Positroids.

$\pi : R_u^w \longrightarrow G(k,n)$ ($\in Fln$) $\qquad \pi(R_u^w) := \Pi_u^w$ positroids.

Recall : [Speyer Prop 1.27] $\quad \Pi = \pi(R_u^w)$ $\quad w$ is k-Grassmannian

$\Pi = \pi(R_{ux}^{wx})$ $\quad x \in S_k \times S_{n-k}$. $\quad \ell(ux) = \ell(u) + \ell(x)$

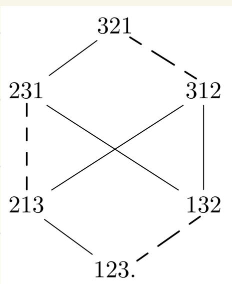
Outline : $\{(u,w) \in Q(k,n)\} \xleftrightarrow{f = u w_k w^{-1}} \{f \in B(k,n)\} \xleftrightarrow{\text{cyclic rk matrix}} \{$Grassmann necklace$\}$

Bounded affine perm. $\qquad I_1 \ldots I_n \in \binom{[n]}{k}$

§1. $Q(k,n)$

Recall. P-Bruhat order. $P = S_k \times S_{n-k}$. define $\preceq_k$. $\quad$: Gale order :

$u \preceq_k w$ if $u = v_0 \lessdot \ldots \lessdot v_t = w$. $\quad v_0[k] \prec \ldots \prec v_t[k]$



e.g. $P = S_1 \times S_2$ $\qquad 123 \preceq_1 312$

$123 \npreceq_1 321$, $123 \npreceq_1 231$

$\pi : R_u^w \to \Pi_u^w$ is birational iff $u \preceq_k w$

Def. $\hat{Q}(k,n)$: (u,w) $\quad u \preceq_k w$ $\quad$ resver containment.

$(u,w) \preceq (u',w')$ if $u \preceq u' \preceq w' \preceq w$

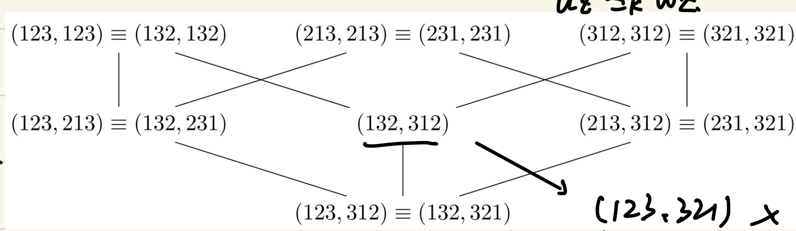
- $Q(k,n) := \hat{Q}(k,n)/\equiv$. $\quad (u,w) \equiv (uz, wz)$. $z \in S_k \times S_{n-k}$

$uz \preceq_k wz$

e.g. S_3.

$P = S_1 \times S_2$

$z = id$ or S_2



$\exists !$ k-Grassmannian w in each "$\equiv$"

$(Q(k,n), \le)$ is a poset. $rk((u,w)) = k(n-k) - (\ell(w) - \ell(u))$

[KnutsonLamSpeyer14]

- $(u,w) \equiv (u',w') \Rightarrow \Pi_u^w = \Pi_{u'}^{w'}$
- $G(k,n) = \bigsqcup_{(u,w)\in Q(k,n)} \mathring{\Pi}_u^w$, $\Pi_u^w = \bigsqcup_{(u',w') \ge (u,w)} \mathring{\Pi}_{u'}^{w'}$

[GaleshinKarpLam22]

- $\mathring{\Pi}_u^w \cap G(k,n)_{\ge 0}$ is open ball. $\dim = \ell(w) - \ell(u)$
  $\Pi_u^w \cap G(k,n)_{\ge 0}$. closed ... $\Big\} \Rightarrow$ form a regular CW decomposition of $G(k,n)_{\ge 0}$

§2. Bounded affine perm. $\quad Q(k,n) \longleftrightarrow B(k,n)$

Def. Affine perm. $\quad \tilde{S}_n := \{ \underset{\text{bijection}}{f: \mathbb{Z} \to \mathbb{Z}} \mid f(i+n) = f(i) + n \} \to$ generator. $s_0, s_1, \dots, s_{n-1}, \pi$

$\pi(i) = i+1$

$disp(f) := \frac{1}{n} \cdot \sum_{i=1}^{n} (f(i) - i) \qquad \tilde{S}_n^k := \{ f \in \tilde{S}_n \mid disp(f) = k \}$

Ex: $disp(f) \in \mathbb{Z}$.

