Deodhar decomposition

Reading seminar on RICHARDSON VARIETIES, PROJECTED RICHARDSON VARIETIES AND POSITROID VARIETIES

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Previously

Definitions and Notations from previous talks

- Identifying the complete flag variety $\mathcal{F}\ell_n$ with GL_n/B_+, the *Schubert cells* are $\mathring{X}_v = (B_- v B_+)/B_+$
- $w[k] \in \binom{[n]}{k}$ is the action of a permutation w on the subset $\{1,\ldots,k\}$
- The *open Bott-Samelson variety* $\mathrm{BS}^\circ(i_1, \ldots, i_a)$ consists of sequences of flags such that $F^{j-1} \xrightarrow{s_{i_j}} F^j$, meaning that they differ only at the i_j-th space.
- The *open Richardson variety* $\mathring{R}_u^w = \mathring{X}_u \cap \mathring{X}^w$

Distinguished subwords

- Let $s_{i_1} \cdots s_{i_a}$ be a word in the simple generators. We take and skip on the digits to form a subword, which is called *distinguished* if we always take the next digit if it reduces the *length* of the partial product.
- e.g. $\underline{s_3}s_2s_1\underline{s_3s_2}$, the second s_3 is forced because it reduces the length of the partial product.
- We represent them in the alphabet $\{s_{i_1}, \ldots, s_{i_a}, \bullet\}$
- e.g. $s_3 \bullet \bullet\ s_3s_2$

Deodhar decomposition

- Given a sequence of flags $(F^0, \ldots, F^a)$ in $\mathrm{BS}^\circ(i_1, \ldots, i_a)$. Let v_j be the permutation such that $F^j \in \mathring{X}_{v^j}$. If $v^{j-1} = v^j$ for some $1 < j \leq a$ then $\ell(v^{j-1}s_{i_j}) > \ell(v^j)$; otherwise, $v^j = v^{j-1}s_{i_j}$.

- In other words, elements of the open Bott-Samelson variety correspond to distinguished subwords of $s_{i_1} \cdots s_{i_a}$, and the v^j's are partial products. We call each piece a *Deodhar piece*, denoted by $\mathscr{D}(x)$ or $\mathscr{D}(v^0, \ldots, v^a)$, where x is a distinguished subword of $s_{i_1} \cdots s_{i_a}$

- Relation with open Richardson: when $s_{i_1} \cdots s_{i_a}$ is a reduced word for w, the disjoint union of the Deodhar pieces of the subword which multiplies to u is $\mathring{R}_u^w$

Example 3.10. Let $n = 3$. The Bott-Samelson variety $\mathrm{BS}(1,2,1)$ is the set of sequences of flags of the form

$$\begin{array}{lll} \mathrm{Span}(e_1) & \subset & \mathrm{Span}(e_1, e_2) \\ L_1 & \subset & \mathrm{Span}(e_1, e_2) \\ L_1 & \subset & P_1 \\ L_2 & \subset & P_1. \end{array}$$

The open subvariety $\mathrm{BS}^\circ(1,2,1)$ imposes that $\mathrm{Span}(e_1) \neq L_1 \neq L_2$ and $\mathrm{Span}(e_1, e_2) \neq P_1$. We observe that L_1 can be recovered from the flag (L_2, P_1) by $L_1 = P_1 \cap \mathrm{Span}(e_1, e_2)$.

We can coordinatize $\mathrm{BS}^\circ(1,2,1)$ by $\mathbb{A}^3$ by sending (t_1, t_2, t_3) to

$$\begin{array}{lll} \mathrm{Span}(e_1) & \subset & \mathrm{Span}(e_1, e_2) \\ \mathrm{Span}(t_1 e_1 + e_2) & \subset & \mathrm{Span}(e_1, e_2) \\ \mathrm{Span}(t_1 e_1 + e_2) & \subset & \mathrm{Span}(t_1 e_1 + e_2, t_2 e_1 + e_3) \\ \mathrm{Span}((t_1 t_3 + t_2) e_1 + t_3 e_2 + e_3) & \subset & \mathrm{Span}(t_1 e_1 + e_2, t_2 e_1 + e_3). \end{array}$$

We rewrite this in terms of matrices; our chain of flags is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B_+, \quad \begin{bmatrix} t_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} B_+, \quad \begin{bmatrix} t_1 & t_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} B_+, \quad \begin{bmatrix} t_1 t_3 + t_2 & t_1 & 1 \\ t_3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} B_+.$$

In the following section, we will often want to use coset representatives for $\mathrm{GL}_2/B_+(2)$ other than $\left[\begin{smallmatrix} t & 1 \\ 1 & 0 \end{smallmatrix}\right]$. We therefore adopt the more general convention: For a word $(s_{i_1}, s_{i_2}, \ldots, s_{i_a})$ and a sequence of elements $(h_1, h_2, \ldots, h_a)$ in GL_2 not lying in $B_+(2)$, let $\mu_{i_1 i_2 \ldots i_a}(h_1, h_2, \ldots, h_a)$ be the sequence of flags

$$\mu_{i_1 i_2 \ldots i_a}(h_1, h_2, \ldots, h_a) := \big(B_+, \rho_{i_1}(h_1) B_+, \rho_{i_1}(h_1) \rho_{i_2}(h_2) B_+, \ldots, \rho_{i_1}(h_1) \rho_{i_2}(h_2) \cdots \rho_{i_a}(h_a) B_+\big)$$

in $\mathrm{BS}^\circ(i_1, i_2, \ldots, i_a)$. We close by remarking on some other choices we could use for h_i:

Example 4.4. Take $n = 3$; let $u = e$ and $w = w_0$; we use the word (s_1, s_2, s_1) for w. There are two subwords of (s_1, s_2, s_1) with product e, namely $(\bullet, \bullet, \bullet)$ and $(s_1, \bullet, s_1)$, and both are distinguished. So $\mathring{R}_e^{w_0}$ is the union of two Deodhar pieces: $\mathcal{D}(\bullet, \bullet, \bullet) = \mathcal{D}_{\mathrm{seq}}(e, e, e, e)$ and $\mathcal{D}(s_1, \bullet, s_1) = \mathcal{D}_{\mathrm{seq}}(e, s_1, s_1, e)$.

