Symmetric Macdonald polynomials

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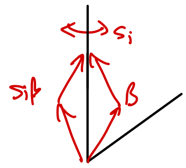

0 Recall

Demazure--Lusztig operators
works for $i = 0$
Y^β
E_λ
fancy way of thinking

$$T_i = ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - id) \qquad i \in I$$

extended to $i=0$, s_0 and α_0

$$\mathbb{Q}_{q,t}[\Lambda] = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}(q,t)\, e^\lambda$$

Y^β, $\beta \in Q^\vee$ = coroot lattice

Nonsym. Macdonald poly E_λ for $\lambda \in \Lambda$

① $Y^\beta(E_\lambda) = q^{-\langle\beta,\lambda\rangle}\, t^{-\langle\beta,\rho_\lambda\rangle} E_\lambda$

② $E_\lambda = e^\lambda + (\text{other})$

"Fancy way" $\mathbb{Q}_{q,t}[Q^\vee] = \bigoplus_{\beta\in Q^\vee} \mathbb{Q}_{q,t} Y^\beta \curvearrowright \mathbb{Q}_{q,t}[\Lambda]$ ($\mathbb{Q}_{q,t}$-linearly)

$\mathbb{Q}_{q,t}[\Lambda] \cong \bigoplus_{\lambda\in\Lambda} \mathbb{Q}_{q,t}\cdot E_\lambda$ as $\mathbb{Q}_{q,t}[Q^\vee]$-module

Questions For two matrices H and E with relation $HE = E(H+2)$, show that they are be simultaneously triangulated. intertwine

If $Hv = \lambda v$, then $H(Ev) = (\lambda+2)Hv$. Then pick an eigenvalue λ such that $\lambda + 2$ is not an eigenvalue, i.e. $Ev = 0$.

If $Hv = \lambda v$, then

$$H(Ev) = E(H+2)v$$
$$= E(\lambda+2)v = (\lambda+2)Ev.$$

$W_\lambda = \{v \in \mathbb{C}^n : Hv = \lambda v\}$

$W_\lambda \xrightarrow{E} W_{\lambda+2}$

Pick any an eigenvalue λ of H

s.t. $\lambda+2$ is not an eigen...

Learning seminar Macdonald

link

notes

1 Intertwine operators

1.1 Bernstein relations

$H_i = t^{1/2} T_i$

Denote $X^\lambda : f \mapsto e^\lambda f$. Then we have

$$H_i Y^\beta - Y^{s_i\beta} H_i = (t^{1/2} - t^{-1/2}) \frac{Y^{s_i\beta} - Y^\beta}{Y^{-\alpha_i^\vee} - 1},$$

$$H_i X^\lambda - X^{s_i\lambda} H_i = (t^{1/2} - t^{-1/2}) \frac{X^{s_i\lambda} - X^\lambda}{X^{\alpha_i^\vee} - 1}.$$

when $i = 0$
diagrams and names
duality

Standard proof. $Y^\beta = H_{t_\beta}$ β dom.

β dom $\Rightarrow$ $s_i\beta + \beta$ dom ($s_i\beta$ is NOT dom)

$Y^{s_i\beta} = Y^{s_i\beta+\beta} / Y^\beta = H_{t_{s_i\beta+\beta}} H_{t_\beta}^{-1}$

$T_i = ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - id) \qquad X^\lambda = e^\lambda$

$$T_i X^\lambda = \left(ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - id)\right) e^\lambda$$
$$= t e^{s_i\lambda} s_i + \frac{t-1}{e^{\alpha_i}-1}(e^{s_i\lambda} s_i - e^\lambda id)$$
$$= \cdots$$

duality $\ddot{H}_W$ double affine Hecke algebra DAHA.

$\hat{H}_W \subseteq \ddot{H}_W \supseteq \hat{H}_{W^\vee}$

affine Hecke algebra $\hat{H}_W$ $\qquad$ $\hat{H}_{W^\vee}$ affine Hecke algebra

$\mathbb{Q}_{q,t}[Q^\vee] \subseteq \hat{H}_W \supseteq H_W \subseteq \hat{H}_{W^\vee} \supseteq \mathbb{Q}_{q,t}[\Lambda]$

Y^β $\quad \mathbb{Q}_{q,t}[Q^\vee] \quad H_W$ finite Hecke algebra $\quad \mathbb{Q}_{q,t}[\Lambda] \quad X^\lambda$

1.2 Intertwine operators

For $i \in I$,

$$\tau_i = T_i + \frac{t-1}{Y^{-\alpha_i^\vee} - 1}.$$

Properties, intertwine + braid relations

Note: $Y^\beta \curvearrowright \mathbb{Q}_{q,t}[\Lambda]$ does not have eigenvalue 1.

