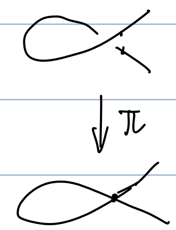
$/\mathbb{C}$

For any singular variety X, we can always find a desingularization

$$\begin{tikzcd} \pi^{-1}(U) \arrow[d, "\simeq"'] \arrow[r, phantom, "\subseteq"] & Y \arrow[d, "\pi"] \\ U \arrow[r, phantom, "\subseteq"] & X \end{tikzcd}$$

Y nonsingular, π birational, U open dense

E.g. 

Zariski's main Thm

When X is a Schubert variety / Richardson variety, a nice choice of Y is a Bott-Samelson / Brick.

1. Def & structure of Bott-Samelson varieties.

$G \supset B \supset T$.

For each simple root $\alpha \iff$ minimal parabolic $P_\alpha \supseteq B$

$$\begin{array}{ccc} G/B & \to & G/P_\alpha \\ \| & & \| \\ X & & X_\alpha \end{array} \qquad \mathbb{P}^1\text{-bundle}$$

$\underline{\alpha} := (\alpha_1, \dots, \alpha_d) \rightsquigarrow$ $G \curvearrowright$ diagonal. $\quad 0 \le i \le d$

big Bott-Samelson variety $Z(\underline{\alpha}) = X \times_{X_{\alpha_1}} X \times_{X_{\alpha_2}} \cdots \times_{X_{\alpha_d}} X \xrightarrow{pr_i} X$

$$\begin{tikzcd} X \times_{X_\alpha} X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X_\alpha \end{tikzcd} \qquad \mathbb{P}^1\text{-bundle}$$

tower of $\mathbb{P}^1$-bundle over X

$\Rightarrow$ nonsingular.

$\dim = \dim X + d$

Example. $G = SL_n$, $X = Fl_n$.

$$Z(\underline{\alpha}) = \left\{ (E_\bullet^{(0)}, \dots, E_\bullet^{(d)}) \;\middle|\; E_k^{(i)} = E_k^{(i-1)} \text{ for all } k \neq j \right\}$$

where $\alpha_i = t_j - t_{j+1} \iff s_j \in S_n$

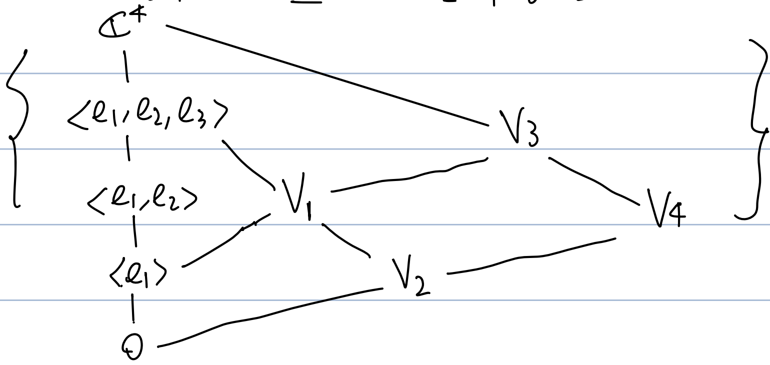
The (small) Bott-Samelson variety $X(\underline{\alpha}) = pr_0^{-1}(eB)$

$X(\underline{\alpha}) = \{eB\} \times_{X_{\alpha_1}} X \times_{X_{\alpha_2}} \cdots \times_{X_{\alpha_d}} X$ tower of $\mathbb{P}^1$-bundle over pt

$\circlearrowright B$ nonsingular of dim d

Example. $G = Sl_4$ $\underline{\alpha} \leftrightarrow s_2 s_1 s_3 s_2$

$X(\underline{\alpha}) =$



Recall: Schubert variety has an affine stratification indexed by T-fixed pts.

$X(\underline{\alpha})$ is similar. :

T-fixed points of $Z(\underline{\alpha})$

Def. An $\underline{\alpha}$-chain of elements of W is a sequence $\underline{v} = (v_0, \cdots, v_d)$ s.t. $\forall i$, either $v_i = v_{i-1}$ or $v_i = v_{i-1} \cdot s_{\alpha_i}$

Exercise: T-fixed points of $Z(\underline{\alpha})$ are $2^d |W|$ points

$Z(\alpha)^T = \{ p_{\underline{v}} = (v_0 B, v_1 B, \cdots, v_d B) \}$, where $\underline{v}$ is a α-chain.

Proof: $v_i B \in X^T \Rightarrow v_i \in W$.