We describe points of $\mathrm{BS}^\circ(1,2,1)$ using two lines, L_1, L_2, and a plane P_1, as in Example 3.10. The Richardson $\mathring{R}_e^{w_0}$ is the open subvariety of $\mathrm{BS}^\circ(1,2,1)$ where L_2 is transverse to $\mathrm{Span}(e_2, e_3)$ and P_1 is transverse to $\mathrm{Span}(e_3)$. In the coordinates of Example 3.10, the open Richardson $\mathring{R}_e^{w_0}$ is the open locus $t_2(t_1 t_3 + t_2) \neq 0$.

The piece $\mathcal{D}_{\mathrm{seq}}(e, e, e, e)$ is the piece where L_1 is transverse to $\mathrm{Span}(e_2, e_3)$; the piece $\mathcal{D}_{\mathrm{seq}}(e, s_1, s_1, e)$ is the piece where $L_1 \subset \mathrm{Span}(e_2, e_3)$. We reinterpret this condition in terms of the flag (L_2, P_1) and in terms of the coordinates (t_1, t_2, t_3). Since $L_1 = P_1 \cap \mathrm{Span}(e_1, e_2)$, the first piece is the piece where $P_1 \cap \mathrm{Span}(e_1, e_2)$ is transverse to $\mathrm{Span}(e_2, e_3)$; equivalently, the first piece is the piece where $\mathrm{Span}(e_2) \not\subset P_1$ and the second piece is the piece where $\mathrm{Span}(e_2) \subset P_1$. In terms of the (t_1, t_2, t_3) coordinates, these pieces are $t_1 \neq 0$ and $t_1 = 0$.

The topology of a Deodhar piece is simple

- Let $m_=, m_\downarrow, m_\uparrow$ be the number of times that our partial products v^j stay put, go down in length, or go up. Then $\mathscr{D}(v^0, \ldots, v^a) \cong \mathbb{G}_m^{m_=} \times \mathbb{A}^{m_\downarrow}$
- We fix everything in a sequence of flags except for the i-th one, i.e. V_i, this is a $\mathbb{P}^1$; there is exactly one point W in this $\mathbb{P}^1$ that increases the length
- If we stay put, we must avoid both V_i, W
- If we go up, we just choose W
- If we go down, then $V_i = W$, and we only need to avoid V_i

Applications

Kazhdan-Lusztig R-polynomials

- The number of $\mathbb{F}_q$ points in the open Richardson variety $\mathring{R}_u^w$ form a polynomial, called the *Kazhdan-Lusztig R-polynomial*, because the number of $\mathbb{F}_q$ points in $\mathbb{A}, \mathbb{G}_m$ do, and the open Richardson is a disjoint union of Deodhar pieces

- $\dim(\mathring{R}_u^w) = m_= + 2m_\downarrow = \ell(w) - \ell(u)$

- The R-polynomials are palindromic up to a sign twist

- *Deodhar torus*: there must exits a maximal one that is Zariski dense with $m_\downarrow = 0$; distinguished sequences like this are called *positive*

$$\rho_i\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

4.2. **Matrix product formulas for Deodhar pieces.** We now turn to the problem of parametrizing the Deodhar pieces. Our primary source is Marsh and Rietsch [MarshRietsch04]. We recall the notation $\rho_i : \mathrm{GL}_2 \to \mathrm{GL}_n$ from Section 3.2. Define:

$$\dot{s}_i = \rho_i\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \qquad \dot{z}_i(t) = \rho_i\left(\begin{bmatrix} t & 1 \\ -1 & 0 \end{bmatrix}\right) \qquad \ddot{z}_i(t) = \rho_i\left(\begin{bmatrix} t & -1 \\ 1 & 0 \end{bmatrix}\right) \qquad y_i(t) = \rho_i\left(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\right).$$

Proposition 3.8. *Let $(s_{i_1}, s_{i_2}, \ldots, s_{i_a})$ be any word in the simple generators of S_n. Map $\mathbb{A}^a$ to $\mathcal{F}\ell_n^{a+1}$ by*

$$(t_1, t_2, \ldots, t_a) \mapsto \left(B_+, z_{i_1}(t_1)B_+, z_{i_1}(t_1)z_{i_2}(t_2)B_+, \ldots, z_{i_1}(t_1)z_{i_2}(t_2)\cdots z_{i_a}(t_a)B_+\right).$$

This is an isomorphism $\mathbb{A}^a \to \mathrm{BS}^\circ(i_1, i_2, \ldots, i_a)$.

$$h_j = \begin{cases} y_{i_j}(t_j) & j \in J_= \\ \dot{s}_{i_j} & j \in J_\uparrow \\ \dot{z}_{i_j}(u_j) & j \in J_\downarrow \end{cases}.$$

Recall the map $\mu_{i_1 i_2 \cdots i_a}$ from $(\mathrm{GL}_2 - B_+(2))^a$ to $\mathrm{BS}^\circ(i_1, i_2, \ldots, i_a)$ introduced in Section 3.2. We will write g^k for the partial product $h_1 h_2 \cdots h_k$, so the image of $\mu_{i_1 \cdots i_a}$ is $(B_+, g^1 B_+, \ldots, g^a B_+)$.

Theorem 4.12. *With the above notation, the map sending $(t_1, t_2, \ldots, t_a)$ to $(B_+, g^1 B_+, \ldots, g^a B_+)$ in $\mathrm{BS}^\circ(i_1, i_2, \ldots, i_a)$ is an isomorphism from $\mathbb{G}_m^{m_=} \times \mathbb{A}^{m_\downarrow}$ to the Deodhar piece $\mathcal{D}_{seq}(v^0, v^1, \ldots, v^a)$.*

Example 4.13. In $\mathcal{F}\ell_3$, consider the word (s_1, s_2, s_1) from Example 3.10. There are two distinguished subexpressions ending in e: (e, e, e, e) and (e, s_1, s_1, e); they correspond to $(J_=, J_\uparrow, J_\downarrow) = (\{1, 2, 3\}, \emptyset, \emptyset)$ and $(\{2\}, \{1\}, \{3\})$ respectively. Since (s_1, s_2, s_1) is reduced, the projection of $\mathrm{BS}^\circ(1, 2, 1)$ onto the last flag $\mathcal{F}\ell_3$ is an isomorphism with its image, so we focus on describing the final flag F^3.

