Kostka - Foulkes Polynomials

- **Outline:**

  Sec. 1: Definition

  Sec. 2: The statement of some results

  Sec. 3: Proof of the combinatorial positive formula

- Reference: ◦ 「Kostka - Foulke polynomials and Macdonald spherical functions」(combinatorial formula)

  ◦ 「A Combinatorial Proof of a Recursion for the q-Kostka Polynomials」(Def of charge)

Sec. 1. Definition of Kosta - Foulkes polynomials.

- Def 1.1: $s_\lambda = \sum_{\mu \in P} K_{\lambda\mu}(t) \, P_\mu(x;t)$ $\quad K_{\lambda\mu}(t) \in \mathbb{Z}[t]$

- Def 1.2: (H_μ is dual basis of $\{P_\mu\}$ w.r.t. the Hall inner product).

$$H_\mu(x;t) = \sum_{\lambda \in P} K_{\lambda\mu}(t) \, s_\lambda$$

- Find a new description of H_μ in order to compute $K_{\lambda\mu}(t)$

The first reference use "Q_μ" for this polynomial

First, extend the definition of Schur polynomials.

- Def 1.3. Let $\delta = (n-1, n-2, \cdots, 0)$, $\lambda = (\lambda_1, \cdots, \lambda_n)$ is composition. Define the Schur polynomials.

$$s_\lambda := \frac{a_{\lambda+\delta}}{a_\delta}.$$

where

$$a_\mu := \sum_{w \in S_n} (-1)^{\ell(w)} w(x^\mu)$$

- Def 1.4. $\forall \alpha \in R^+ := \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \}$.

Define

$$R_\alpha : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$$
$$\lambda \longmapsto \lambda + \alpha.$$

$$(R_{\beta_1} \cdots R_{\beta_\ell}) s_\lambda := s_{R_{\beta_1} \cdots R_{\beta_\ell} \lambda}.$$
$(\beta_1, \cdots, \beta_\ell \in R^+)$.

(·Note : R_α is not an operation on Λ (i.e. sym functions/polynomials))

$(R_{\beta_1} R_{\beta_2}) s_\lambda = s_{R_{\beta_1} R_{\beta_2} \lambda} \neq 0$. not compatible.

- $R_{\beta_2} s_\lambda = 0$. $(R_{\beta_1} R_{\beta_2}) s_\lambda = R_{\beta_1}(R_{\beta_2} s_\lambda) = R_{\beta_1}(0) = 0$

∘ Prop 1.5. $\forall \mu \in P$,

$$H_\mu(x;t) = \left(\prod_{\alpha \in R_+} \frac{1}{1-tR_\alpha} \right) s_\mu.$$

Pf: Consider the t-deformation of the Hall inner product

$$\langle\ ,\ \rangle_t : \mathbb{Z}[t][x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \times \mathbb{Z}[t][x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \to \mathbb{Z}[t]$$

given by $\langle f, g \rangle_t = \frac{1}{|W|} \left[f \bar{g} \cdot \prod_{\alpha \in R} \frac{1-x^\alpha}{1-tx^\alpha} \right]_1$

$R = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$.

where $g = \sum_{\mu \in \mathbb{Z}^n} f_\mu \cdot x^\mu$ $\quad (f_\mu \in \mathbb{Z}[t])$.

let $\bar{g} = \sum_{\mu \in \mathbb{Z}^n} f_\mu \cdot x^{-\mu}$.

and. $[f]_1 :=$ (coef of 1 in f),

(Note: $\langle f, g \rangle_0 = \left[f \bar{g} \cdot \prod_{\alpha \in R} (1-x^\alpha) \right]_1$)

∘ FACT: (without proof).

$$\begin{cases} \langle s_\lambda, s_\mu \rangle_0 = \delta_{\lambda\mu} \quad \left(\Rightarrow \langle\ ,\ \rangle_0 \big|_{\mathbb{Z}[t][x_1, \cdots, x_n]^{S_n}} \text{ is the Hall inner product} \right). \\ \langle P_\lambda, P_\mu \rangle_t = \frac{1}{W_\lambda(t)} \delta_{\lambda\mu}. \end{cases}$$

It suffices to show $K_{\lambda\mu}(t) = \left(\text{coef } s_\lambda \text{ in } \left(\prod_{\alpha \in R^+} \frac{1}{1-tR_\alpha} \right) s_\mu \right)$.

