Plethysm. Quasi-symmetric functions and superization.

- A combinatorial formula for the character of the diagonal coinvariants. Section 2.

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0. Recall

$$\Lambda = \varprojlim \mathbb{Q}[x_1, \cdots x_n]^{S_n}.$$

$\rotatebox{90}{$\in$}$

f : a function over space $\{(x_i)_{i=1}^{\infty} : x_i = 0 \text{ for almost all } i\}$.

For λ - partition.

1° monomial : $\quad m_\lambda = \sum_{\alpha \in S_n \lambda} x^\alpha = \frac{1}{|S_\lambda|} \sum_{w \in S_n} x^{w\lambda}$

2° elementary : $\quad e_\lambda = e_{\lambda_1} \cdot e_{\lambda_2} \cdots \qquad e_r = \sum_{1 \le i_1 < i_2 < \cdots} x_{i_1} x_{i_2} \cdots$

3° complete : $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$ $\quad h_r = \sum_{1 \le i_1 \le i_2 \le \cdots} x_{i_1} x_{i_2} \cdots$

4° power sum: $P_\lambda = P_{\lambda_1} P_{\lambda_2} \cdots$ $\quad P_r = x_1^r + x_2^r + \cdots$

5° Schur : $S_\lambda = \sum_{T \in SSYT(\lambda)} x^T$

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} P_\lambda \qquad \lambda = (1^{m_1} 2^{m_2} \cdots)$$

$$z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots$$

$$e_n = \sum_{\lambda \vdash n} (-1)^{|\lambda| - l(\lambda)} z_\lambda^{-1} P_\lambda$$

involution ω

$$h_\lambda \Leftrightarrow e_\lambda$$
$$P_\lambda \Leftrightarrow (-1)^{|\lambda| - l(\lambda)} P_\lambda$$
$$S_\lambda \Leftrightarrow S_{\lambda'}$$

Hall inner product

$$\langle S_\lambda \, S_\mu \rangle = \delta_{\lambda\mu} = \langle h_\lambda \, m_\mu \rangle.$$

$\langle \omega a \, \omega b \rangle = \langle a \, b \rangle.$

1. Plethysm.

"generalized substitution"

For $f \in \Lambda_{qt} := \mathbb{Q}_{qt} \otimes \Lambda$.

$$f(\bigcirc, \square, \triangle, \diamond, \cdots) = f[\bigcirc + \square + \triangle + \diamond + \cdots].$$

$$f[2x+y+z+4] = f(x, y, z, 1, x, 1, 1, 1, 0, \cdots).$$

Def: Suppose that the symmetric function $f \in \Lambda_{qt}$ is a sum of monomials $f = \sum_{i \geq 1} x^{a^i}$. Given $g \in \Lambda_{qt}$ define the plethysm $g[f]$ by

$$g[f] = g(x^{a^1}, x^{a^2}, \cdots).$$

$X = x_1 + x_2 + \cdots$ $\qquad g[X] = g(x_1, x_2, \cdots) = g$.

$X[f] = x^{a^1} + x^{a^2} + \cdots = f$.

$p_n = x_1^n + x_2^n + \cdots$ $\qquad f[p_n] = f(x_1^n, x_2^n, \cdots) = \sum_{i \geq 1} x^{a^i n} = p_n[f]$ ✓

$f=$

$A \in \Lambda_{q,t}$

Note: $(af+bg)[A] = af[A]+bg[A]$ ✓

$(fg)[A] = f[A]\cdot g[A]$.

Def (Plethysm). For $f, A \in \Lambda_{q,t}$, we define $\varphi: f \mapsto f[A]$ to be the unique map $\Lambda_{q,t} \to \Lambda_{q,t}$ such that

(A) $(cf+gh)[A] = cf[A]+g[A]h[A]$ for $c \in \mathbb{Q}_{q,t}$ $\mathbb{Q}(q,t)$

(P) $p_n[A] = A|_{x_i \mapsto x_i^n,\ q \mapsto q^n,\ t \mapsto t^n}$.

