Nonsymmetric Macdonald polynomials

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1 Demazure–Lusztig operators

$\mathbb{Q}_t[\Lambda] = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}(t)\, e^\lambda$

$\mathbb{Q}_{q,t}[\Lambda] = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}(q,t)\, e^\lambda$ group algebra

$W \curvearrowright \mathbb{Q}_{q,t}[\Lambda]$ Hecke algebra $\curvearrowright \mathbb{Q}_{q,t}[\Lambda]$

The Demazure–Lusztig operators

$$T_i = ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - \mathrm{id}).$$

$i \in I$ $\quad W = \langle s_i \rangle_{i\in I}$

s_i = simple reflection of W

α_i = simple root

$T_i e^\lambda = t e^{s_i\lambda} + \frac{t-1}{e^{\alpha_i}-1}(e^{s_i\lambda} - e^\lambda) \in \mathbb{Q}_t[\Lambda].$

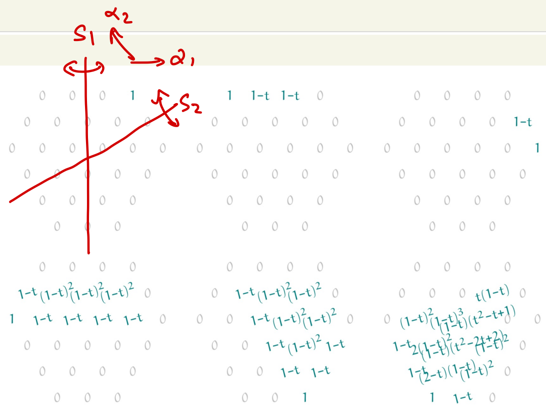
$= \frac{1-t}{e^{\alpha_i}-1}\,\mathrm{id} + \frac{te^{\alpha_i}-1}{e^{\alpha_i}-1}\, s_i$

$(T_i - t)(T_i + 1) = 0$ quadratic relation

$\underbrace{T_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_i\cdots}_{m_{ij}}$ $\quad i \neq j$.

Note: $T_i + 1$ is a symmetrizer

$T_i f = tf \iff s_i f = f$



W $\xrightarrow{\text{intro } s_0}$ W_a
finite $\quad$ affine

Justify $i = 0$

To justify $i=0$, we need:

① action of s_0

$W_a = W \ltimes Q^\vee$ $\quad$ span$(\alpha_i^\vee : i\in I)$ (semi-direct product) $\quad \langle \alpha_i^\vee, \alpha_j\rangle = c_{ij}$.

i.e. $wt_\beta \cdot ut_\gamma = wu\, t_{u^{-1}\beta+\gamma}$

acts on $\Lambda \oplus \mathbb{Z}\delta$ $\quad (q = e^\delta)$

$wt_\beta(\alpha + k\delta) = w\alpha + (k - \langle\beta,\alpha\rangle)\delta$.

$\Rightarrow W_a \curvearrowright \mathbb{Q}_{q,t}[\Lambda]$

$wt_\beta(q^k e^\alpha) = q^{k-\langle\beta,\alpha\rangle} e^{w\alpha}$

② root α_0

θ = highest root of the root system

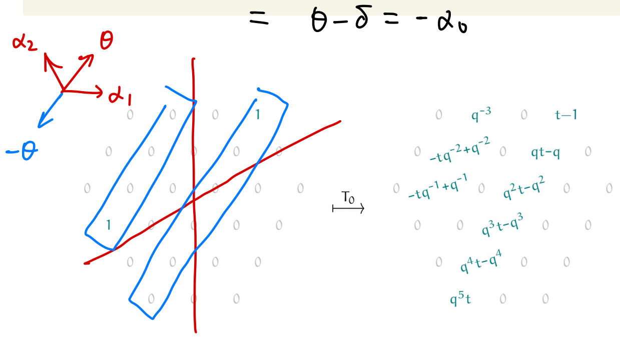
$\alpha_0 = -\theta + \delta$ $\quad (e^{\alpha_0} = qe^{-\theta})$

$s_0 = s_\theta t_{-\theta^\vee}$

$s_0\alpha_0 = s_\theta t_{-\theta^\vee}(-\theta+\delta)$

$= \theta + (1 - \langle -\theta^\vee, -\theta\rangle)\delta$

$= \theta - \delta = -\alpha_0$

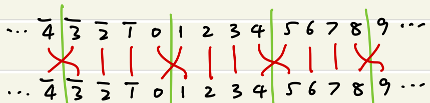


Type A convention

(extended affine Weyl group)

$\tilde{S}_n = \{ \text{bijections } \mathbb{Z} \xrightarrow{f} \mathbb{Z} \mid f(i+n) = f(i) + n \}$.