$\tau_i Y^\lambda = Y^{s_i\lambda} \tau_i$ (intertwine)

$\underbrace{\tau_i \tau_j \cdots}_{m_{ij}} = \underbrace{\tau_j \tau_i \cdots}_{m_{ij}}$ (braid relation)

<u>Rmk</u>.

X^λ: (multiplication by) functions $\qquad X^\lambda$ operators

T_w: operators $\longleftrightarrow$ T_w operators

Y^β: operators $\qquad Y^\beta$ (multiplication by) functions

DL op in "Y"

$$\tau_i = T_i + \frac{t-1}{Y^{-\alpha_i^\vee}-1} = \frac{tY^{-\alpha_i^\vee}-1}{Y^{-\alpha_i^\vee}-1} s_i$$

$\{f(\alpha_i) s_i\}$ satisfies braid relation for any f.

$\tau_i = T_i + \dfrac{t-1}{Y^{-\alpha_i^\vee}-1}$

Theorem If $s_i\lambda > \lambda$ for some $i \in I$, i.e. $\langle \lambda, \alpha_i^\vee \rangle > 0$, then

$$E_{s_i\lambda} = \tau_i E_\lambda = \left(T_i + \frac{t-1}{q^{\langle \lambda, \alpha_i^\vee \rangle} t^{\langle \rho_\lambda, \alpha_i^\vee \rangle} - 1} \right) E_\lambda.$$

① eigenvalue of Y^β at E_λ $= q^{-\langle \beta, \lambda \rangle} t^{-\langle \beta, \rho_\lambda \rangle}$ $(s_i\lambda > \lambda)$

$=$ eigenvalue of $Y^{s_i\beta}$ at $E_{s_i\lambda}$ $= q^{-\langle s_i\beta, s_i\lambda \rangle} t^{-\langle s_i\beta, \rho_{s_i\lambda} \rangle}$

② $Y^{s_i\beta} E_{s_i\lambda} = Y^{s_i\beta} \tau_i E_\lambda = \tau_i Y^\beta E_\lambda$

$= (\cdots)\ \tau_i E_\lambda = (\cdots)\ E_{s_i\lambda}$.

$\Rightarrow \tau_i E_\lambda \in \mathbb{Q}_{q,t} E_{s_i\lambda}$

③ look at the coefficient of $e^{s_i\lambda}$,

$\tau_i E_\lambda = E_{s_i\lambda}$. □

Rmks: There is a version for $i = 0$; while in type A

① If we know E_λ for λ dominant then we know E_λ for all λ.

a) The above formula has a version for $i=0$.

b) For type A, you can use automorphism of root systems. ($(3,4,2,0)$ $\to (\overset{\frown}{1}, 3, 4, 2)$)

Example GL_2

$E_{(1,0)} = x_1$ $\quad T_1(x_1) = x_2$

$E_{(0,1)} = \left(T_1 + \dfrac{t-1}{qt-1}\right) E_{(1,0)}$

$= x_2 + \dfrac{t-1}{qt-1} x_1$

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② $q = 0$, the above formula reduces to

$E_{s_i\lambda}(q=0) = T_i\ E_\lambda(q=0)$

Key polynomials $K_\lambda = E_\lambda(q=\infty, t=\infty)$

$K_{s_i\lambda} = \pi_i K_\lambda$ proof ① recursion ✓ ② initial. ?

? $E_\lambda(q=\infty, t=\infty) \overset{?}{=} x^\lambda$ λ dominant

(this can be seen from "Cherednik inner product")

2 a Symmetric Macdonald

is NOT a non-sym Macdonald poly.

Y^β does not act on $\mathbb{Q}_{q,t}[P]^W$ But $f(Y)$ with f symmetric does. Sketch of the proof.

$E_\lambda \in \mathbb{Q}_{q,t}[\Lambda] \curvearrowleft Y^\beta$

$\mathbb{Q}_{q,t}[\Lambda]^W \cancel{\curvearrowleft} Y^\beta$ $\quad f(Y)$ $\quad f$ symmetric.

<u>Lemma</u> $\mathbb{Q}_{q,t}[Q^\vee] \curvearrowright \mathbb{Q}_{q,t}[\Lambda]$ restricts to

$\mathbb{Q}_{q,t}[Q^\vee]^W \curvearrowright \mathbb{Q}_{q,t}[\Lambda]^W$

<u>Sketch</u> $C_{w_0} = \sum_{w \in W} T_w$.

