LLT polynomials

Lascoux, Leclerc & Thibon (1997)

motivation : the Fock space rep'n of quantum affine algebra $U_q(\widehat{sl_n})$.

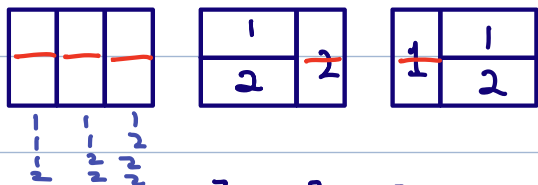
- contains s_λ, $s_{\lambda/\mu}$, Hall-Littlewood pol $H_\mu(X;q)$
- LLT polyn
  - Macdonald polyn. $\widetilde{H}_\mu(X;q,t)$
  - CSF. χ_π (pleth substitution)
  - Diagonal harmonic. DH, DR_n, ...
  - Kazhdan-Luztig pol.

Original def (LLT)

ribbon functions on k-ribbon tableaux $SSYT^{(k)}(\lambda/\mu)$

Def (LLT) $\quad G^{(n)}_{\lambda/\mu}(X;q) = \sum_T q^{spin(T)} X^T$

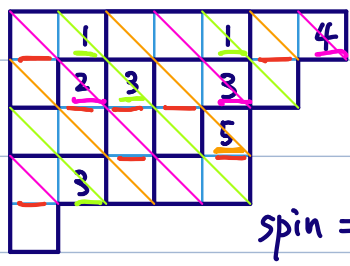
eg $\lambda = 3,3,\ \mu=\phi,\ n=2$



$$G^{(2)}_{33}(x_1,x_2;q) = q^3(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3)+q(\underline{x_1x_2^2+x_1^2x_2})$$

eg $\lambda = 76551$, $\mu=\phi$, $\underline{n=3}$

fill label in upper right corner



spin $=8$

$\Rightarrow\ q^8 x_1^2x_2x_3^3x_4x_5$

$$\text{spin} = \sum_{\xi \text{ ribbon}} (ht(\xi)-1)$$

- n-quotient bijection

shape only depends on λ, μ & n.

- Bylund & Haiman's definition is equivalent to LLT

(see $\underset{\text{Mac.}}{\underline{\text{HHL}}}$ / $\underset{\text{DH.}}{\underline{\text{HHLRU}}}$)

Def Let $\vec{\nu} = (\nu^1, \cdots, \nu^k)$ be a tuple of skew shapes.

for a cell $u : (\overset{\text{row}}{r}, \overset{\text{col}}{c})$, define level is $lvl(u) = r - c$.

define $\boxed{\text{label}}$ is u.

For a $T = (T^1, \cdots, T^k) \in SSYT(\vec{\nu}) = SSYT(\nu^1) \times \cdots \times SSYT(\nu^k)$.

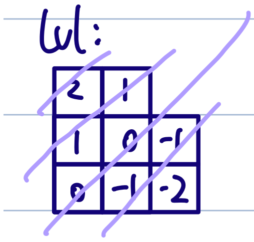
define inversion of T is

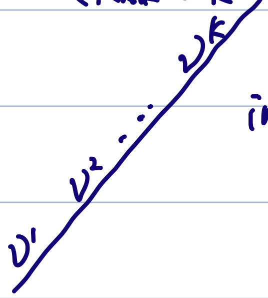
$$inv(T) = \#\left\{ (u,v) \in (\nu^i, \nu^j) \;\middle|\; u < v \ \& \ \begin{array}{l} i<j, \ lvl(u) = lvl(v) \\ \quad\text{or} \\ i>j, \ lvl(u) = lvl(v) - 1 \end{array} \right\}$$

★ $$LLT_{\vec{\nu}}(X;q) = \sum_{T \in SSYT(\vec{\nu})} q^{inv(T)} x^T$$

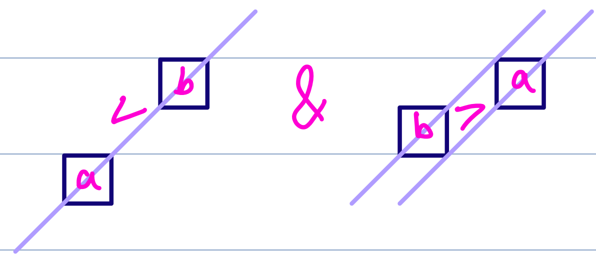
(Rmk : $k=1 \to LLT_{\nu^1}(X;q) = s_{\nu^1}$)

lvl:

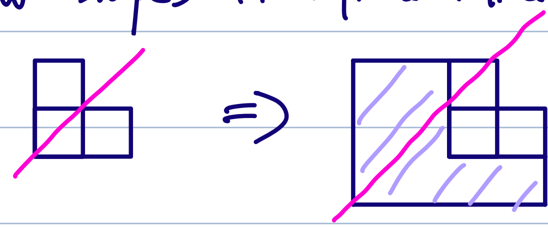




inv:



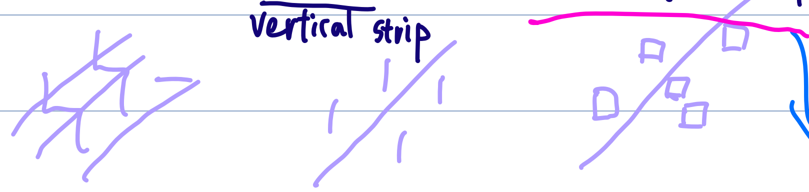
Rmk can place skew shapes in infinite lattice at any hight



- If ν^i's are ribbons / columns / unit cells, then called

ribbon LLT $\supset$ column LLT $\supset$ unicellular LLT. (all important)

vertical strip



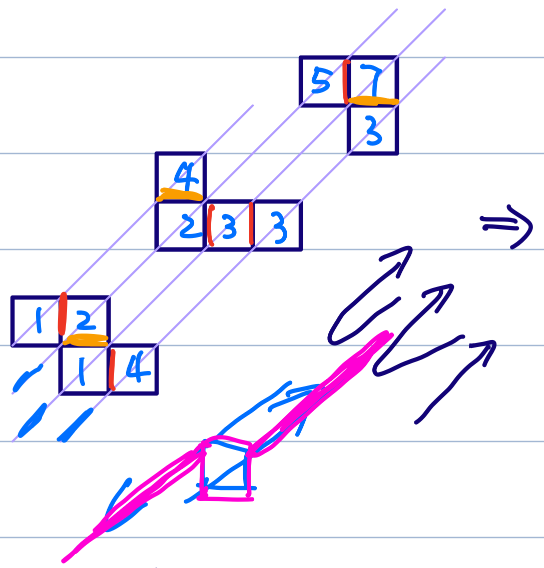
Dyck path without decoration

C_n

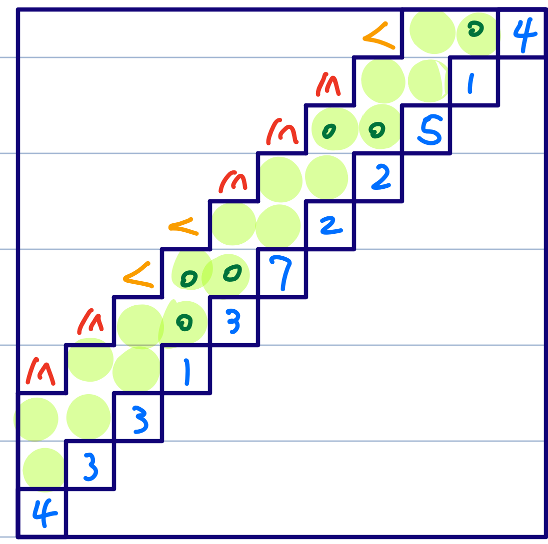
Dyck path LLT : all outer corner decorated w/ $\boxed{<}$

C_n

eg



$\Rightarrow$



inversion $=6$

ribbon LLT can be represented by a Dyck path decorated by $\boxed{<}$ or $\boxed{\wedge}$ on outer corners.

inversion $=$ $\boxed{\text{area}} \to b$, $\downarrow a$

- **Properties** : symmetric (alg & comb proof)
  Schur-pos. (Grojnowski & Haiman, Hecke alg.)
- Fundamental quasi-symmetric fcn expansion.

$$LLT_{\vec{\nu}}(X;q) = \sum_{T \in SYT(\vec{\nu})} q^{inv(T)} F_{pDes(T)}$$

- Schur expansion : open ! (even uncellular case !)

$$LLT_{\pi}(X;q) = \sum_{T \in SYT(n)} q^{\text{stat?}} s_{\lambda(T)} \quad ?$$

- e- positivity : (Alexandersson - Sulzgruber)

$$LLT_{\tilde{\pi}}(X;q+1) = \sum_{S \subseteq area(\pi)} q^{|S|} e_{\underline{\underline{\lambda(S)}}}$$

$\tilde{\pi}$ column

$\lambda(S)$ construction.

