2025/4/24

Main Ref

Haglund - Haiman - Loehr, A combinatorial formula for Macdonald polynomials, 2005.

Part 1. Definition of Macdonald poly

Review.

① two basic operations on

$$\Lambda = \bigoplus_{\lambda} \mathbb{Q}(q,t) \cdot s_\lambda \longrightarrow \text{Schur function}$$

- omega involution

$$\omega : \underset{\substack{\| \\ e_{\lambda_1}\cdot e_{\lambda_2}\cdots}}{e_\lambda} \longmapsto \underset{\substack{\| \\ h_{\lambda_1}\cdot h_{\lambda_2}\cdots}}{h_\lambda} \quad / \quad h_\lambda \mapsto e_\lambda$$

Fact. (i) $\omega(s_\lambda) = s_{\lambda'}$, where λ' denotes the conjugate partition of λ.

(ii) $\omega(P_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} P_\lambda$, where

$|\lambda| = \lambda_1 + \lambda_2 + \cdots$ and $\ell(\lambda) = \#$ of positive parts of λ.

- Hall inner product $\langle \cdot , \cdot \rangle$.

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$$

Fact $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

② plethystic operator

Let A be a formal power series over $\mathbb{Q}(q,t)$

Define $p_k[A] = A\Big|_{\substack{x_i \mapsto x_i^k \\ q \mapsto q^k \\ t \mapsto t^k}}$, $p_k(x) = x_1^k + x_2^k + \cdots$

Suppose that $f \in \Lambda$: $f = \sum_\lambda c_\lambda p_\lambda$

Then $f[A] = \sum_\lambda c_\lambda p_\lambda[A]$

$= \sum_\lambda c_\lambda \prod_{i \geq 1} p_{\lambda_i}[A]$.

Notation

$X = x_1 + x_2 + \cdots = e_1(x)$

$Y = y_1 + y_2 + \cdots = e_1(y)$

easily checked that

$p_k[X] = x_1^k + x_2^k + \cdots = p_k(x)$

$p_k[X+Y] = x_1^k + x_2^k + \cdots + y_1^k + y_2^k + \cdots = p_k(x,y)$

So $f[X] = f(x)$, $f[X+Y] = f(x,y)$.

$p_k[-X] = -x_1^k - x_2^k - \cdots = -p_k(x)$

This implies that $p_\lambda[-X] = (-1)^{\ell(\lambda)} p_\lambda(x)$

$= (-1)^{|\lambda|} \cdot \omega p_\lambda(x)$

If $f \in \Lambda$ is homogeneous of degree d, then

$$f[-X] = (-1)^d\, \omega f(x)$$

Definition of Macdonald poly $\widetilde{H}_\mu(x;q,t)$

For $\lambda, \mu \in \mathrm{Par}(n) = \{\text{partitions of } n\}$,

$\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$ for all $k \geq 1$.

Ex. $\lambda \leq \mu \iff \lambda' \geq \mu'$

$\{\widetilde{H}_\mu(x;q,t) : \mu \in \mathrm{Par}(n)\}$ is determined by

triagularity

(T1) $\widetilde{H}_\mu[X(1-q);q,t] = \sum_{\lambda \geq \mu} a_{\lambda\mu}(q,t)\, s_\lambda$

(T2) $\widetilde{H}_\mu[X(1-t);q,t] = \sum_{\lambda \geq \mu'} b_{\lambda\mu}(q,t)\, s_\lambda$

normalization (N) $\langle \widetilde{H}_\mu(x;q,t), s_{(n)} \rangle = 1$

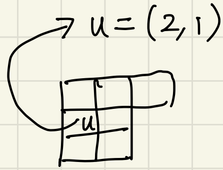
$$p_k[X(1-q)] = (1-q^k)\, p_k(x)$$

$$p_k[X(1-t)] = (1-t^k)\, p_k(x)$$

Part 2. combinatorial formula for $\widetilde{H}_\mu(x;q,t)$

Young diagram of μ

$\mu = (3,2,2)$:



English notation

Indentify μ with

$$\mu = \left\{ (i,j) : \begin{matrix} 1 \leq i \leq \ell(\mu) \\ 1 \leq j \leq \mu_i \end{matrix} \right\}$$

in matrix coordinate

A filling of μ is map

$$\sigma : \mu \longrightarrow \mathbb{Z}_{>0}$$
$$u \longmapsto \sigma(u) \in \mathbb{Z}_{>0}$$

$$\sigma = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 1 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} \end{array}$$

E.g. A filling of μ is a SSYT if the entries are weakly increasing along each row, and strictly increasing down each column.

$$\sigma = \wedge \overset{\leq}{\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} \end{array}} \qquad x^\sigma = \prod_{u \in \mu} x_{\sigma(u)}$$

$$x^\sigma = x_1^2 x_2^2 x_3^2 x_4 .$$

It is well know that

$$S_\mu(x) = \sum_{\sigma \in SSYT(\mu)} x^\sigma$$

back to Macdonald.

$$\tilde{H}_\mu(x;q,t) = \sum_{\sigma : \mu \to \mathbb{Z}_{>0}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$$

- descent of

A box $u \in \mu$ is a descent of σ if $\sigma(u) > \sigma(v)$, where v is the box immediately above u. $\begin{array}{|c|} \hline v \\ \hline u \\ \hline \end{array}$

$Des(\sigma) = \{ \text{ descents of } \sigma \}$

$$\begin{array}{|c|c|c|c|}\hline 4 & 4 & 1 & 3 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 6 & 2 \\ \cline{1-2}\end{array} \qquad maj(\sigma) = 1 + 1 = 2$$

- major of σ

$$\begin{array}{|c|c|c|c|}\hline u & 0 & 0 & 0 \\ \hline \triangle \\ \cline{1-1} \triangle \\ \cline{1-1} \triangle \\ \cline{1-1} \triangle \\ \cline{1-1}\end{array} \qquad \begin{array}{l} arm(u) = 3 \\ leg(u) = 4 \end{array}$$

Define

$$maj(\sigma) = \sum_{u \in Des(\sigma)} (leg(u) + 1)$$

- inversion of σ

Two distinct $u, v \in \mu$ form a attacking pair (u, v), if

(i) u and v lie in the same row and u is to the left of v

$$\begin{array}{c|c|c|c|c}\hline \cdots & u & \cdots & v & \cdots \\ \hline\end{array}$$

(ii) u and v are in two consecutive row s.t. v lies above and strictly to the left of u.

$$\begin{array}{c|c|c|c|c}\hline & v & \cdots & & \\ \hline & \cdots & & u & \\ \hline\end{array}$$

We say $(\sigma(u), \sigma(v))$ is an inversion of σ if

① $\sigma(u) > \sigma(v)$

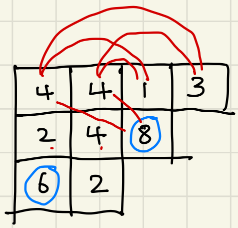
② (u, v) is attaching.

$$\mathrm{Inv}(\sigma) = \{ \text{inversions of } \sigma \}$$

Define

$$\boxed{\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)| - \sum_{u \in \mathrm{Des}(\sigma)} \mathrm{arm}(u)}$$

Ex $\mathrm{inv}(\sigma) \geq 0$.



$$|\mathrm{Inv}(\sigma)| = 4 + 1 + 2$$

$$\sum_{u \in \mathrm{Des}(\sigma)} \mathrm{arm}(u) = 1 + 2$$

$$\mathrm{inv}(\sigma) = 7 - 3 = 4$$

Main Theorem

Let $$C_\mu(x; q, t) = \sum_{\sigma : \mu \to \mathbb{Z}_{>0}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma.$$

Then $$\boxed{\tilde{H}_\mu(x; q, t) = C_\mu(x; q, t)}$$

Part 3. Proof Sketch

Axioms in terms of $m_\lambda(x)$

(T1) $\widetilde{H}_\mu[X(1-q); q,t] = \sum_{\lambda \geq \mu} a_{\lambda\mu}(q.t)\ S_\lambda(x)$