$\mathbb{Q}_{q,t}[\Lambda]^W =$ image of C_{w_0}

$(C_{w_0} f \text{ sym})$

$(f \text{ sym} \Rightarrow C_{w_0} f = (\cdots) f)$

$\mathbb{Q}_{q,t}[Q^\vee]^W$ commutes with C_{w_0}.

$(f(Y)\, C_{w_0} = C_{w_0} f(Y) \text{ if } f \text{ sym})$

ex. $\Rightarrow \mathbb{Q}_{q,t}[Q^\vee]^W \curvearrowright \mathbb{Q}_{q,t}[\Lambda]^W$ □

definition

Symmetric Macdonald poly $P_\lambda \in \mathbb{Q}_{q,t}[\Lambda]^W$ for λ dominant, is the unique poly s.t.

① $f(Y) P_\lambda = f(q^{-\lambda} t^{-\rho}) P^\lambda$ $\quad f \in \mathbb{Q}_{q,t}[Q^\vee]^W$

② $P_\lambda = m_\lambda + (\text{other})$

$f(q^{-\lambda} t^{-\rho}) = f(Y)\big|_{Y^\beta \mapsto q^{-\langle \lambda, \beta \rangle} t^{-\langle \rho, \beta \rangle}}$

$m_\lambda = \sum_{\mu \in W\lambda} e^\mu$ = orbit sum.

= monomial basis.

Lemma For a dominant λ,

$$P_\lambda \in \operatorname{span}(E_\mu : \mu \in W\lambda).$$

$\text{RHS} = \left\{ g \in \mathbb{Q}_{q,t}[\Lambda] : \begin{array}{l} \forall f \in \mathbb{Q}_{q,t}[Q^\vee]^W \\ f(Y) g = f(q^{-\lambda} t^{-\rho}) g \end{array} \right\}$

$\cup\!\!|$

$\operatorname{span}(P_\lambda) = \left\{ g \in \mathbb{Q}_{q,t}[\Lambda]^W : \begin{array}{l} \forall f \in \mathbb{Q}_{q,t}[Q^\vee]^W \\ f(Y) g = f(q^{-\lambda} t^{-\rho}) g \end{array} \right\}$

$\mathbb{Q}_{q,t}[Q^\vee] \curvearrowright \mathbb{Q}_{q,t}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{Q}_{q,t} E_\lambda$

$\cup\!\!|$ $\qquad \|$

$\mathbb{Q}_{q,t}[Q^\vee]^W \curvearrowright \mathbb{Q}_{q,t}[\Lambda] = \bigoplus_{\lambda \text{ dom}} \left(\bigoplus_{\mu \in W\lambda} \mathbb{Q}_{q,t} E_\mu \right)$

$\cup\!\!|$

$\mathbb{Q}_{q,t}[\Lambda]^W = \bigoplus_{\lambda \text{ dom}} \mathbb{Q}_{q,t} P_\mu$.

Example GL_2

$E_{(1,0)} = x_1$

$E_{(0,1)} = \frac{-t+1}{-qt+1} x_1 + x_2$

$\Downarrow$

$P_{(1,0)} = x_1 + x_2$

$E_{(2,0)} = x_1^2 + \left(\frac{-qt+q}{-qt+1}\right) x_1 x_2$

$E_{(0,2)} = \left(\frac{-t+1}{-q^2t+1}\right) x_1^2 + \left(\frac{-qt+q-t+1}{-q^2t+1}\right) x_1 x_2 + x_2^2$

$\Downarrow$

$P_{(2,0)} = \left(1 - \left(\frac{-t+1}{-q^2t+1}\right)\right) E_{(2,0)} + E_{(0,2)}$

$= x_1^2 + \left(\frac{-qt+q-t+1}{-qt+1}\right) x_1 x_2 + x_2^2.$

When λ is dominant

$$P_\lambda = \frac{1}{W_\lambda(t)} \mathrm{Symm}(E_\lambda)$$

left as an exercise.

3 Type A (Sketch)

Theorem P_λ is a symmetric function.

