Applications of Macdonald polynomials

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1 Recall

E_λ λ composition (nonsym)

P_λ λ partition (sym)

defined by DL operator affine Hecke

J_λ integral form of P_λ

H_λ transformed

$\tilde{H}_\lambda$ exchange version

Theorem

For $\mu \vdash n$, the symmetric function

$$\tilde{H}_\mu \in \mathbb{Q}(q,t) \otimes \Lambda$$

variables in Λ: $Z = (z_1, z_2, z_3, \cdots)$

is characterized by

(1) $\tilde{H}_\mu[Z(1-q)] \in \operatorname{span}(s_\lambda : \lambda \geq \mu)$; $\approx J_\mu$

(2) $\tilde{H}_\mu[Z(1-t)] \in \operatorname{span}(s_\lambda : \lambda \geq \mu')$;

(3) $\langle h_n, \tilde{H}_\mu \rangle = 1$.

① $\Lambda \ni f \mapsto f[\frac{Z}{1-q}] = \underset{\| }{f[Z]}\begin{pmatrix} z_1 & z_2 & z_3 & \cdots \\ qz_1 & qz_2 & qz_3 & \cdots \\ q^2z_1 & q^2z_2 & q^2z_3 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$

$\frac{Z}{1-q} = Z + qZ + q^2Z + \cdots$

invertible with inverse $f \mapsto f[Z(1-q)]$

② $f[X-Y] = \sum f_1(x_1, x_2, \cdots)(\omega f_2)(-y_1, -y_2, \cdots)$

$\Delta f = \sum f_1 \otimes f_2$

Take in $X = Z$, $Y = qZ$

$f[Z(1-q)] = \sum f_1(z_1, z_2, \cdots)(\omega f_2)(-qz_1, -qz_2, \cdots)$

$\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$

$f(x_1, x_2, \cdots)$

$s_\lambda \mapsto \sum s_\mu \otimes s_{\lambda/\mu}$

$p_k \mapsto p_k \otimes p_k$

$e_k \mapsto \sum_{r=0}^{k} e_r \otimes e_{k-r}$

2 Symmetric power

For space X, ("variety")

$S^n X = X^n / S_n$ (X^n: n tuple of points in X; S_n: symmetric group)

$= \{\text{multi-sets of } X \text{ of order } n\}$

S_n-orbits of $(x_1, \cdots, x_n) = [x_1] + \cdots + [x_n] \in \mathbb{Z}^{\oplus X}$ 0-cycle

Classical example $\mathbb{C}^n / S_n$

$e: S^n\mathbb{C} \cong \mathbb{C}^n$?

e_i = the i-th ele. sym polynomial.

$\mathbb{C}^n \to S^n\mathbb{C}$, $(x_1, \cdots, x_n) \mapsto (e_1(x), \cdots, e_n(x))$

① surjection $\Longleftrightarrow$ $\mathbb{C}$ is algebraically closed.

② injection $\Longleftrightarrow$ $(\cdots)$

$S^n\mathbb{C} \xleftarrow[\sim]{\text{roots}} \left\{ \begin{matrix} \text{polynomials of the form} \\ x^n + (\text{lower terms}) \end{matrix} \right\} \xrightarrow[\sim]{\text{coefficients}} \mathbb{C}^n$

$\mathrm{Fun}(S^n\mathbb{C}) = \mathbb{C}[x_1, \cdots, x_n]^{S_n}$

$= \mathbb{C}[e_1, \cdots, e_n] = \mathrm{Fun}(\mathbb{C}^n)$

Next step: $\mathbb{C} \rightsquigarrow \mathbb{C}^2$

$S^n\mathbb{C}^2 = (\mathbb{C}^2)^n / S_n$ diagonal invariant.