If either $v_i = v_{i-1}$ or $v_i = v_{i-1} \cdot s_{\alpha_i}$ isn't satisfied.

then $\{v_{i-1}B\} \times_{X_{\alpha_i}} \{v_i B\} = \phi$.

$X(\underline{\alpha})$ has 2^d T-fixed points $\leftrightarrow$ $\underline{v} = (eB, v_1, \cdots, v_d)$ α-chain.

$\updownarrow$

$I = \{ i_1 < \cdots < i_k \} \subseteq \{1, \cdots, d\}$
$= \{ i \mid v_i = v_{i-1} \cdot s_{\alpha_i} \}$

For each $I \subseteq \{1, \dots, d\}$, there is a B-invariant subvariety $X(I) \subseteq X(\underline{\alpha})$

$$X(I) := \{(x_1, \dots, x_d) \in X(\underline{\alpha}) \mid x_j = x_{j-1} \text{ for } j \notin I\}$$

forget repeating components
no info lost.

$$\underline{\alpha}(I) := (\alpha_{i_1}, \dots, \alpha_{i_l}) \qquad X(I) \simeq X(\underline{\alpha}(I))$$

$$X(J) \subseteq X(I) \iff J \subseteq I.$$

$$X(\{1, \dots, d\}) = X(\underline{\alpha}), \quad X(\phi) = pt.$$

$$\mathring{X}(I) := \{(x_1, \dots, x_d) \in X(I) \mid x_i \neq x_{i-1} \quad i \in I\}$$

$$X(I) = \overline{\mathring{X}(I)}$$

Lemma. We have $\mathring{X}(I) \simeq \mathbb{A}^l$, where $l = \# I$

Proof: induction on d, $\mathbb{A}^1$-bundle.

$\rightsquigarrow$ $[X(I)]$ form a basis for $H_T^*(X(\underline{\alpha}))$

Example. $G = SL_n$.

$$z_i(t) := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & t & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \leftarrow i\text{th row.}$$

$$\underline{\alpha} \leftrightarrow (s_{i_1}, s_{i_2}, \dots, s_{i_a})$$

$$\mathbb{A}^a \xrightarrow{\simeq} \mathring{X}(\underline{\alpha})$$

$$(t_1, \dots, t_a) \mapsto (eB, z_{i_1}(t_1)B, z_{i_1}(t_1) z_{i_2}(t_2) B, \dots, z_{i_1}(t_1) z_{i_2}(t_2) \cdots z_{i_a}(t_a) B)$$

right mult. by $z_i(t)$ $\longleftrightarrow$ Column operation

result: varies i-th step E_i in the flag with 1 degree of freedom.

2. How they desingularize Schubert varieties.

$$\begin{tikzcd} X(\underline{\alpha}) \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, equal] \\ Z(\underline{\alpha}) \arrow[r, "pr_d"] & X \end{tikzcd}$$

For each $I \subseteq \{1, \dots, d\} \mapsto \underline{\alpha}$-chain $\underline{v} = (e, v_1, \dots, v_d)$

$f(p_I) = v_d B$ $\qquad v_d = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}}$ is the unique T-fixed pt in the Schub. cell $\mathring{X}_{v_d}$ ← B-stable

T-fixed point

f is closed and B-equivariant, $\Rightarrow f(X(I)) \supseteq X_{v_d}$

Lemma. $w(\underline{\alpha}) \in W \qquad f(X(\underline{\alpha})) = X_{w(\underline{\alpha})}$

$$w(\underline{\alpha}) = \max\{ s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}} \}$$

in Bruhat order.

$w(\underline{\alpha}) = s_{\alpha_1} \cdots s_{\alpha_d} \iff \underline{\alpha}$ reduced.

T-fixed pts

Proof: $f(X(\underline{\alpha}))$ is a Schubert variety X_w.

Demazure product

$$w * s_\alpha = \begin{cases} w s_\alpha & \text{if } \ell(w s_\alpha) > \ell(w) \\ w & \text{otherwise.} \end{cases}$$

Exercise. $w(\underline{\alpha}) = s_{\alpha_1} * \cdots * s_{\alpha_d}$

Lemma. $f: X(\underline{\alpha}) \to X_w$ is birational $\iff \underline{\alpha}$ is a reduced word for $w = w(\underline{\alpha})$

Proof: $\Rightarrow$ ✓.