The corresponding matrix products are

$$\begin{bmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ t_3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{bmatrix} \begin{bmatrix} u_3 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ u_3 & 1 & 0 \\ -t_2 & 0 & 1 \end{bmatrix}$$

with t_1, t_2, $t_3 \in \mathbb{G}_m$ and $u_3 \in \mathbb{A}^1$. These two pieces disjointly cover the open Richardson $\mathring{R}^{321}_{123}$, which is $\Delta_1 \Delta_{12} \Delta_3 \Delta_{23} \neq 0$. The first piece is the open set $\Delta_{13} \neq 0$, and the second piece is the closed set $\Delta_{13} = 0$. The reader is invited to compute these minors and see that they are zero or nonzero as appropriate.

Example 4.14. We give an example with a nonreduced word. We work in $\mathcal{F}\ell_2$ with the word $(s_1, s_1, s_1, s_1, s_1)$. Then a distinguished sequence is a sequence of six e's and s_1's which starts with e and has no consecutive pair of s_1's. As a concrete example, we will take the sequence (e, e, e, s_1, e, s_1), corresponding to the subword $(\bullet, \bullet, s_1, s_1, s_1)$. The matrices ϕ_j are

$$\begin{bmatrix} 1 & 0 \\ t_1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t_2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} u_4 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad t_1 \in \mathbb{G}_m, \;\; u_3 \in \mathbb{A}^1$$

The successive partial products g^j are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t_1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t_1 + t_2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -t_1 - t_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t_1 + t_2 + u_4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -t_1 - t_2 - u_4 \end{bmatrix}.$$

A flag in $\mathcal{F}\ell_2$ is simply a point on the projective line $\mathbb{P}^1$; the sequence of flags $(F^0, F^1, \ldots, F^5)$ in this case is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ t_1 \end{bmatrix}, \begin{bmatrix} 1 \\ t_1 + t_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ t_1 + t_2 + u_4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that consecutive elements of this sequence are always distinct points of $\mathbb{P}^1$; this is the Bott-Samelson condition. Note also that F^0, F^1, F^2 and F^4 are in the Schubert cell $\mathring{X}_e = \{\Delta_1 \neq 0\}$ where as F^3 and F^5 are in the Schubert cell $\mathring{X}_{s_1} = \{\Delta_1 = 0\}$; this is the additional Deodhar condition.

The matrices follow suit of the distinguished sequence

$g^j \in B\_v^j$

- Convenient notation: Let v be a permutation, whose matrix representative has a 1 at every $(v(k), k)$ and zero elsewhere. Let $\dot{v}$ be signed such that every "left-justified" minor is nonnegative. We have $(\dot{vs_i}) = \dot{v}\dot{s}_i$. ($N\_$ seems to be the signed version of $B\_$, but I am not sure.)
- If $v(i) < v(i+1)$, then $\dot{v}y_i(t)\dot{v}^{-1} \in N\_$
- If $v(i) > v(i+1)$, then $\dot{v}(\dot{z}_i(u)\dot{s}_i)\dot{v}^{-1} \in N\_$

Parametrizations of flag varieties

[Marsh, Rietsch]

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & T & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & 0 & 1 \end{bmatrix}$$

$u \mapsto g\dot{z}_i(u)B_+$ is an isomorphism of $\mathbb{A}^1$ to the space of flags $gB_+ \xrightarrow{s_i} F$.

$t \mapsto gy_i(t)B_+$ is an isomorphism of $\mathbb{G}_m$ onto the complement of $g\dot{s}_iB_+$

The $\mathbb{A}^1$ bundle argument from before

Checking $g^jB_+ \in (B_-v^jB_+)/B_+$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & U \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ U & 0 & 1 \end{bmatrix}$$

Inverting the isomorphism $\mathbb{G}_m^{m_=} \times \mathbb{A}^{m_\downarrow} \longrightarrow \mathcal{D}(v^0, v^1, \ldots, v^a)$ is quite complex; see [MarshRietsch04] for the general formula. We will describe the result for the Deodhar torus in the case where $(s_{i_1}, \ldots, s_{i_a})$ is reduced (which is the case which is relevant to Richardsons). We first set up some auxilliary functions, called ***chamber minors***.

Let $(v^0, v^1, \ldots, v^a)$ be the positive sequence for u; it will also be convenient to put $w^j = s_{i_1} s_{i_2} \cdots s_{i_j}$. Because $(s_{i_1}, \ldots, s_{i_a})$ is reduced, at any point $(F^0, \ldots, F^a)$ of $\mathrm{BS}^\circ(i_1, \ldots, i_a)$, the flag F^j is in $\mathring{X}^{w^j}$. By the definition of the Deodhar piece, if $(F^0, \ldots, F^a)$ is in $\mathcal{D}_{\mathrm{seq}}(v^0, \ldots, v^a)$, then $F^j \in \mathring{X}_{v^j}$. So, combining these, $F^j \in \mathring{R}_{v^j}^{w^j}$. This means that the k-th subspace, F^j_k, is in the Grasmmannian Richardson variety $\mathring{R}_{v^j[k]}^{w^j[k]}$. In particular, the Plücker coordinates $\Delta_{v^j[k]}(F^j_k)$ and $\Delta_{w^j[k]}(F^j_k)$ are nonzero. We define the ratio

$$\Phi^j_k = \frac{\Delta_{v^j[k]}(F^j_k)}{\Delta_{w^j[k]}(F^j_k)}$$

to be the (j,k)-***chamber minor***. Since this is a ratio of two Plücker coordinates, it is a well defined invariant of the subspace F^j_k. We can visualize the chamber minors as written in the chambers of the wiring diagram;

Chamber indexing

Why are they called “chamber minors”?