$K_{\lambda\mu}(t) = (\text{coef of } P_\mu \text{ in } s_\lambda). = \langle s_\lambda, W_\mu(t) \cdot P_\mu \rangle_t.$

$$= \left\langle s_\lambda, \; W_\mu(t) \cdot P_\mu \cdot \prod_{\alpha \in R} \frac{1}{1 - t x^\alpha} \right\rangle_0.$$

$$= \left(\text{coef of } s_\lambda \text{ in } \; W_\mu(t) \cdot P_\mu \cdot \prod_{\alpha \in R} \frac{1}{1 - t x^\alpha} \right)$$

Then it remains to show that

$$W_\mu(t) P_\mu \cdot \prod_{\alpha \in R} \frac{1}{1 - t x^\alpha} = \left(\prod_{\alpha \in R^+} \frac{1}{1 - t R_\alpha} \right) s_\mu$$

$\square$

Sec. 2: The combinatorial positive formula for $K_{\lambda\mu}(t)$. (Type A)

(Statement without proof)

Next section!

- Thm 2.1: For partitions λ and μ.

$$K_{\lambda\mu}(t) = \sum_{b \in B(\lambda)_\mu} t^{ch(b)}$$

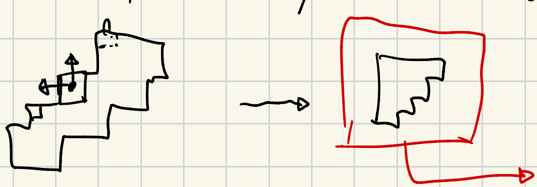
where $B(\lambda)_\mu := \{$ SSYT's of shape λ and weight type $\mu \}$

and $ch(b)$ is the charge statistic of b.

Then we will introduce 3 equivalent definitions of charge. The first two are explicit, while the third one is inductive and mainly used to prove Thm 2.1.

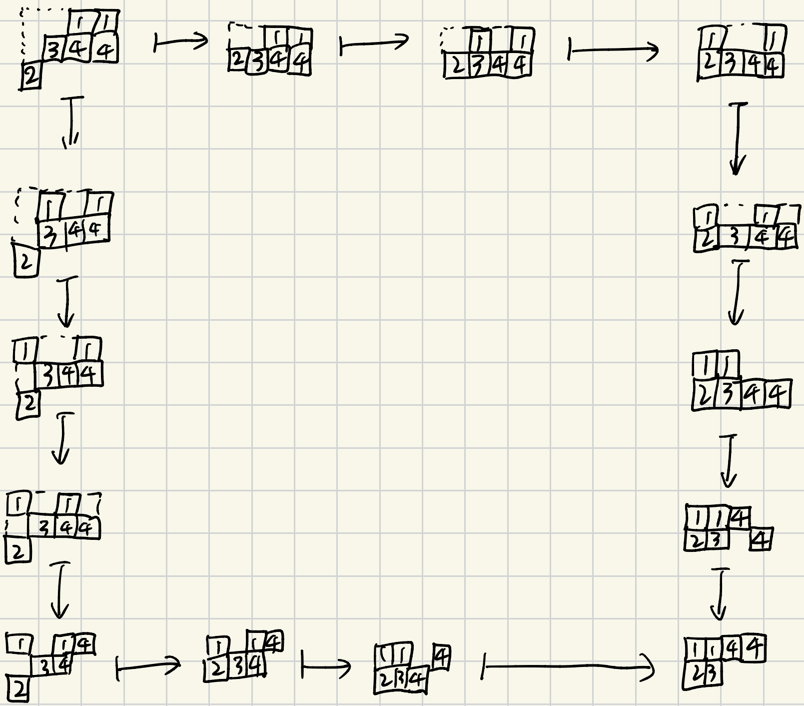
- (Notations and operations of SSYT's)
- $P = \{$partions$\}$, $B(P) := \{$SSYT's of partitions$\}$.
- "jeu de taquin reduction"

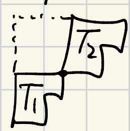
"SSYT of skew shape" $\longmapsto$ "SSYT of partition ship".



unique.

e.g. (jeu de taquin).