e.g. $n = 4$



$\tilde{S}_n \curvearrowright \mathbb{Q}_{q,t}[x_1, \cdots, x_n]$

$f : x_i \longmapsto x_{f(i)}$

$\cdots x_{-n}$	$x_{-n+1} \cdots x_0$	$x_1 \cdots x_n$	$x_{n+1} \cdots x_{2n}$	$x_{2n+1} \cdots$
$\cdots q^2 x_n$	$qx_1 \cdots qx_n$	$x_1 \cdots x_n$	$q^{-1}x_1 \cdots q^{-1}x_n$	$q^{-1}x_1 \cdots$

$(s_0 f)(x_1, \cdots, x_n) = f(qx_n, x_2, \cdots, x_{n-1}, q^{-1}x_1)$

$e^{\alpha_1} = x_1/x_2 \quad \cdots \quad e^{\alpha_{n-1}} = x_{n-1}/x_n$

$e^{\alpha_0} = qx_n/x_1$ $\qquad T_i = ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - \mathrm{id})$

$$(T_0 f)(x_1, \cdots, x_n) = t\, f(qx_n, x_2, \cdots, x_{n-1}, q^{-1}x_1) + \frac{t-1}{qx_n/x_1 - 1}\Big(f(qx_n, x_2, \cdots, x_{n-1}, q^{-1}x_1) - f(x_1, x_2, \cdots, x_{n-1}, x_n) \Big)$$

2 Operators Y^α

$W_a = W \ltimes Q^\vee$ (lattice, normal commutive)

affine Hecke algebra $\supseteq$ a big comm subalgebra $\cong \mathbb{Q}[Q^\vee]$?

Bernstein: Yes.

For $\beta \in Q^\vee$, we can define $Y^\beta \in$ affine Hecke algebra

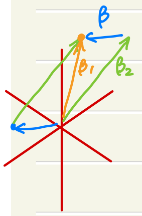
① β dominant $\Rightarrow Y^\beta = t^{-\langle \rho, \beta \rangle} T_{t_\beta}$

$t_\beta = s_{i_1} \cdots s_{i_\ell}$ reduced $T_{t_\beta} = T_{i_1} \cdots T_{i_\ell}$

Note: β dominant $\Rightarrow \ell(t_\beta) = 2\langle \rho, \beta \rangle$.

② $\beta = \beta_1 - \beta_2$ for β_1, β_2 both dominant

$Y^\beta = Y^{\beta_1} (Y^{\beta_2})^{-1}$.



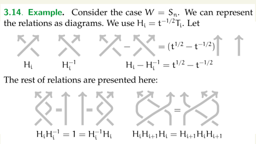
dominant cone $= \mathrm{span}_{\mathbb{R}_{\geq 0}} (\varpi_k^\vee)_{k \in I}$

coroot lattice $\otimes \mathbb{R} = \mathrm{span}_{\mathbb{R}} (\varpi_k^\vee)_{k \in I}$

Rmk $W_a = W \ltimes Q^\vee =$ Coxeter group generated by s_i ($i \in I$) and s_0

($\Downarrow$ s_0, $\Downarrow$ t_β)

Example


3.14. Example. Consider the case $W = S_n$. We can represent the relations as diagrams. We use $H_i = t^{-1/2}T_i$. Let

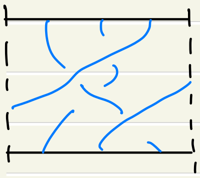
H_i H_i^{-1} $H_i - H_i^{-1} = t^{1/2} - t^{-1/2}$

The rest of relations are presented here:

$H_i H_i^{-1} = 1 = H_i^{-1} H_i$ $H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}$

We used "string diagrams" to "draw" Hecke algebra of type A. ($H_i = t^{-1/2} T_i$).

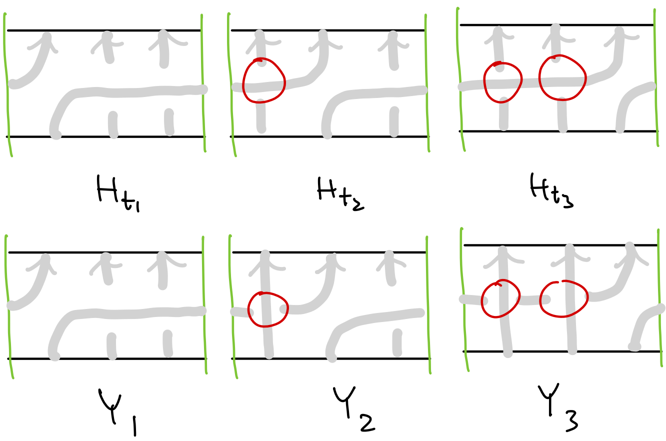
For affine type A, similarly, but draw them on



$Y^{\varepsilon_1} = Y_1$ $t_{\varepsilon_1} = t_1$

$\cdots$ $\cdots$

$Y^{\varepsilon_n} = Y_n$ $t_{\varepsilon_n} = t_n$



Ex. show $Y_1 Y_2 = Y_2 Y_1$, etc.