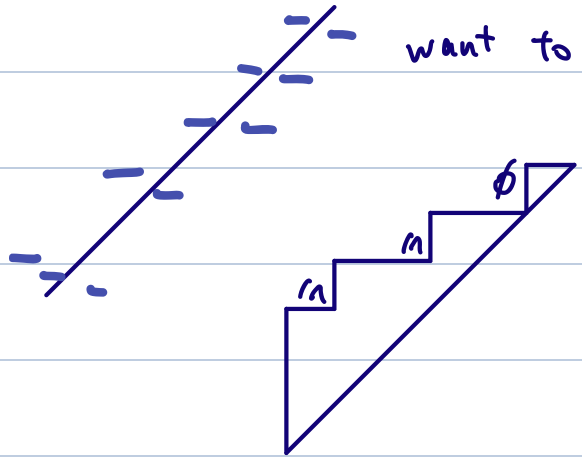
Comb proof of LLT symmetry (HHLRU)

$i^a \; {i+1}^b \longrightarrow i^b \; {i+1}^a$ preserve $\underline{\underline{\text{inv}}}$

case:

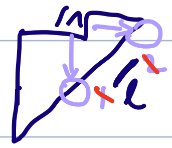
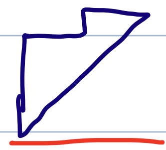
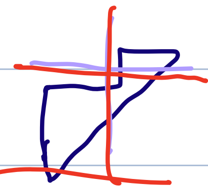
i+1	i+1	i+1	i+1
i	i	i	i

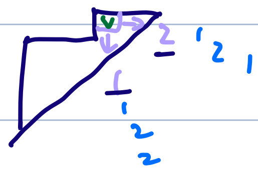
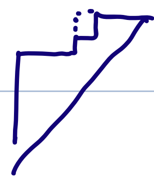
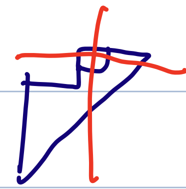
: keep unchanged.



want to change # i's & # i+1's in these rows
keep inv.

i.e. only need to prove x_1, x_2-symmetry of $\boxed{LLT_{rows}(x_1, x_2; q)}$

①  $=$  $- \boxed{x_1 x_2} q^{\ell}$ 

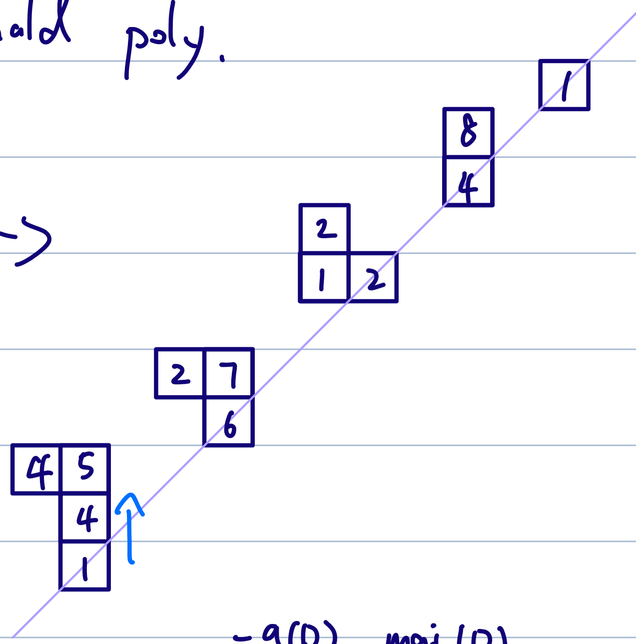
②  $=$  $+ x_1 x_2 (q-1) q^{\ell}$ 

Connection w/ Macdonald poly.