The main idea is to associate such an arrangement (which we call the *ansatz arrangement*) to the pair $\mathbf{v}, \mathbf{w}$. For example, if $\mathbf{w} = (1, s_3, s_3s_2, s_3s_2s_1, s_3s_2s_1s_3, s_3s_2s_1s_3s_2)$, and $\mathbf{v}$ is the distinguished subexpression $(1, s_3, s_3, s_3, 1, s_2)$ for s_2 in $\mathbf{w}$, then the arrangement is as in Figure 1.

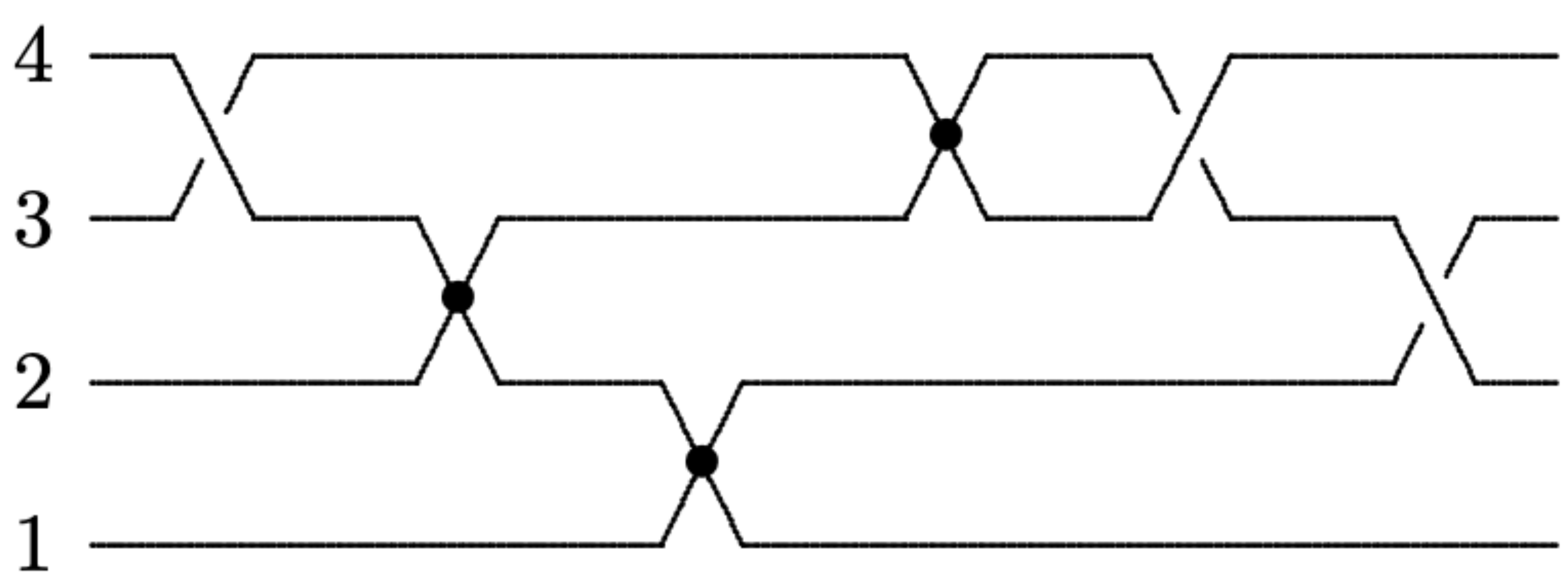


FIGURE 1. Ansatz arrangement (unlabeled) for $\underline{s_3}s_2s_1\underline{s_3s_2}$. Note that $g = \dot{s}_3 y_2(t_2) y_1(t_3) x_3(m_4) \dot{s}_3^{-1} \dot{s}_2$.