- $\forall\, T_1, T_2 \in B(P)$, $T_1 * T_2 :=$ (jeu de taquin reduction of )
- $(B(P), *)$ is a monoid, called the plactic monoid.

(Rmk: If $x \in B(P)$ has only one box,
then $x * T$ = col insertion of x into T.
$T * x$ = row insertion of x into T.)

- Let B^* be the free monoid generated $\mathbb{N}$.

  We have a natural surjection:
$$B^* \twoheadrightarrow B(P).$$
$$w_n \cdots w_1 \longmapsto w_n * \cdots * w_1.$$
  which defines an equivalence on B^*.

  (Rmk: This equivalence is generated by Knuth relations.
$$\begin{cases} x<y\le z \Rightarrow yxz \sim yzx \\ x\le y<z \Rightarrow xzy \sim zxy \end{cases}$$
)

- Recall
$$\begin{tikzcd} B^* \arrow[r, two heads] \arrow[dr, "ch"'] & B(P) \arrow[d, "ch"] \\ & \mathbb{Z}_{\ge 0} \end{tikzcd}$$

  it suffices to
$$\begin{cases} (1) \text{ define } ch: B^* \to \mathbb{Z}_{\ge 0} \text{ s.t. } u \sim v \Rightarrow ch(u) = ch(v) \\ (2) \ \forall\, b \in B(P), \text{ find a preimage of } b \text{ in } B^*. \end{cases}$$

  For (2): Use "reading word" of b.

  (e.g. $b = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 & 5 & \\ \cline{1-5} 4 & 4 & 6 & & & \\ \cline{1-3} \end{array}$ $\quad \text{read}(b) = \underset{\substack{\Uparrow \\ B^*}}{44623345111223}$)

  For (1),

  Step 1. Define $ch(\pi)$ for all $\pi \in B^*$ with $wt(\pi) = (1, \cdots, 1)$ (i.e. $\pi \in S_\infty$)

  Write $\pi = \pi_n \cdots \pi_1$, $c_i(\pi) := \#\{j \mid j \le \pi_i$ and j is to the right of $j+1$ in $\pi\}$

  and $ch(\pi) = \sum_{i=1}^{n} c_i(\pi)$.

e.g.:

$$\pi = \overset{1}{5}\,\overset{2}{6}\,\overset{0}{2}\,\overset{1}{4}\,\overset{0}{1}\,\overset{1}{3}$$

$$ch(\pi) = 1+2+0+1+0+1 = 5.$$

Step 2. Define $ch(w)$ for all $w \in B^*$ with $wt(w) \in P$

We want to decompose w into subwords $\underbrace{w^{(1)}, \dots, w^{(m)}}_{\in \text{Step 1}}$

and then let $ch(w) = \sum_{i=1}^{m} ch(w^{(i)})$

$\boxed{\text{2 ways.}}$ $\Rightarrow$ 2 equiv def!

(From right to left)

e.g. 1: $w = 2^3 1^3 5^1 4^1 2^2 5^2 3^1 1^2 2^1 4^2 7^1 6^1 3^2 1^1$

$w^{(1)} = 5432761.$

$w^{(2)} = 25143.$

$w^{(3)} = 21.$

$$ch(w) = ch(5432761) + ch(25143) + ch(21)$$

$$= \underline{2+3+0} = 5.$$

e.g. 2. (From left to right).

