mathbb{P}^1$

① If $V_1 = \text{span}(e_1)$

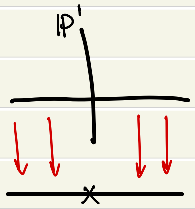
$AV_2 \subseteq V_1 = \text{span}(e_1)$ always true

$\left\{\begin{matrix}\text{The choices} \\ \text{of } V_2\end{matrix}\right\} = \left\{\begin{matrix}\text{2-dim subspaces} \\ \text{between span}(e_1) \\ \text{and } \mathbb{C}^3\end{matrix}\right\} = \left\{\begin{matrix}\text{1-dim subspaces} \\ \text{of } \mathbb{C}^3/\text{span}(e_1)\end{matrix}\right\} = \mathbb{P}^1$

② If $V_1 \neq \text{span}(e_1)$, the only choice is

$V_2 = V_1 \oplus \text{span}(e_1)$

$\{\text{The choices of } V_2\} = \text{point}.$



$\{\text{Choices of } V_1 < V_2\}$

$\{\text{Choices of } V_1\} \cong \mathbb{P}^1$

Theorem We have

$$\text{graded Frob}(H^*(\mathcal{F}\ell_\lambda)) = \tilde{H}_\lambda.$$

$S_n \curvearrowright$ $\quad = \sum_{i \geq 0} \text{Frob}(H^{2i}(\mathcal{F}\ell_\lambda))\, t^i$

Equivalently,

$$\text{gdim}\,\text{Hom}_{S_n}(\text{Specht}_\mu, H^*(\mathcal{F}\ell_\lambda)) = t^{n(\lambda)} K_{\mu\lambda}(t^{-1}).$$

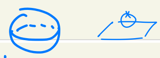
Example

Rmk $H^*(\mathcal{F}\ell_\lambda)$ is a quotient of $\mathbb{Q}[x_1, \cdots, x_n]$ with ideal J_λ.

λ	$\mathcal{F}\ell_\lambda$	dim	$H^{2*}(\mathcal{F}\ell_\lambda)$
(3)	pt	0	tri
(2,1)	$\mathbb{P}^1 \cup_{pt} \mathbb{P}^1$	1	tri $\oplus$ std
(1,1,1)	$\mathcal{F}\ell_3$	3	$\overset{0}{\text{tri}} \oplus \overset{1}{\text{std}} \oplus \overset{2}{\text{std}} \oplus \overset{3}{\text{alt}}$

$$H^*(\mathcal{F}\ell_n) = \frac{\mathbb{Q}[x_1, \cdots, x_n]}{\langle e_k(x) : k = 1, \cdots, n\rangle} \quad \text{coinvariant ring.}$$

$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ $\qquad \mathbb{P}^1 \cup_{pt} \mathbb{P}^1 =$ 

$=$ 

fact: $H^{top}(\mathcal{F}\ell_\lambda) \cong \text{Specht}_\lambda$ $\qquad top = 2 \cdot n(\lambda)$.

expand $\rightarrow$	$P_{(1,1,1)}$	$P_{(2,1)}$	$P_{(3)}$
$s_{(1,1,1)}$	1		
$s_{(2,1)}$	t^2+t	1	
$s_{(3)}$	t^3	t	1

expand $\downarrow$	$H_{(1,1,1)}$	$H_{(2,1)}$	$H_{(3)}$
$s_{(1,1,1)}$	1		
$s_{(2,1)}$	t^2+t	1	
$s_{(3)}$	t^3	t	1

expand $\downarrow$	$\tilde{H}_{(1,1,1)}$	$\tilde{H}_{(2,1)}$	$\tilde{H}_{(3)}$
$s_{(1,1,1)}$	t^3		
$s_{(2,1)}$	t^2+t	t	
$s_{(3)}$	1	1	1

2 Two variations $\quad (\mathbb{F}_q,\ q = p^m)$

Partial flag varieties

$\mathcal{F}\ell_5 = \{0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset \mathbb{C}^5\}$ full flag variety $\quad (\dim V_i = i)$

$\mathcal{F}\ell_5^{(2,1,2)} = \{0 \subset V_2 \subset V_3 \subset \mathbb{C}^5\}$ partial flag variety $\quad$ (steps 2, 1, 2)

$\mathcal{F}\ell_n^\alpha \qquad \alpha = (\alpha_1, \cdots, \alpha_m) \models n$

(composition $\alpha_1 + \cdots + \alpha_m = n$).

Rmk any nilpotent $A \in M_n(F)$ is conjugate to $\text{diag}(J_{\lambda_1}, \cdots, J_{\lambda_k})$ $\lambda \vdash n$ over any field F.

Funny exercise $\#\{A \in M_n(\mathbb{F}_q) : A \text{ nilpotent}\} = ?$.