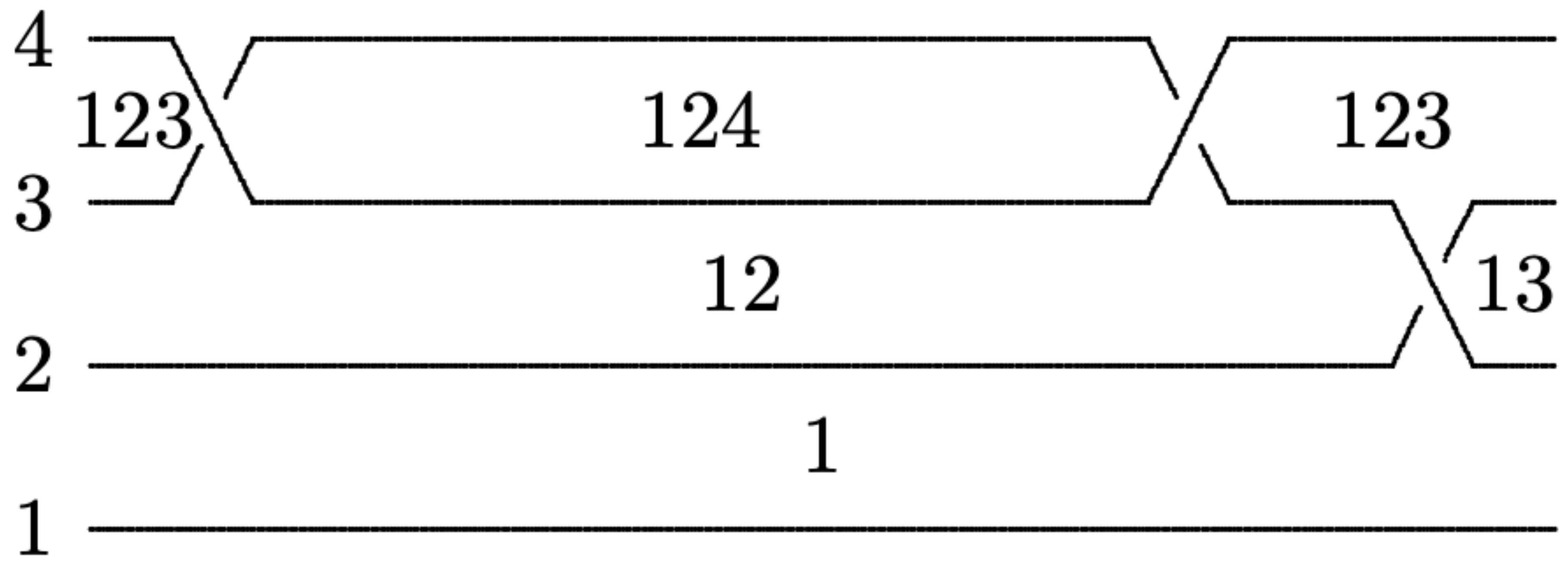


FIGURE 4. Upper arrangement for $\underline{s_3}s_2s_1\underline{s_3s_2}$. Note that $g = \dot{s}_3 y_2(t_2) y_1(t_3) x_3(m_4) \dot{s}_3^{-1} \dot{s}_2$.

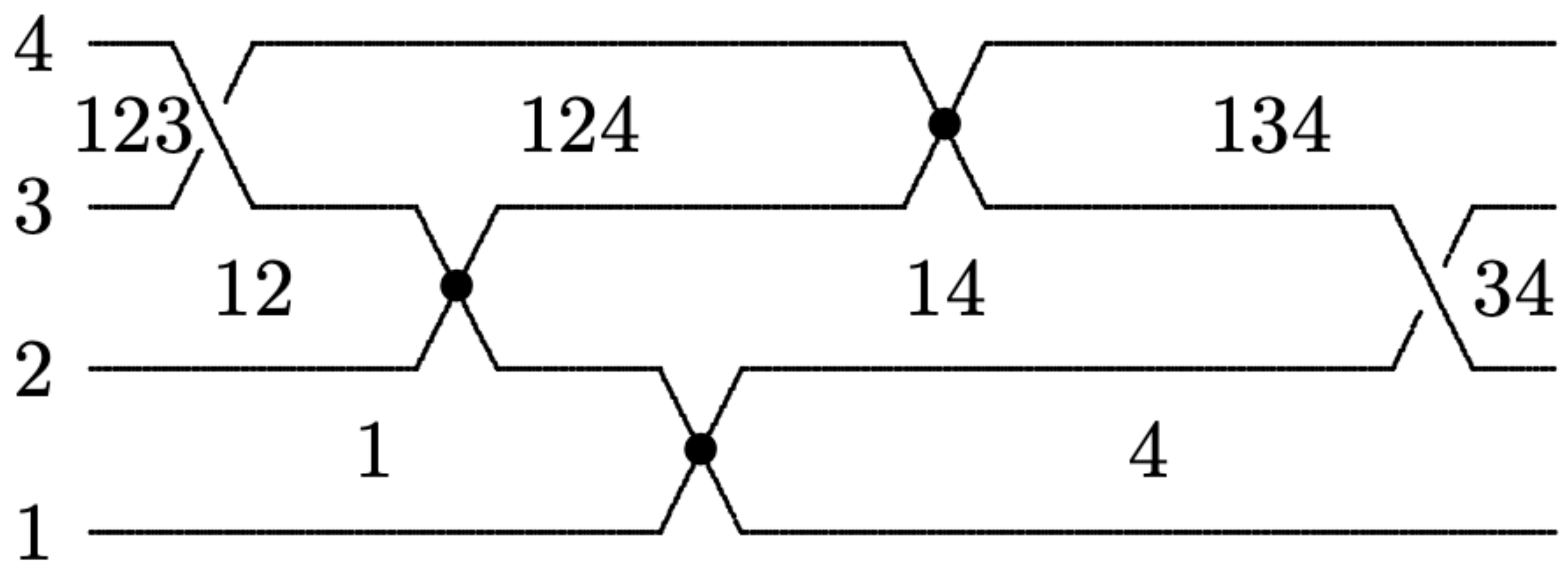


FIGURE 5. Lower arrangement for $\underline{s_3}s_2s_1\underline{s_3s_2}$. Note that $g = \dot{s}_3 y_2(t_2) y_1(t_3) x_3(m_4) \dot{s}_3^{-1} \dot{s}_2$.