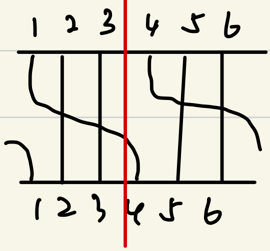
$B(k,n) := \{ f \in \tilde{S}_n^k , \ i \le f(i) \le i+n \}$

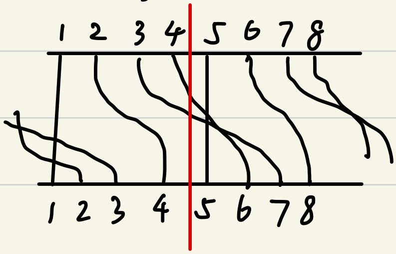
eg $\quad \omega_k(i) := \begin{cases} i+n, & i \equiv 1 \dots k \mod n \\ i. & i \equiv k+1 \dots n \mod n. \end{cases}$

$f = [1476]$

$S_1 \times S_2$

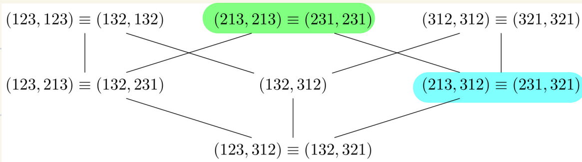
$\omega_k = [423]$

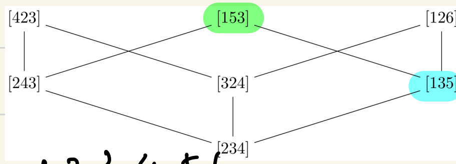


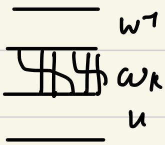


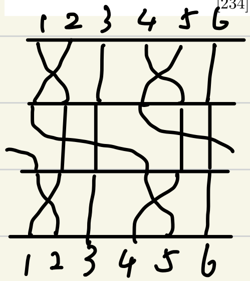
Prop. $\exists$ Bijection : $Q(k,n) \longleftrightarrow B(k,n)$ [Knutson Lam Speyer 13]

$(u,w) \longmapsto u\,\omega_k\,w^{-1} := f_{uw}$









$[153] = 213\cdot\omega_k\cdot(213)^{-1}$
$= 231\cdot\omega_k\cdot(231)^{-1}$

Sketch of proof.

① $a = w^{-1}(i)$ $\quad f_{uw}(i) = \begin{cases} u(a) & a>k \\ u(a)+n & a\le k \end{cases}$ $\quad k \le_k w$ $\Rightarrow f_{uw}$ is "bounded" $(f_{uw}(i) \in [i, i+n])$

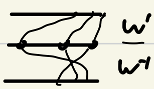
② Well-defined. $(u',w') = (uz, wz)$. $z \in S_k \times S_{n-k}$.

$f_{u'w'} = u'\omega_k w'^{-1} = u\cdot \underline{z\,\omega_k\,z^{-1}}\cdot w^{-1} = u\cdot\underline{\omega_k}\cdot w^{-1} = f_{uw}$.

③. Injective: $f_{uw} = f_{u'w'} \Rightarrow (u,w) \equiv (u',w')$

$\forall i.\quad u\cdot w^{-1}(i) = u'w'^{-1}(i)$ &. $w^{-1}(i) > k \Leftrightarrow w'^{-1}(i) > k$.

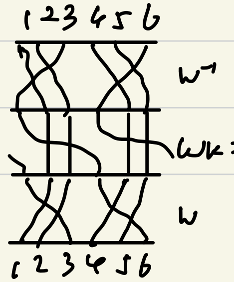
$w^{-1}\cdot w' \in S_k \times S_{n-k}$



④. $\forall f \in B(k,n)$. find (u,w) st. $u\,\omega_k\,w^{-1} = f$.