In general, if $A = A(z, y, q, x \cdots)$ is any function, $f = f(\underline{z}, y, q, x \cdots)$ is any function sysmetric in z, we define $f[A]$ by

(A) $(cf+gh)[A] = cf[A]+g[A]h[A]$ for $c \in \mathbb{Q}_{q,t}$

(P) $p_n[A] = A(z^n, y^n, q^n, x^n)$.

eg: 1° For any $f \in \Lambda_{grt}$.

$X = x_1 + x_2 \cdots$. $\underline{f[X] = f}$

(A) $f \mapsto f[X] = f$ id

(P) $P_r[X] = X|_{x_i \mapsto x_i^r} = x_1^r + x_2^r + \cdots = P_r$.

2°. $X = x_1 + x_2 + \cdots + x_n$ $f[X] = f(x_1, \cdots x_n)$.

eg: For $X = x_1 + x_2 + \cdots$ consider $f[-X]$.

$$f = \sum c_\lambda h_\lambda$$

$$f[-X] = \sum c_\lambda h_\lambda[-X]$$

$$= \sum c_\lambda e_\lambda(-x_1, -x_2, \cdots)$$

$$= \omega f(-x_1, -x_2, \cdots).$$

$$\underline{h_n[-X]} = \sum_{\lambda \vdash n} z_\lambda^{-1} \underline{P_\lambda}[-X]$$

$$= \sum_{\lambda \vdash n} z_\lambda^{-1} \prod_{i=1}^{\ell(\lambda)} P_{\lambda_i}[-X]$$

$$= \sum_{\lambda \vdash n} z_\lambda^{-1} \prod_{i=1}^{\ell(\lambda)} -X[P_{\lambda_i}]$$

$$= \underline{\sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} z_\lambda^{-1} P_\lambda} = (-1)^n e_n = \underline{e_n(-x_1, -x_2 \cdots)}.$$

eg: Recall the coproduct $\Lambda \to \Lambda \otimes \Lambda$.

$$f(x,y) := f(x_1, x_2, \dots, y_1, y_2, \dots)$$
$$= \sum f_1(x_1, x_2, \dots)\, f_2(y_1, y_2, \dots).$$

Setting $X = x_1 + x_2 + \cdots$ $\quad Y = y_1 + y_2 + \cdots$

$$\underline{f[X+Y]} = f(x_1, x_2, \dots, y_1, y_2, \dots)$$
$$= \sum f_1[X]\, f_2[Y]$$
$$= \sum f_1(x_1, x_2, \dots)\, f_2(y_1, y_2, \dots).$$

$$\varphi : f \mapsto f[X+Y].$$

$$p_r[X+Y] = p_r[X] + p_r[Y].$$

eg. Setting $Z = z_1 + z_2 + \cdots$ $W = w_1 + w_2 + \cdots$

$\omega^{W} f[Z+W]$, consider as a symmetric function in W variables with functions of Z as coefficient.

f

$$\omega^{W} f[Z+W] \big|_{z^{\mu} m_{\eta}} = \langle f, e_{\eta} h_{\mu} \rangle.$$

$\hookrightarrow$. $f|_{z^{\mu}} = \langle f, h_{\mu} \rangle$.

$$\langle \omega^{W} f[Z+W], h_{\mu} h_{\eta} \rangle = \langle f \; e_{\eta} h_{\mu} \rangle.$$

2. Quasi-symmetric function.

Def: If for all composition $(\alpha_1, \cdots \alpha_k)$, $f|_{x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}} = f|_{x_{j_1}^{\alpha_1} \cdots x_{j_k}^{\alpha_k}}$, then

QSym.

$i_1 < \cdots < i_k$ $\quad j_1 < \cdots j_k$.

we call $f \in \mathbb{Q}[x_1, \cdots x_n]$ is a quasi-symmetric function.