$\mathrm{Fun}(S^n\mathbb{C}^2) = \mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_n]^{S_n}$

S_n acts "diagonally"

$w\, x_i = x_{w(i)}$

$w\, y_i = y_{w(i)}$

Example

$(n=1)$ $S^n\mathbb{C}^2 = \mathbb{C}^2$

$(n=2)$ $\mathbb{C}[x_1, x_2, y_1, y_2] = R[x, y]$

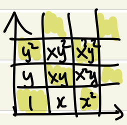
$\cup\!\mathsf{I}$

$R = \mathbb{C}[x_1 + x_2, y_1 + y_2]$

$x = x_1 - x_2$, $y = y_1 - y_2$ $x \mapsto -x$, $y \mapsto -y$

$R[x,y]^{S_2} = \{f(x,y) \in R[x,y] : f(-x,-y) = f(x,y)\}$

$= \operatorname{span}_R(x^a y^b : a + b \in 2\mathbb{Z})$



$= R[a,b,c]/(bc - a^2)$

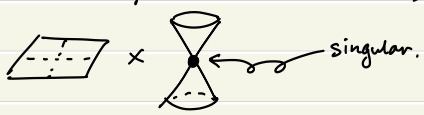
$a = xy$

$b = x^2$

$c = y^2$

$= \mathbb{C}[x_1+x_2, y_1+y_2] \otimes \mathbb{C}[a,b,c]/(bc-a^2)$

$S^2\mathbb{C}^2 \cong \mathbb{C}^2 \times \{(a,b,c) \in \mathbb{C}^3 : bc = a^2\}$.



3 Hilbert schemes

def
Hilbert{Chow morphism

$H_n = \mathrm{Hilb}^n \mathbb{C}^2$ ideal.
$= \{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \}$

$\pi : H_n \to S^n\mathbb{C}^2$ Hilbert-Chow morphism.

$I \longmapsto [\mathbb{C}[x,y]/I]$ "support with multiplicity"

$\dim \mathbb{C}[x,y]/I = n$
x, y comm

multiplication by x $= \begin{bmatrix} x_1 & \cdots & * \\ & \ddots & \vdots \\ & & x_n \end{bmatrix}$ multiset $[(x_1,y_1)] + \cdots + [(x_n,y_n)] \in S^n\mathbb{C}^2$.

multiplication by y $= \begin{bmatrix} y_1 & \cdots & * \\ & \ddots & \vdots \\ & & y_n \end{bmatrix}$ does not depend on the choice of basis.

Th. The Hilbert scheme H_n is a smooth variety, and Hilbert-Chow morphism π is a resolution of singularities. $S^n\mathbb{C}^2 \supseteq (\mathbb{C}^2)^n_{\text{distinct}}/S_n$

- $\mathrm{Hilb}^n \mathbb{A}^d$ is smooth if and only if $d \le 2$ or $n \le 3$.
- $\mathrm{Hilb}^n \mathbb{A}^d$ is irreducible for all d and $n \le 7$, see [163].
- $\mathrm{Hilb}^n \mathbb{A}^3$ is irreducible for $n \le 11$, see [52] and the references therein. See also [116, 233] for $n = 9, 10$.
- $\mathrm{Hilb}^n \mathbb{A}^3$ is reducible for $n \ge 78$, see [125].
- $\mathrm{Hilb}^{13}(\mathbb{A}^6)$ is nonreduced by Szachniewicz's work [218] .

Example

$S^n\mathbb{C}^2$ for $n = 1, 2$.

$(n=1)$ $H_1 = \{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = 1 \}$
$= \mathbb{C}^2$. must be a maximal ideal.
$m_p = \langle x - x_1, y - y_1 \rangle \overset{1:1}{\longleftrightarrow} P = (x_1, y_1) \in \mathbb{C}^2$.

$H_1 \xrightarrow{\sim} S^1\mathbb{C}^2$

$(n=2)$ $H_2 = \{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = 2 \}$.
$\downarrow$ $\qquad I \mapsto [p] + [q]$.
$S^2\mathbb{C}^2$

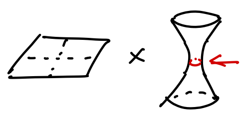
① $p \neq q$, $I = m_p \cdot m_q = m_p \cap m_q$.

② $p = q$, $I \supseteq m_p^2$.

Consider $p = (0,0)$. $m_p = \langle x, y \rangle$

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \\ y^2 & xy^2 & x^2y^2 & \cdots \\ y & xy & x^2y & \cdots \\ 1 & x & x^2 & \cdots \end{array} \qquad \begin{array}{cccc} \vdots & \vdots & \vdots & \\ y^2 & xy^2 & x^2y^2 & \cdots \\ y & xy & x^2y & \cdots \\ 1 & x & x^2 & \cdots \end{array}$$
$\qquad m_p \qquad\qquad\qquad m_p^2$

The choice of I = choice of one-dim subspace of $\mathbb{C}x \oplus \mathbb{C}y$.
$= \mathbb{P}^1$.

