Background on Richardson varieties

Preparation for Speyer's survey

Mingzhi Yang

2025

symmetric groups and related combinatorics

- $\binom{[n]}{k}$ denotes the set of k-element subsets of $[n] := \{1, 2, \cdots, n\}$.
- $\preceq$ denotes the partial order on $\binom{[n]}{k}$ by defining

$$\{i_1 < i_2 < \cdots < i_k\} \preceq \{j_1 < j_2 < \cdots < j_k\} \Leftrightarrow i_a \leq j_a, a \in [k].$$

- For $w \in S_n$ and $k \in [n]$, $w[k] := \{w(1), w(2), \cdots, w(k)\} \in \binom{[n]}{k}$.

Definition

The Bruhat order(or strong order) is the partial order on S_n defined by

$$u \preceq w \Longleftrightarrow u[k] \preceq w[k], \forall k \in [n].$$

Theorem

Each of the following statements is equivalent to $u \preceq w$

1. *For all* $1 < i, j < n$, *we have* $\sharp([i] \cap u[j]) \geq \sharp([i] \cap w[j])$.
2. *There is a reduced word* $s_{j_1} s_{j_2} \cdots s_{j_\ell}$ *for* w *and a subword* $s_{j_{a_1}} s_{j_{a_2}} \cdots s_{j_{a_m}}$ *with product* u.
3. *For every reduced word* $s_{j_1} s_{j_2} \cdots s_{j_\ell}$ *for* w, *there is a subword* $s_{j_{a_1}} s_{j_{a_2}} \cdots s_{j_{a_m}}$ *with product* u.

Idea of proof: Bruhat order can be generated by Bruhat cover $u \lessdot ut_{ab} \Leftrightarrow \ell(ut_{ab}) = \ell(u) + 1$; Braid relations.

Demazure product

Definition

The Demazure product is the unique associative multiplication $* : S_n \times S_n \to S_n$ *such that*

$$s_i * w = \begin{cases} s_i w & \ell(s_i w) = \ell(w) + 1 \\ w & \ell(s_i w) = \ell(w) - 1 \end{cases}$$

and

$$w * s_i = \begin{cases} w s_i & \ell(w s_i) = \ell(w) + 1 \\ w & \ell(w s_i) = \ell(w) - 1 \end{cases}$$

For well-definedness, check braid relations. Associativity, $w * v * s_i$

Remark (uniqueness?)

$\mathbf{id} * \mathbf{id} = \mathbf{id}$?

group notations

Let GL_n be the group of $n \times n$ invertibe matrices, T be the group of diagonal matrices, B_- be the Borel subgroup of lower triangular matrices, $U_- \subseteq B_-$ be the subgroup of lower triangular unipotent matrices,

Example

$$
\underbrace{\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix}}_{T}, \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & * & 1 \end{pmatrix}}_{U_-} \subseteq \underbrace{\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{pmatrix}}_{B_-}
$$

We will also use B_+ for the opposite Borel subgroup and $U_+ \subseteq B_+$ for the group of upper triangular unipotent matrices.

We embed S_n into GL_n by sending $w \in S_n$ to the permutation matrix which has ones in positions $(w(j), j)$ and zeros everywhere else.

Example

$w = 54312 = s_4 s_3 s_2 s_1 s_4 s_3 s_2 s_4 s_3 \in S_5$, the corresponding permuation matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Grassmannians, Flag varieties, Plücker coordinates

- We write $G(k,n)$ for the Grassmannian.

$$G(k,n) := \{V \leq \mathbb{C}^n | \dim V = k\}.$$

- Let M be an $n \times k$ matrix, we write $\Delta_{i_1 i_2 \cdots i_k}(M)$ to be the minor of M in the rows indexed by $\{i_1, i_2, \cdots, i_k\}$. For an $n \times n$ matrix g, we take $\Delta_{i_1 i_2 \cdots i_k}(g)$ to be the minor of g in the rows indexed by $\{i_1, i_2, \cdots, i_k\}$ and the k leftmost columns.
- If M is of maximal rank, then M can be viewed as a point in $G(k,n)$, $\Delta_{i_1 i_2 \cdots i_k}(M)$ is called Plücker coordinate.

- The complete flag variety $\mathcal{F}\ell_n$ is defined as

$$\mathcal{F}\ell_n := \{V_1 \leq V_2 \leq \cdots \leq V_{n-1} \leq \mathbb{C}^n | \dim V_k = k, \forall k \in [n-1]\}.$$

- $\mathcal{F}\ell_n$ can be viewed as a subvariety of $\prod_{k=1}^{n-1} G(k,n)$, so for each subset $I \subset [n]$ with $0 < \sharp I < n$, there is a Plücker coordinate Δ_I.
- $\mathcal{F}\ell_n$ can be identified with GL_n/B_+, sending gB_+ to the flag $V_\bullet$ with V_k being the span of the leftmost k column vectors of g.
- There is a natural projection map $\mathcal{F}\ell_n \longrightarrow G(k,n)$. In matrix language, this is the map sending a $n \times n$ matrix g to an $n \times k$ matrix M by taking the leftmost k columns.

Bruhat decomposition, Schubert cells, Schubert varieties

The decomposition $GL_n = \sqcup_{w \in S_n} B_- w B_+$ is called the Bruhat decomposition of GL_n.

Proposition

A matrix $g \in GL_n$ lies in $B_- w B_+$ if and only if, for each $0 < i, j < n$, the upper-left $i \times j$ submatrix of g has rank $\sharp([i] \cap w[j])$.