Strong composition. $\alpha = (\alpha_1, \cdots \alpha_k)$ $\forall \alpha_i > 0$ $\quad 1 \le i \le k$.

$$|\alpha| = \alpha_1 + \cdots + \alpha_k.$$

$$\ell(\alpha) = k.$$

$$\alpha \Leftrightarrow (|\alpha|, S = \{\alpha_1, \alpha_1 + \alpha_2, \cdots \alpha_1 + \cdots + \alpha_{k-1}\}).$$

$$4131 \Leftrightarrow (9, \{4\ 5\ 8\}).$$

$$0\ 0\ 0\ 0 | 0 | 0\ 0\ 0 | 0$$

1. Monomial quasi-symmetric functions.

For a strong composition α,

$$M_\alpha = \sum_{\beta^+ = \alpha} x^\beta$$

β^+ → delete all 0's in β

→ filling of α such that $\begin{cases} a \; b \Rightarrow a = b \\ a \,|\, b \Rightarrow a < b \end{cases}$

$$M_{4131} = \sum_{a<b<c<d} x_a^4 x_b^1 x_c^3 x_d^1$$

2. Fundamental (Gessel) quasi-symmetric function.

For a strong composition α

$$F_\alpha = \sum_{\beta \vDash \alpha} M_\beta$$

β refines α.

$$(\beta_1 \cdots \beta_n)$$

$$\alpha_1 \ \alpha_2 \ \cdots \ \alpha_k .$$

(4131)

$(1, \cdots 1)$ $\quad (\underbrace{1\,3}_{4}\,1\,3\,1)$.

filling of α, such that $\begin{cases} ⓐⓑ \Rightarrow a \leq b. \\ ⓐ|ⓑ \Rightarrow a < b. \end{cases}$

$$\circ\ \circ | \circ\ \circ\ \circ | \circ | \circ\ \circ\ \circ | \circ$$

$\alpha \Leftrightarrow (n, S)$.

$F_{n,S} := F_\alpha$

$$F_{4131} = \sum_{i_1 \leq i_2 \leq i_3 \leq i_4 < i_5 < i_6 \leq i_7 \leq i_8 < i_9} x_{i_1} \cdots x_{i_9}.$$

$$= \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_n \\ a \in S \Rightarrow i_a < i_{a+1}}} x_{i_1} \cdots x_{i_n}$$

$$= \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_n \\ i_a = i_{a+1} \Rightarrow a \notin S}} x_{i_1} \cdots x_{i_n}.$$

Note $\quad F_{1^n} = e^n \quad , F_n = h_n$.

involution

$w: QSym \to QSym$.

$F_\alpha \mapsto F_{\alpha'}$ ← dual composition of α.

$M_\alpha \mapsto (-1)^{\ell(\alpha)} \sum_{\alpha \vDash \beta} M_\beta$.

o o o o | o | o o o | o

o | o | o | o o o | o | o o

1 1 1 3 1 2

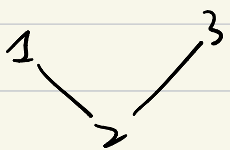
eg: (P-partition)

Definition: Assume we have a partial order P on $[n]$. A P-partition is a function $P \xrightarrow{T} \mathbb{Z}_{>0}$ with the following properties.

1. $a <_P b$ and $a < b$. $\Rightarrow T(a) \le T(b)$

2. $a <_P b$ and $a > b$ $\Rightarrow T(a) < T(b)$.

eg.



$2 <_P 1 \quad 2 > 1 \Rightarrow T(2) < T(1)$

$2 <_P 3. \quad 2 < 3 \Rightarrow T(2) \le T(3)$.