$\underbrace{(u\cdot w^{-1})}_{\in S_n.}\cdot(\underline{w\cdot\omega_k\cdot w^{-1}}) = f$

$[126] =$



$w = 312.\ u = 213$

$(213, 312) \equiv (231, 321)$

$f = [135]$ $\quad w\,\omega_k\,w^{-1} = [126]$

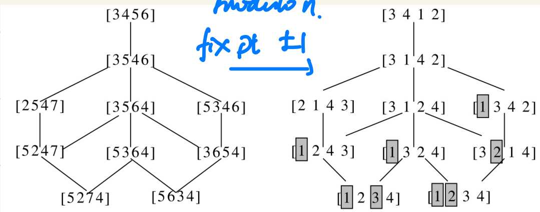
$\downarrow \qquad\qquad\qquad \downarrow$

$132 \xleftarrow{u\cdot w^{-1} = s_2} 123$

Rmk: $B(k,n) \longleftrightarrow$ "Decorated permutation" [Postnikov 06]

modulo n.

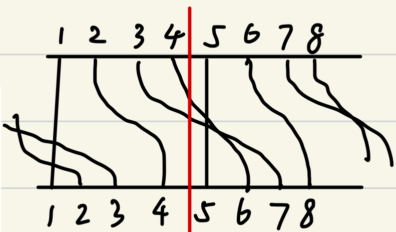
fix pt ± 1



§ 3. Cyclic rank matrix.

Def. $r_{ij}(f) := k - \#\{a<i,\ f(a)>j\}$

e.g. $f = [1476]$



$$
\begin{array}{c}
\begin{array}{cccccccc} & 2 & & & & & & \end{array} \\
\left[\begin{array}{cccccccc}
\boxed{0} & 0 & & & & & & \\
1 & 1 & 0 & & & & & \\
2 & 2 & 1 & 0 & & & & \\
2 & \boxed{2} & 2 & 1 & 0 & \boxed{-1} & & \\
2 & 2 & 2 & 1 & \boxed{0} & 0 & & \\
2 & 2 & 2 & \boxed{1} & 1 & 1 & 0 & \\
2 & 2 & \boxed{2} & 2 & 2 & 2 & 1 & 0 \\
2 & 2 & 2 & 2 & 2 & \boxed{2} & 2 & 1
\end{array}\right]
\end{array}
$$

$r_{11} = 2 - \#\{a<1.\ f(a)>1\}$

$\to r_{64} = 2 - \#\{a<6.\ f(a)>4\}$

$f(i)$ 1 4 7 6 5. . —

Prop: cyclic rank matrix $(r_{ij}) \longleftrightarrow B(k,n)$

$f(i) = j$ iff.

	i	$i+1$
$j-1$	$s+1$	s
j	$s+1$	$s+1$

Th. $L \in G(k,n)$ $L \in \mathring{\Pi}_f$ iff projection of L onto $\mathrm{Span}\{e_i \ldots e_j\}$ $i<j$ (modulo n) has dim $r_{ij}(f)$. $\forall i,j$.

e.g. $f=[1476]$. $L=(v_1,v_2,v_3,v_4)\in \mathring{\Pi}_{[1476]}$

$i=j=1$. $r_{11}=0$ $\quad \dim\langle v_1\rangle=0 \Rightarrow v_1=0$.

$i=0, j=2$. $r_{02}=1$ $\quad \dim\langle v_0,v_1,v_2\rangle=1 \Rightarrow v_0 \,//\, v_2$.

$L=(0,v_2,v_3,\lambda v_2)\in \mathring{\Pi}_{[1476]}$

$L=(v_1,\dots,v_n)\in G(k,n)$. $\quad L\in \mathring{\Pi}_f$. find "$f$":

$f(i)=\min\{j\geq i \,.\, v_i\in \mathrm{Span}\langle v_{i+1}\dots v_j\rangle\}$ $\quad$ if $v_i=0$. $f(i)=i$

$f(1)=1$. $\quad f(2)=4$. $\quad f(3)=7$. $\quad f(4)=6$.