Example

Let $w = 54312 \in S_5$. A matrix g lies in $B_- w B_+$ looks like

$$\begin{pmatrix} 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & * & 1 \\ 0 & 0 & 1 & * & * \\ 0 & 1 & * & * & * \\ 1 & * & * & * & * \end{pmatrix}$$

The Schubert cells in $\mathcal{F}\ell_n$ are defined as $\mathring{X}_w := B_- w B_+ / B_+$. Or equivalently,

$$\mathring{X}_w = \{V_\bullet \in \mathcal{F}\ell_n | \dim(E_i \cap V_j) = \sharp([i] \cap w[j])\}.$$

Example

Let $w = 54312 \in S_5$, then a point in $\mathring{X}_w$ can be represented as

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Schubert variety $X_w := \overline{B_- w B_+} / B_+$ is the closure of the Schubert cell. We have

$$X_w = \sqcup_{w \preceq u} \mathring{X}_u.$$

Example

Let $w = 54312 \in S_5$, then

$$X_w = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Grassmannian Schbuert varieties

- Let $I \in \binom{[n]}{k}$, we write X_I for the subvariety of $G(k,n)$ consisting of k-planes V where $\Delta_J(V) = 0$ for $I \not\leq J$.
- $\mathring{X}_I$ be the open subvariety of X_I where $\Delta_I \neq 0$.

Example

Let $I = \{2,4\} \in \binom{[5]}{2}$, then points in $\mathring{X}_I$ can be presented by 5×2 matrices

$$\mathring{X}_{24} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ * & 0 \\ * & 1 \\ * & * \end{pmatrix} \in G(2,5)$$

We see that $\Delta_{12} = \Delta_{13} = \Delta_{14} = \Delta_{15} = \Delta_{23} = 0$, $\Delta_{24} = 1$.

Let $X_u \subseteq \mathcal{F}\ell_n$, then its projection in $G(k,n)$ has image in $X_{u[k]}$.

Richardson varieties

We use $\mathring{X}^w := B_+ w B_+ / B_+$. If $u \preceq w$, then the open Richardson varieties is defined as

$$\mathring{X}_u^w := \mathring{X}_u \cap \mathring{X}^w = B_- u B_+ / B_+ \cap B_+ w B_+ / B_+.$$

Similarly, the closed Richardson variety is defined as the intersection of Schubert varieties.

$$X_u^w := X_u \cap X^w.$$

Remark

We also use $\mathring{R}_u^w$ and R_u^w for (open) Richardson varieties.

Remark

Schubert varieties are special cases of Richardson varieties. Not true for Schubert cells.

Proposition

We have $R_u^w = \sqcup_{u \preceq u' \preceq w' \preceq w} \mathring{R}_{u'}^{w'}$ *and* $\dim R_u^w = \dim \mathring{R}_u^w = \ell(w) - \ell(u)$.

Ref. Deodhar85-*On some geometric aspects of Bruhat orders.*

Proposition

If $u \preceq w$*, then the open Richardson variety* $\mathring{R}_u^w$ *is a smooth irreducible affine variety. The Richardson variety is an irreducible projective variety.* R_u^w *is normal and Cohen-Macaulay with rational singularities.*

Ref. Richardson92, Brion-Lakshmibai03, Knutson-Lam-Speyer14, Billey-Coskun12.

Projected Richardson varieties

Fix a partial flag manifold $\mathcal{F}\ell_n(k_1, k_2, \cdots, k_p)$ and let π be the projection map

$$\pi : \mathcal{F}\ell_n \to \mathcal{F}\ell_n(k_1, k_2, \cdots, k_p).$$

For $u \preceq w$ in S_n, we define $\prod_u^w := \pi(R_u^w)$ and call it a projected Richardson variety.

Remark

In the case $\{k_1, k_2 \cdots, k_p\} = \{k\}$, a projected Richardson variety in Grassmannian $G(k, n)$ also called a positroid variety.

P-Bruhat order

Write W_P for the Young subgroup $S_{k_1} \times S_{k_2-k_1} \times \cdots \times S_{n-k_p}$. We write $u \lessdot v$ and say v covers u if $u \preceq v$ and $\ell(v) = \ell(u) + 1$.

Definition

We write $u \lessdot_P v$ and say v P-cover u if $u \lessdot v$ and $vW_P \neq uW_P$. We define the P-Bruhat order to be the transitive closure of the P-covering relation and write $\preceq_P$.

Remark

In the case $\{k_1, k_2 \cdots, k_p\} = \{k\}$, the P-Bruhat order is called k-Bruhat order by Bergeron-Sottile98. In fact, $u \lessdot_k v$ if and only if $v = ut_{ab}$ with $\ell(v) = \ell(u) + 1$, $a \leq k < b$.

Proposition

The map $\pi : R_u^w \to \prod_u^w$ is birational if and only if $u \preceq_P w$.

Proposition

For every projected Richardson variety $\prod_u^w$, we can find $u \preceq u' \preceq_P w' \preceq w$ such that $\prod_u^w = \prod_{u'}^{w'}$.

Let W^P be the set of minimal representatives of W/W_P.

Lemma

If $w \in W^P$, then $u \preceq w$ if and only if $u \preceq_P w$.

Proposition

Each projected Richardson can be represented in exactly one way as $\pi(R_u^w)$ with $w \in W^P$ and $u \preceq w$.

Open projected Richardson varieties

The open projected Richardson variety $\mathring{\prod}_u^w$ is the open subvariety of $\prod_u^w$ where we remove all proper sub-projected Richardson varieties of $\prod_u^w$.

Proposition

If $u \preceq_P w$, so that $\pi : R_u^w \to \prod_u^w$ is birational, then $\pi : \mathring{R}_u^w \to \mathring{\prod}_u^w$ is an isomorphism.

Question

What is $\pi(\mathring{R}_u^w)$ if $u \preceq w$ holds but $u \preceq_P w$ fails?

Thank you