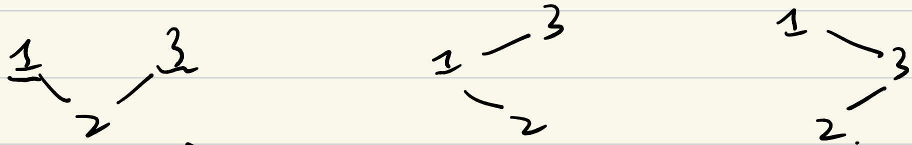
$\mathcal{A}(P)$: the set of (P,w)-partition. where $w: P\to[n]$ be a partition.

eg: $w\uparrow$, P-partition is weakly increasing map $T:P\to[\infty]$. ← property 1

$w\downarrow$, P-partition is strictly increasing map $T:P\to[\infty]$ ← property 2

prop: $\mathcal{A}(P) = \bigsqcup_{P'} \mathcal{A}(P')$ P' is linear extension of P.

eg:



$$\left\{ T \;\middle|\; \begin{matrix} T(2)<T(1) \\ T(2)\le T(3) \end{matrix} \right\} = \{T \mid T(2)<T(1)\le T(3)\} \sqcup \{T \mid T(2)\le T(3)<T(1)\}$$

$T: P\to[\infty]$.

$$F_P(x) = \sum_{T\in\mathcal{A}(P)} x_T \,, \quad x_T := \prod_{i\in P} x_{T(i)}.$$

P' is a chain $\{w_1 <_P w_2 < \cdots <_P w_n\}$

$\text{des}P' := \{ i\in[n-1] \;\; w_i > w_{i+1} \}$.

$$= \sum_{\substack{i_1\cdots i_n \\ a<_P b,\ a<b \Rightarrow i_a\le i_b \\ a<_P b,\ a>b \Rightarrow i_a< i_b}} x_{i_1}\cdots x_{i_n} = \sum_{P'} F_{P'}. = \sum_{P'} F_{n,\text{des}P'}$$

$$wF_P = \sum_{\substack{i_1\cdots i_n \\ a<_P b,\ a<b \Rightarrow i_a<i_b \\ a<_P b,\ a>b \Rightarrow i_a\le i_b}} x_{i_1}\cdots x_{i_n} = F_Q.$$

for Q, the order $a <_Q b$ iff $n-a <_P n-b$.

wS_λ.

$$S_\lambda = \sum_{\underbrace{T\in SSYT(\lambda)}} z^T$$

↳ SSYT is a P-partition.

$$\begin{array}{ccccc} 2 & 2 & 5 & 5 & 20 \\ 5 & 10 & 20 & & \\ 10 & & & & \end{array} \longrightarrow \begin{array}{ccccc} 1 & 2 & 4 & 5 & 9 \\ 3 & 7 & 8 & & \\ 6 & & & & \end{array}.$$

For each $T\in$ SSYT define $S = std(T) \in SSYT(\lambda)$ such that

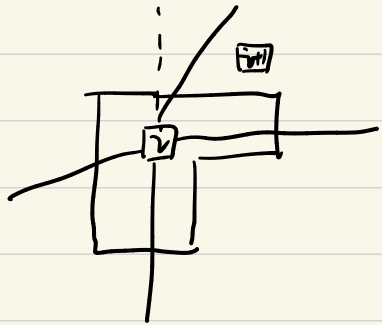
- if $T(\square_1) < T(\square_2)$, then $S(\square_1) < S(\square_2)$
- if $T(\square_1) = T(\square_2)$ and $\square_1$ is left to the $\square_2$, then $S(\square_1) < S(\square_2)$.

$$s_\lambda = \sum_{S \in SYT} F_{n,\,\mathrm{des}(S)}$$

$\hookrightarrow$ $\mathrm{des} = \{ i \in [n-1] : \boxed{i+1} \text{ is lower than } \boxed{i} \}$.

$$\omega s_\lambda = \sum_{S \in SYT} F_{n,\,[n-1]\setminus \mathrm{des}(S)}$$

$$= s_{\lambda'}$$



3. Superization.

SSYT: $T: \lambda \to A_+ := \{1 < 2 < \cdots\}$. weakly increasing on each row
strictly increasing on each column.

$SSYT(\lambda) := \{$ semistandard tableau $T: \lambda \to A_+ \}$.

"super" version.