Example In SL_2, we have $t_{\alpha^\vee} = s_0 s$, so

$$Y = Y^{\alpha^\vee} = t^{-1} T_0 T.$$

(↑ dom; $T_0 T = T_{t_{\alpha^\vee}}$)

Remark: GL_n v.s. SL_n $\quad x^2 = e^\alpha$

$x = e^{\varpi}$, ϖ = fundamental weight

"Forget from GL_2 to SL_2" = "Specialize $x_1 \mapsto x$, $x_2 = x^{-1}$"

$e^{-\lambda}$	$e^{-\lambda+\alpha}$	$\cdots$	$e^{\lambda-\alpha}$	e^{λ}	$\xrightarrow{T}$	$e^{-\lambda}$	$e^{-\lambda+\alpha}$	$\cdots$	$e^{\lambda-\alpha}$	e^{λ}
0	0	$\cdots$	0	1		1	$1-t$	$\cdots$	$1-t$	0
1	0	$\cdots$	0	0		$t-1$	$t-1$	$\cdots$	$t-1$	t

$e^{-\lambda}$	$e^{-\lambda+\alpha}$	$\cdots$	$e^{\lambda-\alpha}$	e^{λ}	$\xrightarrow{T_0}$	$e^{-\lambda}$	$e^{-\lambda+\alpha}$	$\cdots$	$e^{\lambda-\alpha}$	e^{λ}
0	0	$\cdots$	0	1		$q^m t$	$q^{m-1}(t-1)$	$\cdots$	$q(t-1)$	$t-1$
1	0	$\cdots$	0	0		0	$q^{-1}(1-t)$	$\cdots$	$q^{-m+1}(1-t)$	q^{-m}

e.g. $T x^9 = x^{-9} + (1-t) x^{-7} + \cdots + (1-t) x^7$

$T x^{-9} = (t-1) x^{-9} + (t-1) x^{-7} + \cdots + (t-1) x^7 + t x^9$

$T_0 x^9 = q^9 t x^{-9} + \cdots + q(t-1) x^7 + (t-1) x^9$

$1 \xrightarrow{T} t \xrightarrow{T_0} t^2 \xrightarrow{t^{-1}} t$ $\quad Y(1) = t \cdot 1$

$s 1 = 1$

$x \xrightarrow{T} x^{-1} \xrightarrow{T_0} q^{-1} x \xrightarrow{t^{-1}} t^{-1} q^{-1} x$ $\quad Y(x) = t^{-1} q^{-1} x$

$x^{-1} \xrightarrow{T} (t-1) x^{-1} + t x \xrightarrow{T_0} (t-1) q^{-1} x + t(qt x^{-1} + (t-1)x)$

$\xrightarrow{t^{-1}} q t x^{-1} + ((t-1) + (1-t^{-1}) q^{-1}) x$ $\quad Y(x^{-1}) = q t x^{-1} + (\cdots)$

$x^2 \xrightarrow{T} x^{-2} + (\cdots) x^0 \xrightarrow{T_0} q^{-2} x^2 + (\cdots) x^0$

$\xrightarrow{t^{-1}} q^{-2} t^{-1} x^2 + (\cdots) x^0$ $\quad Y(x^2) = q^{-2} t^{-1} x^2 + (\cdots) x$

$1 \mapsto t$

$x \mapsto q^{-1} t^{-1} x$

$x^{-1} \mapsto q t x^{-1} + (t - 1 + q^{-1} - q^{-1} t^{-1}) x$

$x^2 \mapsto (-t + 1 - q^{-1} + q^{-1} t^{-1}) + q^{-2} t^{-1} x^2$

$x^{-2} \mapsto q^2 t x^{-2} + (qt - q + t - 1 - q^{-1} t + 2q^{-1} - q^{-1} t^{-1})$

$+ (t - 1 + q^{-2} - q^{-2} t^{-1}) x^2$

$x^3 \mapsto (\cdots)$

Observation:

1. parity $Y^{\alpha^\vee}(x^{\mathrm{odd}}) \in \mathrm{span}(x^{\mathrm{odd}})$