(T2) $\widetilde{H}_\mu[X(1-t); q,t] = \sum_{\lambda \geq \mu'} b_{\lambda\mu}(q.t)\ S_\lambda(x)$

(N) $\langle \widetilde{H}_\mu(X; q,t),\ S_{(n)}(X) \rangle = 1$

$f[-X] = (-1)^d \omega f(x)$ ✓

$\lambda \leq \mu \iff \lambda' \geq \mu'$ ✓

$S_\lambda \in \mathbb{Z}\{m_\rho : \rho \leq \lambda\}$, $m_\lambda \in \mathbb{Z}\{S_\rho : \rho \leq \lambda\}$

$\Downarrow$

(A1) $\widetilde{H}_\mu[X(q-1); q,t] = \sum_{\rho \leq \mu'} C_{\rho\mu}(q.t)\ m_\rho(x)$

(A2) $\widetilde{H}_\mu[X(t-1); q,t] = \sum_{\rho \leq \mu} d_{\rho\mu}(q.t)\ m_\rho(x)$

(N) $\langle \widetilde{H}_\mu(X; q,t),\ S_{(n)}(x) \rangle = 1$

Prove that $C_\mu(x;q,t)$ satisfies (A1), (A2), (N).

Step 1. Show that $C_\mu(x;q,t)$ is symmetric in x.

Establish a connection with LLT symmetric fuction.

For $D \sqsubseteq \mu$, write

$$F_{\mu,D}(x;q) = \sum_{\substack{\mathrm{Des}(\sigma)=D \\ \sigma:\mu\to\mathbb{Z}_{>0}}} q^{|\mathrm{Inv}(\sigma)|} x^{\sigma}$$

Then

$$C_\mu(x;q,t) = \sum_{D\sqsubseteq\mu} q^{-\mathrm{arm}(D)}\, t^{\mathrm{maj}(D)}\, F_{\mu,D}(x;q),$$

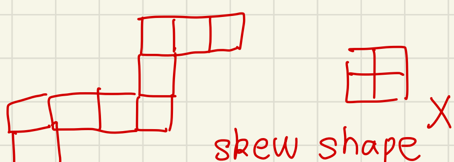
where

$$\mathrm{arm}(D) = \sum_{u\in D} \mathrm{arm}(u),$$

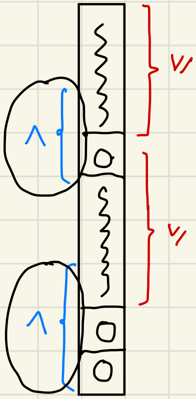
$$\mathrm{maj}(D) = \sum_{u\in D} (\mathrm{leg}(u)+1).$$

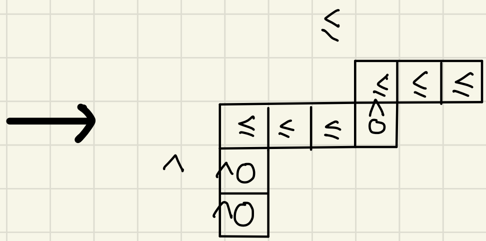
We next explain that $F_{\mu,D}(x;q)$ is LLT.

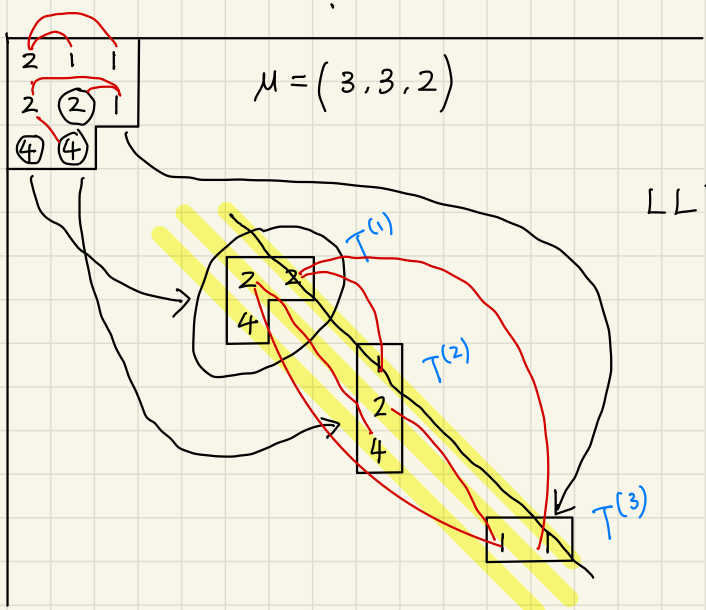
The key idea is to identify each column of σ as a SSYT of a ribbon shape.