$H_2 =$  $\leftarrow \mathbb{P}^1$

4 Punctured Hilbert schemes

definition
Garsia{-Haiman algebra
difference of scheme and variety

actually flat
$$\begin{array}{ccc} X_n & \longrightarrow & (\mathbb{C}^2)^n \\ \rho \downarrow & & \downarrow {}^*/S_n \\ H_n & \xrightarrow{\pi} & S^n\mathbb{C}^2 \end{array}$$

$X_n = \{ (I, p_1, \ldots, p_n) : \pi(I) = [p_1] + \cdots + [p_n] \}$ punctured Hilbert scheme.
S_n acts on it

For each $\mu \vdash n$

 $\overset{1:1}{\longleftrightarrow}$ $I_\mu =$

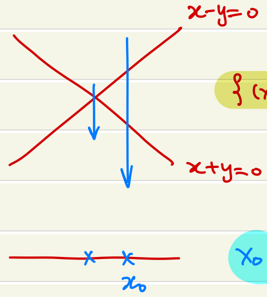
$(4,2,1)$ $\qquad\qquad \cap$
(French notation) $\qquad H_n$

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \\ y^3 & xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & \cdots \\ y^2 & xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & \cdots \\ y & xy & x^2y & x^3y & x^4y & \cdots \\ 1 & x & x^2 & x^3 & x^4 & \cdots \end{array}$$

Garsia-Haiman algebra: bi-graded $\deg x = (1,0)$ $\deg y = (0,1)$

$R_\mu = \mathrm{Fun}[\underbrace{\rho^{-1}(I_\mu)}]$.
as a scheme, not a variety.

Rmk. As an S_n-rep, R_μ can be described by certain determinants.



$\{ (x,y) \in \mathbb{C}^2 : x^2 = y^2 \} \subset \mathbb{C}^2$

$\downarrow \qquad\qquad \downarrow pr_1$

$x_0 \in \mathbb{C}$ $\qquad\qquad \mathbb{C}$

$\mathbb{C}[x,y]/(x^2 - y^2) \leftarrow \mathbb{C}[x,y]/(x^2-y^2, x-x_0)$

$\uparrow \cup \qquad\qquad \downarrow$ $\qquad$ Fun(fibre at x_0)

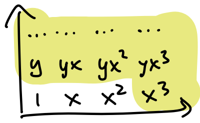
$\mathbb{C}[x] \longrightarrow\!\!\!\!\rightarrow \mathbb{C}[x]/(x-x_0)$

When $x_0 \neq 0$, $\mathbb{C}[x,y]/(x^2-y^2, x-x_0)$
$= \mathbb{C}[y]/(y^2 - x_0^2)$
$= \underbrace{\mathbb{C}[y]/(y-x_0) \oplus \mathbb{C}[y]/(y+x_0)}_{2\ \dim}$.

When $x_0 = 0$, $\mathbb{C}[x,y]/(x^2-y^2, x-x_0)$
$= \mathbb{C}[y]/(y^2) = \underbrace{\mathbb{C} \oplus \mathbb{C}y}_{2\ \dim.}$.

Example

e.g. $n=3$



Let us compute $I_{(n)} = \langle x^n, y \rangle$.

$$p^{-1}(I_{(n)}) \overset{\text{as a set}}{=\!=} \{ (I_{(n)}, p_1, \cdots, p_n) : \overset{n[0]}{\pi(I_{(n)})} = [p_1]+\cdots+[p_n] \}$$

$= \mathrm{pt}$ must be 0.

Parameters $e = (e_1, \cdots, e_n)$

$c = (c_1, \cdots, c_n)$

$$I_{e,c} = \left\langle \begin{matrix} x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n \\ y - (c_1 x^{n-1} + \cdots + c_n) \end{matrix} \right\rangle$$

$$\mathbb{C}^{2n} = \{(e,c)\} \overset{(*)}{\hookrightarrow} H_n \leftarrow \dim = 2n$$

$\Rightarrow$ (*) gives a neighborhood of $I_{(n)}$.