$w = 2^2 1^2 5^1 4^1 2^3 5^2 3^1 1^3 2^1 4^2 7^1 6^1 3^2 1^1$

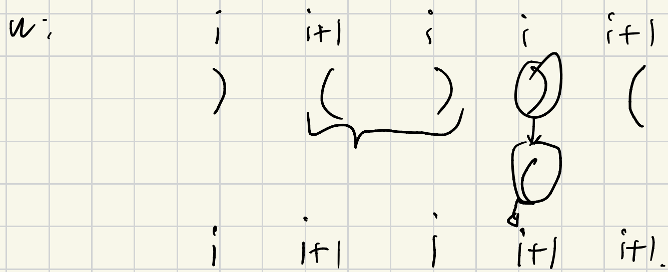
$$ch(w) = ch(5432761) + ch(21543) + ch(21)$$

$$= \underline{2+3+0} = 5.$$

- Step 3: Define ch for general words $w \in B^*$.

$S_\infty \circlearrowright B^*$. $\forall u \in S_\infty, w \in B^*, ch(u \cdot w) = ch(w).$

σ_i interchanges μ_i and μ_{i+1} in $wt(w)=\mu$.



e.g.: $w = 2\,4\,1\,3\,2\,3\,5\,2\,4\,3\,1\,2\,3\,2$ $\qquad wt(w) = (25421)$

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \sigma_1$

$\sigma_1 w = 2\,4\,1\,3\,1\,3\,5\,2\,4\,3\,1\,1\,3\,1$ $\qquad wt(\sigma_1 w) = (52421)$

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \sigma_2$

$\sigma_2\sigma_1 w = 2\,4\,1\,2\,1\,3\,5\,2\,4\,2\,1\,1\,3\,1.$ $\qquad wt(\sigma_2\sigma_1 w) = (54221)$

$ch(w) = \underline{ch(\sigma_2\sigma_1 w)}$

Therefore, we obtain two equivalent (explicit) definitions of charge.

Finally, we introduce an inductive definition of charge:

- Definition 2.2 (Inductive def of ch). Fix $n \in \mathbb{N}$

Let $B(P)_{\geq} = \bigcup_{i=1}^{n} B(P)_{\geq i}$

$$B(P)_{\geq i} := \left\{ \text{SSYT } b \;\middle|\; \begin{array}{l} wt(b) = (\mu_1, \cdots, \mu_n) \\ \mu_1 = \cdots = \mu_{i-1} = 0 \\ \mu_i \geq \cdots \geq \mu_n \geq 0 \end{array} \right\}.$$

Let $i^k = \underbrace{\boxed{i}\boxed{i}\cdots\boxed{i}}_{k \text{ boxes.}}$

Charge is the <u>unique</u> function $ch : B(P)_{\geq} \to \mathbb{Z}_{\geq 0}$,

s.t. (a) $ch(\emptyset) = 0$

(b) if $T \in B(P)_{\geq (i+1)}$ and $T * i^{\mu_i} \in B(P)_{\geq i}$

$$ch(T * i^{\mu_i}) = ch(T).$$

(c) if $T \in B(P)_{\geq i}$ and $x \neq i$.

$$ch(T * x) = ch(x * T) + 1.$$

- Prop 2.3. (Alain Lascoux and Marcel-Paul Schützenberger 「Sur une conjecture de H.O. Foulkes」)

$$K_{\lambda\mu}(t) = \begin{cases} 0, & \text{if } |\lambda| \neq |\mu| \text{ or } \lambda \ntrianglerighteq \mu. \\ \text{a monic polynomial of degree } n(\mu) - n(\lambda), & \text{if } \lambda \trianglerighteq \mu \text{ and } |\lambda| = |\mu|. \end{cases}$$

Sec. (Sketch) Proof of Thm 2.1.

Prop 1.5. indicates that $\forall \mu \in P$,

$$\left(\prod_{1\le i<j\le n} \frac{1}{1-tR_{ij}}\right) s_\mu = \sum_{\lambda\in P} K_{\lambda\mu}(t)\, s_\lambda$$

$\Big\downarrow$ Cancellation

$|\mu|\geq 0$,

$w\circ\mu = w(\underline{\mu+\delta})-\delta$ distinct $\quad w=t_{ij}$ $\quad s_\mu=-s_\mu \Rightarrow s_\mu=0$

- Prop 3.1 (The straightening law for Schur functions)

  $\forall \mu\in\mathbb{Z}^n,\ w\in S_n$.

$$s_\mu = (-1)^{\ell(w)} s_{w\circ\mu} \quad \text{where } w\circ\mu = w(\mu+\underset{\underset{(n-1,\cdots,1,0)}{\parallel}}{\delta})-\delta.$$

(Notations: $P := \{\text{partitions}\}$.