consider any column of σ

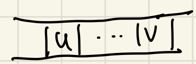




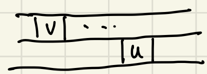


LLT

inversion



$\sigma(u) > \sigma(v)$



$$\mu \longleftrightarrow (\nu^{(1)}, \dots, \nu^{(k)}) = \nu(\mu, D)$$

$$\sigma \longleftrightarrow (T^{(1)}, \dots, T^{(k)}), \quad \text{where } k = \mu_1$$

$\mathrm{Inv}(\sigma) = \{ (T^{(i)}(u), T^{(j)}(v)) :$ $i<j$, u and v lie on the same diagonal $\}$

$\sqcup \ \{ (T^{(i)}(u), T^{(j)}(v)) :$ $i>j$, v lies on the diagonal right above u $\}$

This is just the inversion set in the definition of a LLT $G_{\nu(\mu,D)}(x;q)$

Prop. For $D \subseteq \mu$, we have

$$F_{\mu,D}(x;q) = G_{\nu(\mu,D)}(x;q).$$

LLT

In particular, $F_{\mu,D}(x;q)$ is symmetric.

So $C_\mu(x;q,t)$ is symmetric in x.

Step 2. Express $C_\mu[X(q-1);q,t]$ and $C_\mu[X(t-1);q,t]$ via "super" filling.

Consider the super alphabet

$$A = \mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0} \qquad \bar{i} = -i$$
$$= \{1,2,\dots\} \sqcup \{\bar{1},\bar{2},\dots\}$$

We shall fix any chosen total order on A.

A super filling of μ is a map

$$\sigma : \mu \to A$$

For $x, y \in \mathcal{A}$, let

$$I(x,y) = \begin{cases} 1, & \text{if } x > y \text{ or } x = y \in \mathbb{Z}_{<0}, \\ 0, & \text{if } x < y \text{ or } x = y \in \mathbb{Z}_{>0}. \end{cases}$$

- $u \in \mu$ is a descent of σ if

$$\begin{array}{|c|}\hline v \\ \hline u \\ \hline \end{array} \qquad I(\sigma(u), \sigma(v)) = 1$$

descent

- $(\sigma(u), \sigma(v))$ is an inversion of σ if

$$I(\sigma(u), \sigma(v)) = 1 \text{ and } (u, v) \text{ is an}$$

attacking pair.

super filling

Then $\mathrm{inv}(\sigma)$ and $\mathrm{maj}(\sigma)$ are defined in the same way as ordinary fillings.

Prop. quasisymmetric function technique.

$$C_\mu[x(q-1); q, t] = \sum_{\sigma : \mu \to \mathcal{A}} (-1)^{m(\sigma)} q^{p(\sigma) + \mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^{|\sigma|},$$

$$C_\mu[x(t-1); q, t] = \sum_{\sigma : \mu \to \mathcal{A}} (-1)^{m(\sigma)} q^{\mathrm{inv}(\sigma)} t^{p(\sigma) + \mathrm{maj}(\sigma)} x^{|\sigma|},$$

any order on $\mathcal{A}$

where $m(\sigma) = |\{u \in \mu .\ \sigma(u) \in \mathbb{Z}_{<0}\}|$,

$p(\sigma) = \left| \{ u \in \mu .\ \sigma(u) \in \mathbb{Z}_{>0} \} \right|$, and $x^{|\sigma|} = \prod_{u \in \mu} x_{|\sigma(u)|}$

Step 3. Prove (A1), (A2), (N).