$\Leftarrow$

$U \subseteq B$ unipotent $\qquad$ e.g. $SL_n \qquad U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$

$U(w) := \dot{w} U^- \dot{w}^{-1} \cap U \subseteq B$

$\mathring{X}_w$ is a free $U(w)$-orbit in X. $\qquad U(w) \xrightarrow{\sim} \mathring{X}_w$

$u \mapsto u.wB$

f: $U(w)$-equivariant $\Rightarrow \mathring{X}(\underline{\alpha})$ is a free $U(w)$-orbit.

$\Rightarrow f: \mathring{X}(\underline{\alpha}) \xrightarrow{\sim} \mathring{X}(w)$ is an isomorphism.

$G/B \to G/P$.

$X_{w_{min}} \xrightarrow[\text{birational}]{} Y_{w_{min}}$

Example. $\underline{\alpha} = (\alpha, \alpha)$. $\quad X(\underline{\alpha}) = \{eB\} \times_{X_\alpha} X \times_{X_\alpha} X \simeq \mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr] \arrow[d] & & X \arrow[d] \\
\mathbb{P}^1 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X_\alpha \\
\{eB\} \arrow[r] & X_\alpha &
\end{tikzcd}$$

$\to$ factors thru $\{eB\}$

$S_\alpha * S_\alpha = S_\alpha$. $\quad X_{S_\alpha} \simeq \mathbb{P}^1$.

Subvarieties $X(J) \subseteq X(\underline{\alpha})$.

$X(\{1,2\}) = X(\underline{\alpha})$

$X(\{1\}) =$ the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$

$\simeq X(\alpha) \xrightarrow{\simeq} X_{S_\alpha} \simeq \mathbb{P}^1$.

1st α.

$X(\{2\}) = \{eB\} \times \mathbb{P}^1 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$

$\simeq X(\alpha) \xrightarrow{\simeq} X_{S_\alpha} \simeq \mathbb{P}^1$

2nd α

$X(\phi) = pt \xrightarrow{\simeq} X_e \simeq pt$.

* Alternative definition of B-S

$\underline{\alpha} = (\alpha_1, \dots, \alpha_d)$.

$B^d \curvearrowright P_{\alpha_1} \times \cdots \times P_{\alpha_d}$ by $(b_1, \dots, b_d).(p_1, \dots, p_d) := (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \dots, b_{d-1} p_d b_d^{-1})$

$B \curvearrowright P_{\alpha_1} \times \cdots \times P_{\alpha_d} / B^d \xrightarrow{\simeq} X(\underline{\alpha})$ $\quad$ $B \curvearrowright$ diagonal $\quad$ B-equiv.

$[p_1, \dots, p_d] \mapsto (eB, p_1 B, p_1 p_2 B, \dots, p_1 \cdots p_d B)$

$\rotatebox{90}{$\in$} \qquad \rotatebox{90}{$\in$}$

$\{[p_1,\dots,p_d] \mid p_i B = eB \text{ for } i \notin I\} \longleftrightarrow X(I)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \cup$

$[\varepsilon_1,\dots,\varepsilon_d]$, where $\longleftrightarrow p_I$

$$\varepsilon_i = \begin{cases} \dot{e} & i \in I \\ \dot{s}_{\alpha_i} & i \notin I \end{cases}$$

Proof, induction on d. $d=0$ ✓.

$$\begin{tikzcd} {[p_1,\dots,p_d]} \arrow[r, mapsto] & p_1\cdots p_d B \\ P_{\alpha_1}\times\cdots\times P_{\alpha_d}/B^d \arrow[r] \arrow[d] & X \arrow[d] \\ P_{\alpha_1}\times\cdots\times P_{\alpha_{d-1}}/B^{d-1} \arrow[r] & X_{\alpha_d} \\ {[p_1,\dots,p_{d-1}]} \arrow[r, mapsto] & p_1\cdots p_{d-1} P_{\alpha_d} \end{tikzcd}$$

commute, pullback of $\mathbb{P}^1$-bundle

Good cohom. property.

$Y = X(\underline{\alpha}) \xrightarrow{\pi} X = X_w$. $\qquad \pi_*[Y] = [X]$ in H^*.

$$\pi_*[\mathcal{O}_Y] := \sum_{i\geq 0}(-1)^i[R^i\pi_*\mathcal{O}_Y] = [\pi_*\mathcal{O}_Y] = [\mathcal{O}_X]. \text{ in } K\text{-theory.}$$

3. Brick var.

[Kumar–Schwede].
Desingularization ~~of~~ $X^v \subseteq X = G/B$.
Pick a reduced word $\underline{\beta}$ for $w_0 v$.
$f: X(\underline{\beta}) \longrightarrow X_{w_0 v}$ from before.