$\forall \lambda,\mu\in P,\ \mathcal{S},\mathcal{W}\subseteq P$,

$B(\lambda) := \{\text{SSYT } T \mid sh(T)=\lambda\}$

$B(\lambda)_\mu := \{\text{SSYT } T \mid sh(T)=\lambda\ \ wt(T)=\mu\}$

$B(\mathcal{S})_\mathcal{W} := \{\text{SSYT } T \mid sh(T)\in\mathcal{S},\ wt(T)\in\mathcal{W}\}$.

$\forall \mu,\gamma\in P$ and $r\in\mathbb{Z}_{\geq 0}$, let

$\gamma\otimes(r) := \{\lambda\in P \mid \lambda/\gamma$ is a horiz strip of length $r\}$

$(B(r)\otimes B(\gamma))_\mu := \{\text{pairs } v\otimes T \mid v\in B(r),\ T\in B(\gamma).\ wt(v)+wt(T)=\mu\}$

$(B(\gamma)\otimes B(r))_\mu := \{\text{pairs } T\otimes v, \cdots\}$

- Lem: 3.2 (Tableau version of the Pieri rule)

Let $\gamma, \mu, \tau \in \mathcal{P}$, $r, s \in \mathbb{Z}_{\geq 0}$,

$$\begin{array}{ccc} \big(B(r) \otimes B(\gamma)\big)_{\mu} & \xrightarrow{1:1} & B(\gamma \otimes (r))_{\mu} \\ v \otimes T & \longmapsto & v * T. \end{array}$$

and

$$\begin{array}{ccc} \big(B(\gamma) \otimes B(s)\big)_{\tau} & \xrightarrow{1:1} & B(\gamma \otimes (s))_{\mu} \\ T \otimes u & \longmapsto & T * u. \end{array}$$

- Pf of Thm 2.1.

$\mu = (\mu_1, \cdots, \mu_n)$.

By induction on n and pair-wise cancellations.

For $n+1$, add x_0, allow to fill 0 into SSYT.

W.T.S. $$H_{(\mu_0, \underset{\substack{\| \\ (\mu_1, \cdots, \mu_n)}}{\mu})} = \sum_{p \in B(\nu)_{(\mu_0, \mu)}} t^{ch(b)} s_{\nu}$$

$$H_{(\mu_0, \mu)} = \Big(\prod_{0 \leq i < j \leq n} \frac{1}{1 - tR_{ij}} \Big) s_{(\mu_0, \mu)}$$

$$= \Big(\prod_{j=1}^{n} \frac{1}{1 - tR_{0j}} \Big) \Big(\prod_{1 \leq i < j \leq n} \frac{1}{1 - tR_{ij}} \Big) s_{(\mu_0, \underline{\mu})}$$

$$= \Big(\prod_{j=1}^{n} \frac{1}{1 - tR_{0j}} \Big) \Big(\sum_{\lambda \in \mathcal{P}} K_{\lambda\mu}(t) s_{(\mu_0, \underbrace{\lambda}_{})} \Big)$$

$$= \sum_{\lambda \in P} K_{\lambda,\mu}(t) \sum_{r \in \mathbb{Z}_{\geq 0}} t^r \sum_{\substack{k_1,\dots,k_n \in \mathbb{Z}_{\geq 0} \\ k_1+\cdots+k_n = r}} S_{(\mu_0+r,\, \lambda-(k_1,\dots,k_n))}$$

Let $\gamma = \lambda - (k_1,\dots,k_n)$ s.t. λ/γ is not a horizontal strip.

Let m be minimal s.t. $\lambda_m - k_m < \lambda_{m+1}$.