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & X_n|_{\mathbb{C}^{2n}} \arrow[r, phantom, "\subset"] \arrow[d] & X_n \arrow[r] \arrow[d, "\rho"] & (\mathbb{C}^2)^n \arrow[d, "/S_n"] \\ 0 \in & \mathbb{C}^{2n} \arrow[r, hook] & H_n \arrow[r, "\pi"] & S^n\mathbb{C}^2 \end{tikzcd}$$

$(e,c) \longmapsto I_{e,c}$

$$\pi(I_{e,c}) = [(x_1,y_1)] + \cdots + [(x_n,y_n)].$$

$\{\{x_1, \cdots, x_n\}\} =$ solutions of $x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n$ $\Updownarrow$ $e_k(x_1,\cdots,x_n) = e_k$.

each $y_i = c_1 x_i^{n-1} + \cdots + c_n$.

$$X_n|_{\mathbb{C}^{2n}} = \left\{ (e,c,(x_1,y_1),\cdots,(x_n,y_n)) : \begin{matrix} e_k(x_1,\cdots,x_n) = e_k \\ y_i = c_1 x_i^{n-1} + \cdots + c_n \end{matrix} \right\}$$

$$= \{ (e,c,x_1,\cdots,x_n) : e_k(x_1,\cdots,x_n) = e_k \}$$

$$\mathrm{Fun}(\ ) = \frac{\mathbb{C}[e_1,\cdots,e_n,c_1,\cdots,c_n,x_1,\cdots,x_n]}{\langle e_k(x_1,\cdots,x_n) - e_k \rangle}$$

$$\mathrm{Fun}(p^{-1}(I_{(n)})) = \mathrm{Fun}(\ ) \Big/ \left\langle \begin{matrix} e_1, \cdots, e_n \\ c_1, \cdots, c_n \end{matrix} \right\rangle$$

$$= \mathbb{C}[x_1,\cdots,x_n] / \langle e_k(x_1,\cdots,x_n) \rangle .$$

$\wr\!\wr$ Coinvariant algebra

$\wr\!\wr$ $H^*(\mathrm{Fl}_n)$

Theorem

$$\text{grd-Frob } R_\mu = \tilde{H}_\mu .$$

Example

Consider $n=2$.

$x_1^2 = x_1(x_1+x_2) - x_1x_2$

$$\square\square \quad R_\mu = \mathbb{C}[x_1,x_2] \Big/ \left\langle \begin{matrix} x_1+x_2 \\ x_1x_2 \end{matrix} \right\rangle$$

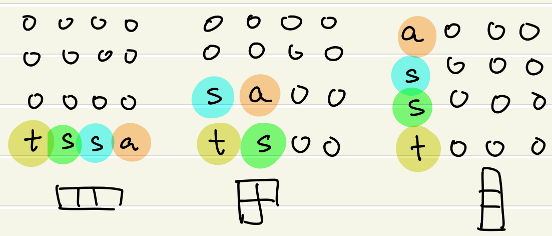
$$= \mathbb{C} \oplus \mathbb{C}x_1 \qquad x_1 = -x_2$$

$$\begin{matrix}\square\\\square\end{matrix} \quad R_\mu = \mathbb{C}[y_1,y_2] \Big/ \left\langle \begin{matrix} y_1+y_2 \\ y_1y_2 \end{matrix} \right\rangle$$

$$= \mathbb{C} \oplus \mathbb{C}y_1 \qquad y_1 = -y_2$$

$$\tilde{H}_{\square\square} = h_2 + q e_2 \qquad \tilde{H}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = h_2 + t e_2$$

$(n=3)$ $t=$ tri $s=$ std $a=$ alt



	$S_{(1,1,1)}$ alt	$S_{(2,1)}$ std	$S_{(3)}$ tri
$\tilde{H}_{(1,1,1)}$	t^3	t^2+t	1
$\tilde{H}_{(2,1)}$	qt	$q+t$	1
$\tilde{H}_{(3)}$	q^3	q^2+q	1

<u>Rmk</u> $\tilde{H}_\mu(q=1, t=1) = p_1^n \qquad \mu \vdash n$

(Thm $\Longleftrightarrow$ $n!$ conjecture)