For a nilpotent matrix A of type λ.

$$\mathcal{F}\ell_\lambda^\alpha = \{V_\bullet \in \mathcal{F}\ell^\alpha : AV_{s(i)} \subset V_{s(i-1)}\}$$

$$\Updownarrow$$

$$\mathcal{F}\ell_{(\lambda)}^\alpha = \{V_\bullet \in \mathcal{F}\ell^\alpha : AV_{s(i)} \subset V_{s(i)}\}$$

e.g. $s(1) = \alpha_1$
$s(2) = \alpha_1 + \alpha_2$
$s(3) = \alpha_1 + \alpha_2 + \alpha_3$
$\cdots$

the dimensions appeared in $\mathcal{F}\ell^\alpha$

$Fl^\alpha_\lambda = \{ V_\bullet \in Fl^\alpha : A V_{s(i)} \subset V_{s(i-1)} \}$

Theorem We have

$$\#(\mathcal{F}\ell^\alpha_\lambda(\mathbb{F}_q)) = t^{-d}\langle e_\mu, \tilde{H}_\lambda\rangle|_{t\mapsto q}.$$

$d = \dim Fl_n - \dim Fl^\alpha = \binom{\alpha_1'}{2} + \dots + \binom{\alpha_k'}{2}$

e.g. $\alpha = (2,1,2)$ oo|o|oo $\quad \alpha' = (1,3,1)$ o|ooo|o

$\mu = \mathrm{sort}(\alpha)$ e.g. $\mathrm{sort}(2,1,2) = (2,2,1)$

Example

	$\{0\subset\mathbb{C}^3\} = Fl^{(3)}_\lambda$	$\{0\subset V_1\subset\mathbb{C}^3\} = Fl^{(1,2)}_\lambda$	$\{0\subset V_2\subset\mathbb{C}^3\} = Fl^{(2,1)}_\lambda$	$Fl_3 = Fl^{(1,1,1)}_\lambda$
(3)	$\emptyset$	$\emptyset$	$\emptyset$	pt
(2,1)	$\emptyset$	pt	pt	$\mathbb{P}^1 \cup_{pt} \mathbb{P}^1$
(1,1,1)	pt	$Fl^{(1,2)}$	$Fl^{(2,1)}$	Fl_3

$0 \xleftarrow{A} e_1 \xleftarrow{A} e_2 \qquad A V_1 = 0$

$0 \xleftarrow{A} e_3 \qquad \mathrm{span}(e) = A\mathbb{C}^3 \subseteq V_1$

$0 \xleftarrow{A} e_1 \xleftarrow{A} e_2 \qquad \left.\begin{array}{l} A V_2 = 0 \\ A\mathbb{C}^3 \subseteq V_2 \end{array}\right\} \Rightarrow V_2 = \mathrm{span}(e_1, e_2)$

$0 \xleftarrow{A} e_3$

$\langle -,- \rangle$	$e_{(3)}$	$e_{(2,1)}$	$e_{(1,1,1)}$
$\tilde H_{(3)}$	0	0	1
$\tilde H_{(2,1)}$	0	t	$2t+1$
$\tilde H_{(1,1,1)}$	t^3	t^3+t^2+t	t^3+2t^2+2t+1

Learning seminar Macdonald

link notes

$Fl^\alpha_{(\lambda)} = \{ V_\bullet \in Fl^\alpha : A V_{s(i)} \subset V_{s(i)} \}$

Theorem We have

$$\#(\mathcal{F}\ell^\alpha_{(\lambda)}(\mathbb{F}_q)) = \langle h_\mu, \tilde{H}_\lambda\rangle|_{t\mapsto q}$$

Example

	$\{0\subset\mathbb{C}^3\} = Fl^{(3)}_\lambda$	$\{0\subset V_1\subset\mathbb{C}^3\} = Fl^{(1,2)}_\lambda$	$\{0\subset V_2\subset\mathbb{C}^3\} = Fl^{(2,1)}_\lambda$	$Fl_3 = Fl^{(1,1,1)}_\lambda$
(3)	pt	pt	pt	pt
(2,1)	pt	$\mathbb{P}^1$	$\mathbb{P}^1$	$\mathbb{P}^1 \cup_{pt} \mathbb{P}^1$
(1,1,1)	pt	$Fl^{(1,2)}$	$Fl^{(2,1)}$	Fl_3

$\langle -,- \rangle$	$h_{(3)}$	$h_{(2,1)}$	$h_{(1,1,1)}$
$\tilde H_{(3)}$	1	1	1
$\tilde H_{(2,1)}$	1	$t+1$	$2t+1$
$\tilde H_{(1,1,1)}$	1	t^2+t+1	$\underbrace{t^3+2t^2+2t+1}_{W(t) \text{ for } W = S_3}$

Remark

For general $A \in M_n(\mathbb{F}_q)$

$$\#\{ V \in Fl^\alpha : A V_{s(i)} \subseteq V_{s(i)} \}$$

we know the answer for nilpotent A.

Formulate a generating function

$$\sum_{\alpha \models n} \#\{ V \in Fl^\alpha : A V_{s(i)} \subseteq V_{s(i)} \} x^\alpha$$

$$= \tilde H_{\lambda_1}[p_{d_1}] \cdots \tilde H_{\lambda_k}[p_{d_k}]$$