§4. Grassmann necklace.

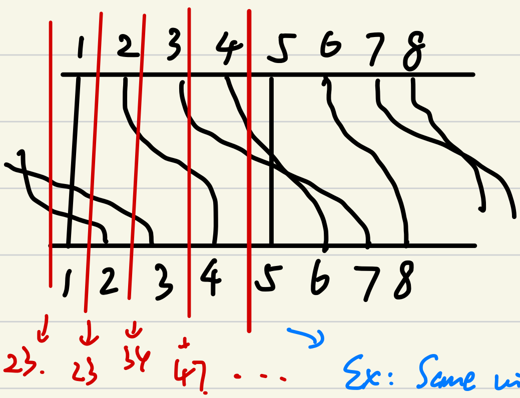
$f \longrightarrow r_{ij}(f)$

$\downarrow$ $\quad \tilde{I}_i := \{j : r_{ij}-r_{ij-1}=1\}$

$\tilde{I}_1=\{23\}$. $\tilde{I}_2=\{23\}$. $\tilde{I}_3=\{34\}$. $\tilde{I}_4=\{47\}$ $\tilde{I}_5=\{67\}$ ......

$\downarrow$ modulo n. $\quad$ "$I_i\setminus\{i\}\subseteq I_{i+1}$ $(I_n\setminus\{n\}\subseteq I_1)$"

$I_1=\{23\}$. $I_2=\{23\}$ $I_3=\{34\}$. $I_4=\{34\}$ $\quad$ Grassmann necklace.



modulo n.

$$\begin{cases} i\notin I_i . \ I_{i+1}=I_i & f(i)=i \\ i\in I_i \quad I_{i+1}=I_i & f(i)=i+n \\ i\in I_i . \quad I_{i+1}\neq I_i . & I_{i+1}=I_i\setminus\{i\}\cup\{j\} \end{cases}$$

$$\Rightarrow f(i)=\begin{cases} j & j>i \\ j+n & j<i \end{cases}$$

Ex: Same with our definition from cyclic rk matrix.

" permuted Schubert cell"
shifted

[KnutsonLamSpeyer13] [Suho Oh 11]

$L \in G(k,n)$. $L \in \mathring{\Pi}_f$ $(I_1 \ldots I_n)$ $L \in \sigma^{j-1} \mathring{X}_{\rho^{-j+1}(I_j)}$ $\mathring{\Pi}_f \subseteq \bigcap \sigma^{j-1} \mathring{X}_{\rho^{-j+1}(I_j)}$

$\rho: 1 \to 2 \to 3 \ldots \to n \to 1$

$\sigma: \mathbb{A}^n$ $e_1 \to e_2 \to \ldots \to e_n \to e_1$

$$\mathring{\Pi}_f = \bigcap_{j=1}^{n} \sigma^{j-1} \mathring{X}_{\rho^{-j+1}(I_j)}$$

$$\Pi_{[1476]} = \mathring{X}_{23} \cap \sigma \cdot \mathring{X}_{12} \cap \sigma^2 \cdot \mathring{X}_{12} \cap \sigma^3 \cdot \mathring{X}_{14}$$

$(\mathring{X}_I := \{ V.\ \Delta_I(V) \neq 0 \})$

§5. Cohomology class.

Q: Cohomology class of R_u^w in Fl_n.

A: $X_u \pitchfork X^w \Rightarrow [R_u^w] = [X_u] \cdot [X^w]$

Q: Cohomology class of Π_u^w in $G(k,n)$

A: Affine Stanley Sym. func. F_f.

Def. $v = s_{i_1} \ldots s_{i_k}$. cyclically decreasing

① $i_1 \ldots i_k$ distinct

② Both i. $i+1$ appears, then $i+1$ before i.

$$F_f = \sum_{f = v_1 \ldots v_k} x_1^{l(v_1)} \cdots x_k^{l(v_k)}$$

$l(f) = l(v_1) + \ldots + l(v_k)$

v_i is cyc. decreasing.

fact: F_f is sym func.

[Lam06]. Affine Stanley sym func.