Super tabelau: $T: \lambda \to A_\pm := \{1 < \bar{1} < 2 < \bar{2} < \cdots\}$. weakly increasing on each row and column.

such that: $a = i$ form horizontal strip
$a = \bar{i}$ form vertical strip.

$$\begin{array}{ccccc} 1 & 1 & \bar{1} & 2 & 2 \\ \bar{1} & 2 & \bar{2} & & \\ \bar{1} & \bar{2} & & & \end{array}$$

$SSYT_\pm(\lambda) = \{$ super tableaux $T: \lambda \to A_\pm \}$.

f: symmetric function in z.

$$s_\lambda = \sum_{T \in SSYT(\lambda)} z^T$$

"super" version

Setting $Z = z_1 + z_2 + \cdots$ $W = w_1 + w_2 + \cdots$

$z_1 + z_2 + \cdots$

$$\underbrace{\omega^W f[Z+W]}$$

$$\tilde{S}_\lambda(z, w) = \sum_{T \in SSYT_\pm(\lambda)} z^T$$

$$\underbrace{\omega^W S_\lambda[Z+W]} = \sum \langle \underbrace{S_\lambda\, h_\mu e_\nu} \rangle z^\mu w^\nu$$

$$z^T$$

Gessel's quasi-symmetric function.

$$F_{n,S}(z) = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \Rightarrow i \notin S}} z_{a_1} \cdots z_{a_n} \quad : \quad S_\lambda = \sum_{T \in SYT(\lambda)} F_{|\lambda|\, des(T)}(z).$$

"super" version

$$\tilde{F}_{nS}(z,w) = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \in A_+ \Rightarrow i \notin S \\ a_i = a_{i+1} \in A_- \Rightarrow i \in S}} z_{a_1} z_{a_2} \cdots z_{a_n}.$$

$\hookrightarrow \{\bar{1} < \bar{2} < \cdots\}$

$$\tilde{S}_\lambda(z,w) = \sum_{T \in SYT(\lambda)} \tilde{F}_{|\lambda|\, des(T)}(z,w)$$

$$\parallel$$

$$\sum_{T \in SSYT_\pm(\lambda)} z^T$$

✓

$$\begin{array}{lllll} 1 & 1 & \bar{1} & 2 & 2 \\ \bar{1} & 2 & \bar{2} & & \\ \bar{1} & \bar{2} & & & \end{array} \longrightarrow \begin{array}{lllll} 1 & 2 & 3 & 7 & 8 \\ 4 & 6 & 9 & & \\ 5 & 10. & & & \end{array}$$

$$T : \lambda \to A_{\pm} \xrightarrow{std} S : \lambda \to \{1, 2, \cdots n = |\lambda|\}.$$

Such that

1. $T \circ S^{-1}$ weakly increasing function. $1\,1\,\bar{1}\,\bar{1}\,\bar{1}\,2\,2\,2\,\bar{2}\,\bar{2}$.
2. if $T\circ S^{-1}(j) = T\circ S^{-1}(j+1) = \cdots = T\circ S^{-1}(k) = a$ then.

$$\{j, j+1, \cdots k-1\} \cap des(S) = \begin{cases} \phi & a \text{ positive} \\ \{j, \cdots k-1\} & a \text{ negative.} \end{cases}$$

T', satisfy 1, 2. $T = T' \circ S \xrightarrow{std} S$.

$$\tilde{F}_{|\lambda|\, des(S)}(z, w). = \sum_{std(T) = S} z^T$$

$$\sum_{T \in SYT(\lambda)} \tilde{F}_{|\lambda|\, des(S)}(z, w) = \sum_{T \in SYT(\lambda)} \sum_{std(T) = S} z^T = \sum_{T \in SSYT_{\pm}(\lambda)} z^T$$

Coro: if $f(z) = \sum_s c_s F_{n,s}(z)$, then $\tilde{f}(z,w) = \sum_s c_s \widetilde{F}_{n,s}(z,w)$.