$B^- \curvearrowright X(\underline{\beta}) \longrightarrow X^v = w_0 X_{w_0 v} \qquad b \odot z := (\dot{w}_0 b^- \dot{w}_0).z$

$\qquad z \longmapsto w_0 f(z) \qquad\qquad B^-$-equiv.

$X_w^v = X_w \cap X^v \qquad \underline{\alpha}$ reduced word for w.

$$\begin{tikzcd}
Z := X(\alpha)\times_X X(\beta) \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & X(\beta) \arrow[d] \\
& X_w^\vee \arrow[r, hook] \arrow[d, hook] & X^\vee \arrow[d, hook] \\
X(\alpha) \arrow[r] & X_w \arrow[r, hook] & X
\end{tikzcd}$$

Kleiman transversality:

G irred. alg. grp $/ k = \bar{k}$. X transitive G-var.

$f: Y \to X$, $g: Z \to X$ morphism of irred. vars.

For $s \in G$, let $s.Y$ denote Y with $Y \to X$, $y \mapsto s.f(y)$

a) $\exists$ dense open $U \subseteq G$ s.t.

$\forall s \in U$, $s.Y \times_X Z$ has pure dim $= \dim Y + \dim Z - \dim X$.

b). Assume char $k = 0$, Y, Z nonsingular.

$\exists$ dense open $U \subseteq G$ s.t. $\forall s \in U$, $s.Y \times_X Z$ is nonsingular.

$\Rightarrow$ $g.X(\alpha) \times_X X(\beta)$ smooth of $\dim = \dim X_w^\vee$ for $g \in G$ general.

$B^- B \overset{\text{dense}}{\subseteq} G$

$\Rightarrow$ Can take $g = b^- b \in B^- B$, $\quad b^- b . X(\alpha) \times_X X(\beta) \simeq X(\alpha) \times_X X(\beta)$

$$\begin{tikzcd}
Z = X(\alpha)\times_X X(\beta) \arrow[r] & X_w^\vee \\
\mathring{Z} := \mathring{X}(\alpha)\times_X \mathring{X}(\beta) \arrow[u, hook] \arrow[r, "\simeq"] & \mathring{X}_w^\vee \arrow[u, hook]
\end{tikzcd}$$

Good. cohom. ppty.

$\underline{\alpha} = (\alpha_1, \dots, \alpha_d)$ $\quad$ $\underline{\beta} = (\beta_1, \dots, \beta_e)$

reduced word for

w $\qquad\qquad$ $w_0 v$.

$$X(\underline{\alpha}) = \{eB\} \times_{X_{\alpha_1}} X \times_{X_{\alpha_2}} \cdots \times_{X_{\alpha_d}} X$$

$X(\underline{\beta}) \simeq X \times_{X_{\beta_e}} X \times_{X_{\beta_{e-1}}} \cdots \times \{eB\}$. $\quad$ $\underline{\beta}' = (\beta_e, \dots, \beta_1)$ reduced word for $v^{-1} w_0$

$$\begin{tikzcd} X(\underline{\beta}) \arrow[d, "w_0 pr_0"'] \arrow[r, "\simeq"] & X \times_{X_{\beta_e}} \cdots \times_{X_{\beta_1}} \{w_0 B\} \arrow[dl, "pr_0"] \\ X & \end{tikzcd}$$

mult. by w_0 everywhere

$$Z = X(\underline{\alpha}) \times_X X(\underline{\beta}) \simeq \{eB\} \times_{X_{\alpha_1}} \cdots \times_{X_{\alpha_d}} X \times_{X_{\beta_e}} \cdots \times_{X_{\beta_1}} \{w_0 B\}$$

This is the fiber of $X(\underline{\alpha}, \underline{\beta}') = \{eB\} \times_{X_{\alpha_1}} \cdots \times_{X_{\alpha_d}} X \times_{X_{\beta_e}} \cdots \times_{X_{\beta_1}} X$

over $w_0 B$

$$\begin{tikzcd} X(\underline{\alpha}, \underline{\beta}') \arrow[d, "pr_{d+e}"'] \arrow[dr, "pr_d"] & \\ X & X_w^v \end{tikzcd}$$

Example SL_4.

$\alpha \Leftrightarrow (s_1, s_2, s_3, s_1, s_2)$ $\quad$ w.

$v = s_1 s_2$

$w_0 v$ $\quad$ $\underline{\beta} \Leftrightarrow (s_1, s_2, s_1, s_3)$

$\qquad\quad$ $\underline{\beta}' \Leftrightarrow (s_3, s_1, s_2, s_1)$