① check (N)

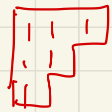
$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$

$$\langle C_\mu(x;q,t), \underbrace{S_{(n)}}_{= h_n} \rangle = \langle C_\mu(x;q,t), h_n \rangle$$

$$= \sum_\lambda c_{\mu\lambda} m_\lambda \qquad m_{(n)} = x_1^n$$

$$= 1 \iff [x_1^n]\, C_\mu(x;q,t) = 1$$

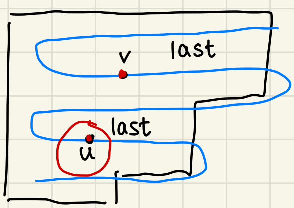
clear!



② check (A1) choose order $\{ 1 < \bar{1} < 2 < \bar{2} < \cdots \}$ of A

Construct a sign-reversing, weight-preserving involution Φ_1 on super fillings $\sigma : \mu \to A$, which cancels out all terms involving x^p if $p \nsubseteq \mu'$

- $\nexists$ attacking pair (u, v) s.t. $|\sigma(u)| = |\sigma(v)|$,
  define $\Phi_1(\sigma) = \sigma$ fixed point
- $\exists$ attacking pair (u, v) s.t. $|\sigma(u)| = |\sigma(v)|$



$|\sigma(u)| = |\sigma(v)|$ is smallest

$$\Phi_1(\sigma) \begin{cases} u \mapsto \overline{\sigma(u)}, \\ w \mapsto \sigma(w) \quad \text{for } w \neq u. \end{cases}$$

Lemma. Φ_1 is sign-reversing and weight preserving.

$$C_\mu[X(q-1);\, q, t] = \sum_{\Phi_1(\sigma)=\sigma} (-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^{|\sigma|}$$ check.

fixed point.

For σ s.t. $\Phi_1(\sigma)=\sigma$, no repeated $|\sigma(u)|$ in each row 推到左侧

$\Rightarrow \quad x^{|\sigma|} = x_1^{p_1} x_2^{p_2} \cdots \qquad p_1 + \cdots + p_k \le \mu_1' + \cdots + \mu_k' \quad \forall\, k \ge 1.$

fixed point $\qquad p \le \mu'$

③ check (A2). Use the order $\{\underbrace{1<2<3<\cdots}_{\text{off}} < \underbrace{\bar{3}<\bar{2}<\bar{1}}_{A}\}$

Define Φ_2:

- If $|\sigma(u)| \ge i$ for all $\overset{u}{\overset{\parallel}{(i,j)}} \in \mu$, then $\Phi(\sigma)=\sigma$ fixed point
- Otherwise, let a be the smallest $a = |\sigma(u)| < i$.

Then choose the first u such that $|\sigma(u)| = a$.
reading order.

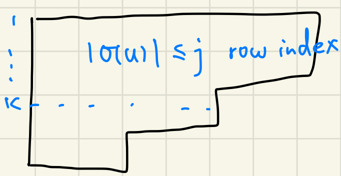
$$\Phi_2(\sigma) \begin{cases} u \mapsto \overline{\sigma(u)} \\ w \mapsto \sigma(w) \quad \text{for } w \ne u. \end{cases}$$

check carefully.

Lemma. Φ_2 is sign-reversing and weight-preserving.

$$C_\mu[X(t-1);\, q, t] = \sum_{\Phi_1(\sigma)=\sigma} (-1)^{m(\sigma)} q^{\mathrm{inv}(\sigma)} t^{p(\sigma)+\mathrm{maj}(\sigma)} x^{|\sigma|}.$$

fixed point

$\sigma =$ 

$x^p = x_1^{p_1} x_2^{p_2} \cdots$

$p_1 + \cdots + p_k \le \mu_1 + \cdots + \mu_k$

$p \le \mu$