Type A, S_n, $\tilde{S}_n$, H_n, $\tilde{H}_n$

$Y_1, \dots, Y_n \in \tilde{H}_n$ $\quad Y_1 + \cdots + Y_n \in \mathbb{Q}_{q,t}[Q^\vee]^W$ $\quad \mathbb{Z}^n$

① already has distinct eigenvalues on $P_\lambda(x_1, \dots, x_n)$

② can be computed explicitly known as "Macdonald operator"

$$\sum_{i=1}^{n} \left(\prod_{j \neq i} \frac{t^{1/2} x_i - t^{-1/2} x_j}{x_i - x_j} \right) d_i^{-1}$$

$d_i f(\dots, x_i, \dots) = f(\dots, q x_i, \dots)$

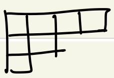
To take $n \to \infty$ limit, we need a modification

$$\frac{1}{q^{-1}-1}\left(t^{\frac{n-1}{2}} (Y_1 + \cdots + Y_n) - (t^1 + \cdots + t^n) \right)$$

Its eigenvalue at P_μ

$$\sum_{(i,j)\in\mu} \bar{q}^{\,j-1} t^{i-1}$$

$(\bar{q} = q^{-1})$

e.g. 

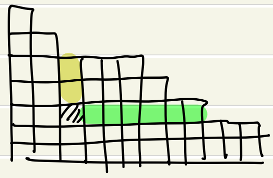
$$\sum \left\{ \begin{matrix} 1, \bar{q}, \bar{q}^2, \bar{q}^3, \\ t, \bar{q}t \\ t^2, \end{matrix} \right\}$$

$\Rightarrow P_\mu \in \Lambda.$

3.1 Other Macdonald functions

$$J_\lambda = P_\lambda \cdot \prod_{\square \in \lambda} \left(1 - q^{a(\square)} t^{\ell(\square)+1}\right)$$

"integral form"



$a(▨) = 6$

$\ell(▨) = 2$

$$H_\lambda = J_\lambda\left[\frac{Z}{1-t}\right] = J_\lambda \begin{pmatrix} x_1, t x_1, t^2 x_1, \cdots \\ x_2, t x_2, t^2 x_2, \cdots \\ \vdots \quad \vdots \quad \vdots \quad \ddots \end{pmatrix}$$

"transformed"

"Z" $= x_1 + x_2 + \cdots$ all variables

$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$

$p_k = x_1^k + x_2^k + x_3^k + \cdots$

$p_k\left[\frac{Z}{1-t}\right] = x_1^k + (t x_1)^k + (t^2 x_1)^k + \cdots$

$+ x_2^k + (t x_2)^k + (t^2 x_2)^k + \cdots$

$+ \cdots + \cdots + \cdots + \cdots$

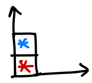
$= p_k + t^k p_k + t^{2k} p_k + \cdots$

$= \frac{1}{1-t^k} p_k$

$f \mapsto f\left[\frac{Z}{1-t}\right]$ invertible

$$\tilde{H}_\lambda = t^{n(\lambda)} H_\lambda|_{t \leftrightarrow t^{-1}}.$$

balance q and t

Example  $a(▨) = 0$, $a(▨) = 0$, $\ell(▨) = 1$, $\ell(▨) = 0$ $\quad n(\lambda) = 1$

$P_{(1,1)} = x_1 x_2 + \cdots = e_2 = m_{(1,1)}.$

$J_{(1,1)} = (1-t)(1-t^2) P_{(1,1)}$

$= (t-1)(t^2-1) \frac{p_1^2 - p_2}{2}$

$$H_{(1,1)} = (t-1)(t^2-1)\left(\frac{\left(\frac{1}{1-t}\right)^2 p_1 - \frac{1}{1-t^2} p_2}{2} \right)$$

$= \frac{t+1}{2} p_1 + \frac{t-1}{2} p_2 = (t+1) m_{(1,1)} + t m_{(2)}.$

$\tilde{H}_{(1,1)} = t\left((t^{-1}+1) m_{(1,1)} + t^{-1} m_{(2)}\right)$

$= (t+1) m_{(1,1)} + m_{(2)}.$

$= t s_{(1,1)} + s_{(2)}.$

$P_{(2,0)} = m_{(1,1)} + \frac{(q+1)(t-1)}{qt-1} m_{(2)}.$

$J_{(2,0)} = (\cdots)$

$H_{(2,0)} = (q+1) m_{(1,1)} + m_{(2)}$

$\tilde{H}_{(2,0)} = (q+1) m_{(1,1)} + m_{(2)}$

$= q s_{(1,1)} + s_{(2)}.$

In general: $\tilde{H}_\lambda|_{q \leftrightarrow t} = \tilde{H}_{\lambda'}$