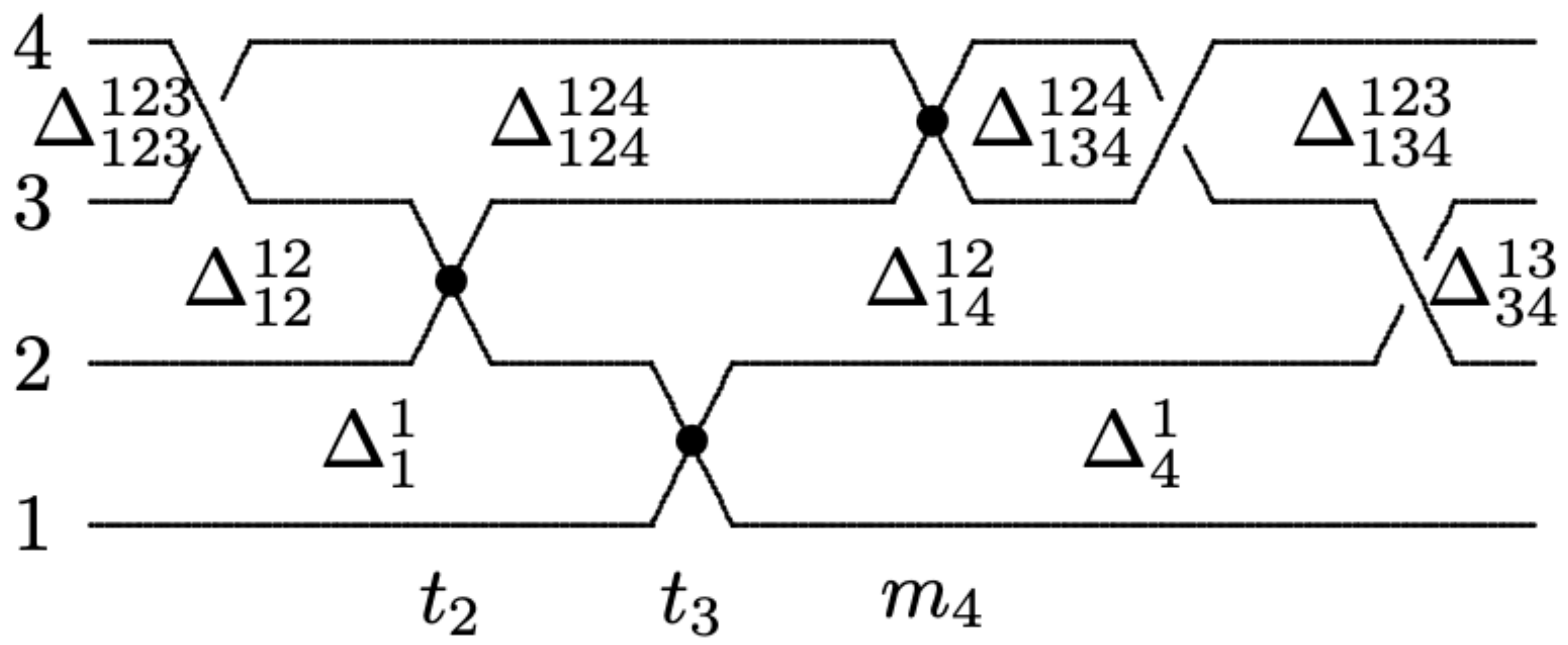


FIGURE 6. Ansatz arrangement for $\underline{s_3}s_2s_1\underline{s_3s_2}$. Note that $g = \dot{s}_3 y_2(t_2) y_1(t_3) x_3(m_4) \dot{s}_3^{-1} \dot{s}_2$.

convenience. The ansatz arrangement can then be used to compute the coefficients t_k and m_k as follows. Suppose $k \in J_{\mathbf{v}}^- \cup J_{\mathbf{v}}^\circ$. Let A_k, B_k, C_k and D_k be the minors labelling the chambers surrounding the singular point in the ansatz arrangement corresponding to k, with A_k and D_k above and below it, and B_k and C_k on the same horizontal level (see Figure 7). It is easy to check that Theorem 7.1 implies that, for $k \in J_{\mathbf{v}}^\circ$,

$$t_k = \frac{A_k(z)D_k(z)}{B_k(z)C_k(z)},$$

and, for $k \in J_{\mathbf{v}}^-$,

$$r_k = \frac{B_k(z)C_k(z)}{A_k(z)D_k(z)}.$$

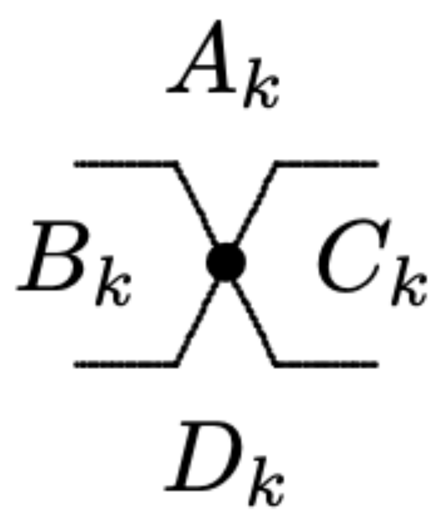


FIGURE 7. Chambers surrounding the singular point corresponding to $k \in J_{\mathbf{v}}^- \cup J_{\mathbf{v}}^\circ$.

Theorem 7.1. *(Generalized Chamber Ansatz)*
Let $B = z\dot{w} \cdot B^+ \in \mathcal{R}_{v,w}$, where $z \in U^+$, $v, w \in W$ and $v \leq w$. Let $\mathbf{w} = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ be a reduced expression for w with factors $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$. Then B lies in a Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$, where $\mathbf{v} = (v_{(0)}, v_{(1)}, \ldots, v_{(n)})$ is a distinguished subexpression for v in $\mathbf{w}$. By Proposition 5.2, there is $g \in G_{\mathbf{v},\mathbf{w}}$ such that $B = g \cdot B^+$. By Definition 5.1 we can write $g = g_1 g_2 \cdots g_n \in U^- \dot{v} \cap B^- \dot{w} B^+$, where

$$g_k = \begin{cases} y_{i_k}(t_k) & k \in J^{\circ}_{\mathbf{v}}, \\ \dot{s}_{i_k} & k \in J^{+}_{\mathbf{v}}, \\ x_{i_k}(m_k)\dot{s}_{i_k}^{-1} & k \in J^{-}_{\mathbf{v}}. \end{cases}$$

For each k, let $g_{(k)} = g_1 g_2 \cdots g_k$ denote the partial product. Then the following hold.

(1) *For $k \in J^{\circ}_{\mathbf{v}}$, we have:*

$$t_k = \frac{\prod_{j \neq i_k} \Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}{\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}$$

(2) *For $k \in J^{-}_{\mathbf{v}}$, we have:*

$$m_k = \frac{\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}{\prod_{j \neq i_k} \Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}} - \Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)}).$$

Theorem 4.16. *With the above notation, let $j \in J_=$, so that $v^{j-1} = v^j$, and set $k = i_j$. Then*

$$t_k = \frac{\Phi^j_{k+1}\Phi^j_{k-1}}{\Phi^{j-1}_k \Phi^j_k}.$$

We remark that $F^{j-1}_{k\pm1} = F^j_{k\pm1}$, so we could switch the superscripts in the numerator to $j-1$ without effecting the formula. Visually, these are the minors for the four chambers surrounding the j-th crossing of the wiring diagram.

Not a stratification!