   $Y^{\alpha^\vee}(x^{\mathrm{even}}) \in \mathrm{span}(x^{\mathrm{even}})$

2. automorphism (root system automorphism)

3. Triangular

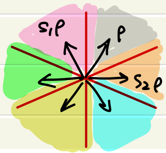
   "$\cdots > x^3 > x^{-1} > x$"

   "$\cdots > x^{-2} > x^2 > x^0$"

Theorem There is a certain order over Λ such that ($Q^\vee \ni \beta$, $\Lambda \ni \lambda$)

$$Y^\beta(e^\lambda) = q^{-\langle\beta,\lambda\rangle} t^{-\langle\beta,\rho_\lambda\rangle} e^\lambda + (\text{lower terms}),$$

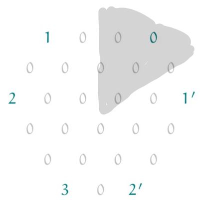
$$\rho_\lambda = \frac{1}{2}\sum_{\alpha>0}\begin{cases}\alpha, & \langle\alpha,\lambda\rangle>0,\\ -\alpha, & \langle\alpha,\lambda\rangle\le 0.\end{cases}$$



The order can be chosen to be

$$\mu\le\lambda \iff \begin{cases}\mu^+ >_{dom} \lambda^+, & W\mu \ne W\lambda,\\ \mu \le_{dom} \lambda, & W\mu = W\lambda.\end{cases}$$

(μ^+ unique dominant weight in $W\mu$; λ^+ unique dominant weight in $W\lambda$)



any gray $<$

$< 0 <$ both 1 & 1′ $<$

$<$ both 2 & 2′ < 3

Actually, it suffices to choose the Bruhat order ($\Lambda \cong$ extended affine Weyl group for the dual root system / finite Weyl group)

Example

SL_2: $x^0 < x^2 < x^{-2} < x^4 < x^{-4} < x^6 < x^{-6} < \cdots$

$x < x^{-1} < x^3 < x^{-3} < x^5 < x^{-5} < \cdots$

$Y^{\alpha^\vee}(1) = t$

$Y^{\alpha^\vee}(x^m) = q^{-m} t^{-1} x^m + \text{lower}$

$Y^{\alpha^\vee}(x^{-m}) = q^m t\, x^{-m} + \text{lower}$ $(m>0)$

$$\begin{bmatrix}1\\x^2\\x^{-2}\\x^4\\x^{-4}\\\vdots\end{bmatrix}\quad \begin{matrix}[1 & x^2 & x^{-2} & x^4 & x^{-4} & \cdots]\end{matrix}\quad \begin{bmatrix}t &&&&& \\ * & q^{-2}t^{-1} &&&& \\ * & * & q^2t &&& \\ * & * & * & q^{-4}t^{-1} && \\ * & * & * & * & q^4t & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

Observation:

$\{t,\ q^{-2}t^{-1},\ q^2t,\ q^{-4}t^{-1},\ q^4t, \cdots\}$

all distinct.

3 Macdonald polynomials

The non-symmetric Macdonald polynomial $E_\lambda \in \mathbb{Q}_{q,t}[\Lambda]$, weight $\lambda\in\Lambda$ is the unique polynomial such that

$$Y^\beta(E_\lambda) = q^{-\langle\beta,\lambda\rangle} t^{-\langle\beta,\rho_\lambda\rangle} E_\lambda$$

(distinct as linear functions in β)

$$E_\lambda = e^\lambda + (\text{other terms}).$$

Rmk. $A, B \in M_n(\mathbb{C})$ commute

A, B diagonalizable

$\iff$ A, B diagonalizable simultaneously.

Example

For SL_2, $E_{2\varpi} = x^2 + C$ $\quad C = (\text{const})$

$Y^{\alpha^\vee}(x^2) = (-t+1-q^{-1}+q^{-1}t^{-1}) + q^{-2}t^{-1}x^2$

$Y^{\alpha^\vee}(1) = t$

$Y^{\alpha^\vee}(E_{2\varpi}) = (-t+1-q^{-1}+q^{-1}t^{-1}) + q^{-2}t^{-1}x^2 + tC$

$= q^{-2}t^{-1}(x^2+C)$

$q^{-2}t^{-1}C = (-t+1-q^{-1}+q^{-1}t^{-1}) + tC$

$$C = \frac{-qt+q}{-qt+1}$$

$$E_{2\varpi} = x^2 + \frac{-qt+q}{-qt+1}$$

Eg. $E_0 = 1$ $\qquad E_\varpi = x$

Ex. $E_{-\varpi} = x^{-1} + \frac{-t+1}{-qt+1}x.$

Learning seminar Macdonald

link $\qquad$ notes