Consider $\gamma \longleftrightarrow \tilde{\gamma}$.

where $\tilde{\gamma} := s_m \circ \gamma$

Prop 3.1 $\Rightarrow$ $S_{(\mu_0+r,\, \gamma)} = -S_{(\mu_0+r,\, \tilde{\gamma})}$

$$\Rightarrow H_{(\mu_0,\mu)} = \sum_{\lambda \in P} \sum_{r \in \mathbb{Z}_{\geq 0}} t^r K_{\lambda\mu}(t) \sum_{\substack{\gamma \in P \\ \lambda \in \gamma \otimes (r)}} S_{(\mu_0+r,\, \gamma)}$$

$$= \sum_{\gamma,\, r} \sum_{b \in B(\gamma \otimes (r))_\mu} t^{r+ch(b)} S_{(\mu_0+r,\, \gamma)}$$

$$= \sum_{\gamma,\, r} \sum_{v \otimes T \in (B(r) \otimes B(\gamma))_\mu} t^{r+\underbrace{ch(v*T)}} S_{(\mu_0+r,\, \gamma)}$$

$$ch(v * T * 0^{\mu_0})$$

$$\parallel$$

$$ch(T * 0^{\mu_0} * v) - r$$

(By inductive def of charge)

$$= \sum_{\gamma, r} \sum_{v \otimes T \in (B(r) \otimes B(\gamma))_\mu} t^{ch(T * 0^{\mu_0} * v)} s_{(\mu_0 + r, \gamma)}$$

Let $p = T * \underline{0^{\mu_0} * v}$. $\lambda = sh(p)$.

(Idea: In general $\lambda_1 \geq \mu_0 + r$.

If $\lambda_1 = \mu_0 + r$, Great!

$\lambda = (\mu_0 + r, \gamma)$, yielding all the terms that we want!)

Then we have to cancellate the remaining terms through pair-wise matching as follows.

$\exists ! d \in \mathbb{N}$, s.t. $\begin{cases} \mu_0 + r + d > \lambda_d \\ \mu_0 + r + d - 1 \leq \lambda_{d-1} \end{cases}$

用"$\lambda_1 \geq \mu_0 + r$"的临界版本

If $d = 1$. ✓. These terms are exactly what we want!

If $d > 1$. cancellation all of these terms!

$$= \sum_{\substack{(\gamma, r, T, v) \\ v \otimes T \in (B(r) \otimes B(\gamma))_\mu}} t^{ch(T * 0^{\mu_0} * v)} s_{(\mu_0 + r, \gamma)}$$

Construct $(\tilde{\gamma}, \tilde{r}, \tilde{T}, \tilde{v})$ as follows.

Let $\boxed{\tilde{\gamma}} = (\gamma_1, \cdots, \gamma_{d-2}, \mu_0 + r + d - 1, \gamma_d, \cdots, \gamma_n)$

$\mu_0 + \tilde{r} + d - 1 = \gamma_{d-1}$.

Claim: $\exists !\, \tilde{v} \otimes \tilde{T} \in (B(\tilde{r}) \otimes B(\tilde{\gamma}))_\mu$,

s.t. $T * 0^{\mu_0} * v = \tilde{T} * 0^{\mu_0} * \tilde{v}$.

$(\gamma, r, T, v) \longleftrightarrow (\tilde{\gamma}, \tilde{r}, \tilde{T}, \tilde{v})$

s.t. $T * 0^{\mu_0} * v = \tilde{T} * 0^{\mu_0} * \tilde{v}$.

and $s_{(\mu_0 + r, \gamma)} \underset{\Uparrow}{=} -s_{(\mu_0 + \tilde{r}, \tilde{\gamma})}$.

(By Prop 3.1,

$(\mu_0 + \tilde{r}, \tilde{\gamma}) = (s_0 \cdots s_{d-3} s_{d-2} s_{d-3} \cdots s_0) \circ (\mu_0 + r, \gamma)$)

$\Rightarrow$ Cancellation!

The remaining terms (w.r.t. $d = 1$) is exactly what we want.

$\square$