The closure of a Deodhar piece is not a union of Deodhar pieces

It is easier to give a counterexample in an open Bott-Samelson variety, without the assumption that $(s_{i_1}, s_{i_2}, \ldots, s_{i_a})$ is reduced. Let x be a word in $\{1, 2, \ldots, n-1\}$ and let a and b be distinct distinguished subwords of x. Then aa^R and bb^R will be distinguished subwords of xx^R (for the identity). Suppose that

(1) $\overline{\mathcal{D}(a)} \supset \mathcal{D}(b)$ in $\mathrm{BS}^\circ(x)$ but
(2) $\dim \mathcal{D}(aa^R) \leq \dim \mathcal{D}(bb^R)$ in $\mathrm{BS}^\circ(xx^R)$.

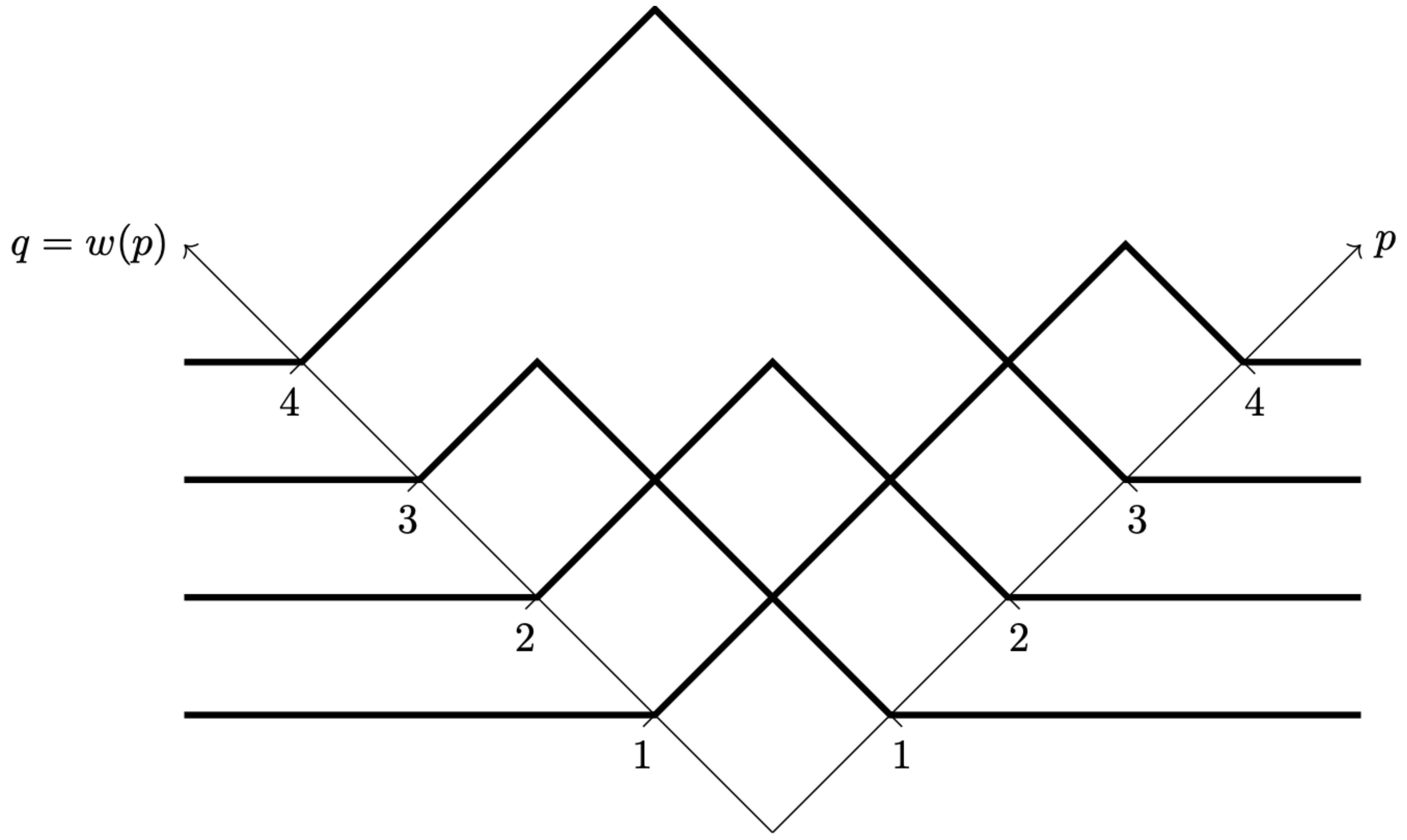
A concrete example is $x = (1, 2, 1)$, $a = (1, \bullet, 1)$, $b = (1, 2, \bullet)$. To check the first condition, we describe points of $\mathrm{BS}^\circ(1, 2, 1)$ using two lines, L_1, L_2, and a plane P_1, as in Example 3.10. Then $\mathcal{D}(1, \bullet, 1)$ is the subvariety where $L_1 = \mathrm{Span}(e_2)$ and $P_1 \neq \mathrm{Span}(e_2, e_3)$, and that $\mathcal{D}(1, 2, \bullet)$ is the subvariety where $L_1 = \mathrm{Span}(e_2)$, $P_1 = \mathrm{Span}(e_2, e_3)$ and $L_2 \neq e_3$. It is easy to see from this description that $\overline{\mathcal{D}(1, \bullet, 1)}$ contains $\mathcal{D}(1, 2, \bullet)$. (Concretely, $\overline{\mathcal{D}(1, \bullet, 1)}$ is the locus where $L_1 = \mathrm{Span}(e_2)$.) To check the second condition, note that $\mathcal{D}(1, \bullet, 1, 1\bullet, 1) \cong \mathcal{D}(1, 2, \bullet, \bullet, 2, 1) \cong \mathbb{G}_m^2 \times \mathbb{G}_a^2$.