μ :

1	6	2	4	1
4	7	1	8	
5	2	2		
4				

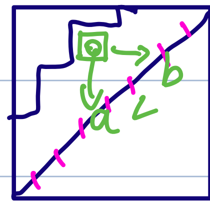
$\longrightarrow$



$$\tilde{H}_\mu[X; q, t] = \sum_{D:\text{ possible descent set}} q^{-a(D)} t^{maj(D)} LLT_D(X; q) \qquad \blacksquare$$

$\hookrightarrow$ symmetric $\blacksquare$

Connection w/ CSF



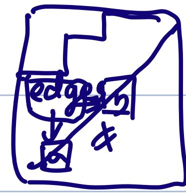
- Let π be a Dyck path.

  let $LLT_\pi(X;q)$ is the corresponding unicellular LLT:

  $$LLT_\pi(X;q) = \sum_{w \in \mathbb{Z}_{>0}^n} q^{\mathrm{inv}_\pi(w)} x^w$$

- $X_\pi(X;q)$ is the corresponding unit interval graph CSF

  $$X_\pi(X;q) = \sum_{\substack{w \in \mathbb{Z}_{>0}^n \\ \text{non attacking}}} q^{\mathrm{inv}_\pi(w)} x^w$$



Thm (Carlsson–Mellit 15')

$$LLT_\pi(\underbrace{(q-1)X}_{\text{plethysm}\; qX-X};q) = (q-1)^n \cdot \boxed{X_\pi(X;q)}$$

$\boxed{(*)}$ L=R

- What is $LLT_\pi(X-Y)$? $X=\{x_1,x_2,\cdots\}$ $Y=\{y_1,y_2,\cdots\}$
- by Haglund,

  Let $\mathcal{A} = \mathbb{Z}_{>0} \cup \mathbb{Z}_{<0}$.

  $$z_i = \begin{cases} x_i & i>0 \\ -y_{|i|} & i<0 \end{cases}$$

- we impose any total order on $\mathcal{A}$, call "<"

  Then define: $\mathrm{inv}(w) = \#\{(a,b) : \boxed{a} \to b \;\&\; (a<b \text{ or } a=b<0)\}$

  $$LLT_\pi(X-Y) = \sum_{w \in \mathcal{A}^n} q^{\mathrm{inv}(w)} z^w$$

Proof of (*): we define the order:

$$-1 < 1 < -2 < 2 < -3 < 3 < \cdots$$

Thus if $|i|<|j|$ then $i<j$.

Let $X = \{qx_1, qx_2, \cdots\}$ $Y = \{x_1, x_2, \cdots\}$

LHS of (*) is $LLT_\pi(qX-X) = \sum_{w \in \mathcal{A}^n} q^{\mathrm{inv}(w)} z^w$

where $z_i = \begin{cases} qx_i & i>0 \\ -x_i & i<0 \end{cases}$

We divide A^n into 2 disjoint sets $W_1 \sqcup W_2$.

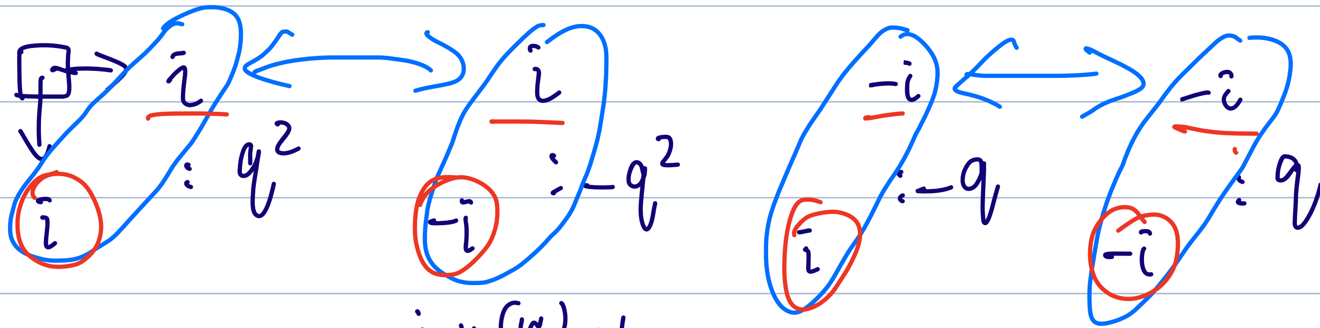
$W_1 = \{ w \in A^n : |w| \text{ non-attacking}$

$W_2 = A^n \setminus W_1 \quad (\exists \ \square \to b, \ \downarrow a \quad |a| = |b|)$

It's clear that :

$$\sum_{w \in W_1} q^{\mathrm{inv}(w)} z^w = (q-1)^n \chi_\pi(X;q)$$

WTS : $\sum_{w \in W_2} q^{\mathrm{inv}(w)} z^w = 0$ by involution.



$\Rightarrow \sum_{w \in W_2} q^{\mathrm{inv}(w)} z^w = 0.$ ▨