Proposition 4.19. *For x, a and b as above, we have $\overline{\mathcal{D}(aa^R)} \cap \mathcal{D}(bb^R) \neq \emptyset$, but $\overline{\mathcal{D}(aa^R)} \not\supseteq \mathcal{D}(bb^R)$. Thus, the Deodhar pieces do not form a stratification of $\mathrm{BS}^\circ(xx^R)$.*

Proposition 4.20. *For x, a and b as above, we have $\overline{\mathcal{D}(a \bullet \bullet a^R)} \cap \mathcal{D}(b \bullet \bullet b^R) \neq \emptyset$, but $\overline{\mathcal{D}(a \bullet \bullet a^R)} \not\supseteq \mathcal{D}(b \bullet \bullet b^R)$. Thus, the Deodhar pieces do not form a stratification of $\mathrm{BS}^\circ(x s_1 s_n x^R)$.*

Examples

That are explicit!



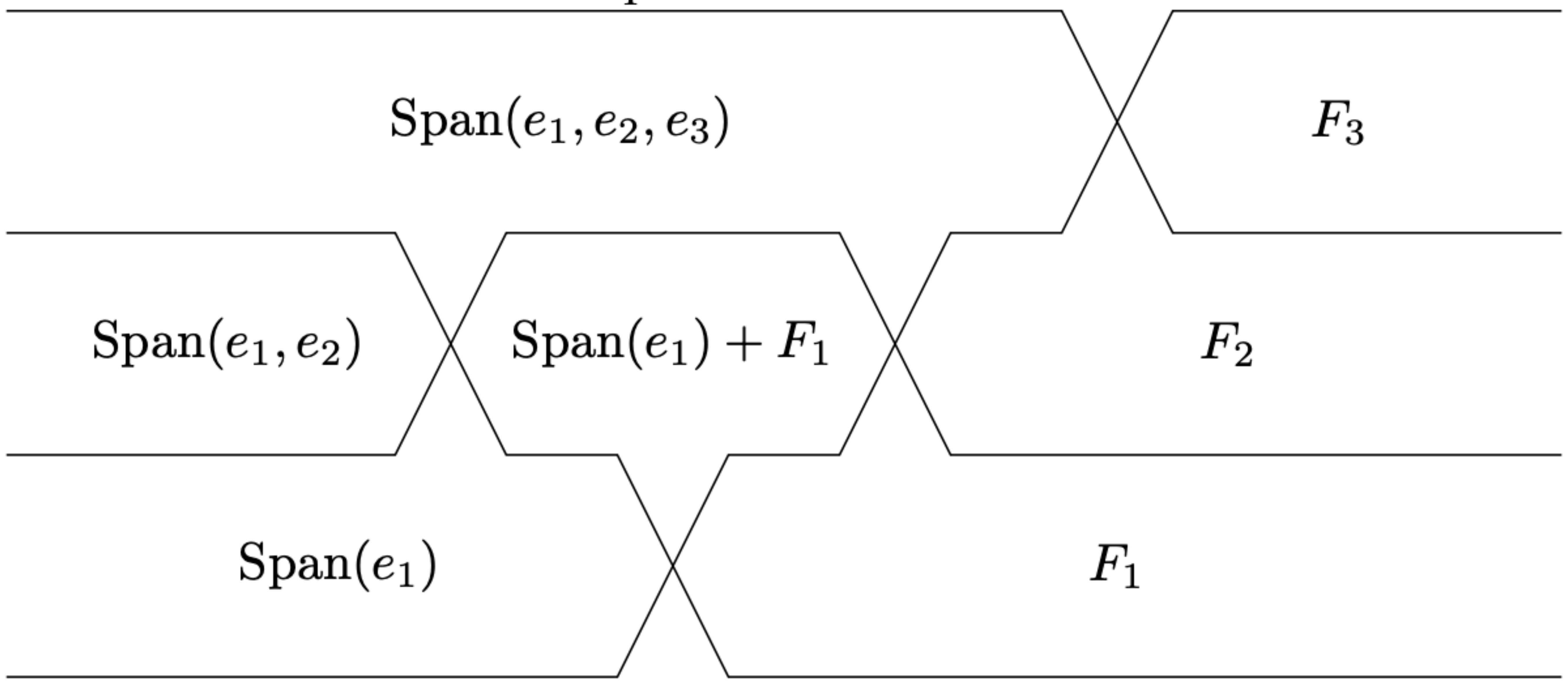
There might be some Le-diagram business under the rug?

Let C be a chamber of the unipeak diagram and consider the geometric construction of the unipeak word above. The top of C consists of two line segments, one with slope 1 and one with slope -1. Let the line segment with slope 1 come from the wire σ_i and let the line segment with slope -1 come from wire σ_j; we have $i \leq j$. We will say that (i, j) is the ***roof*** of C.

Let $(s_{i_1}, s_{i_2}, \ldots, s_{i_a})$ be the unipeak word for w and let $(F_1, F_2, \ldots, F_{n-1})$ be a flag in $\mathring{X}^w$. Since $(s_{i_1}, s_{i_2}, \ldots, s_{i_a})$ is reduced, there is a unique chain of flags $(F^0, F^1, \ldots, F^a)$ in $\mathrm{BS}^\circ(i_1, i_2, \ldots, i_a)$ ending with $F^a = (F_1, F_2, \ldots, F_{n-1})$, and thus a unique labeling of the chambers by subspaces.

Proposition 4.24. *In the above notation, the subspace in chamber C is* $\mathrm{Span}(e_1, e_2, \ldots, e_{i-1}) + F_{w^{-1}(j)-1}$.

Example 4.25. We take the chambers of Example 3.3 and fill them as described here:



We give the analogous formula for univalley wiring diagrams. If C is a chamber of a univalley wiring diagram then there are two wires running along the bottom of C; let σ_i be the decreasing wire and let σ_j be the increasing wire. Then the subspace in C is $\mathrm{Span}(e_1, e_2, \ldots, e_i) \cap F_{w(j)}$.

Using this, we can give an explicit description of the Deodhar strata for a unipeak wiring diagram.

Proposition 4.26. *Let w be a permutation, let $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ be the unipeak wiring diagram for w and let F be a flag in $\mathring{X}^w$. Then the knowledge of which Deodhar piece F is in is equivalent to the knowledge, for all $i \leq i'$ and all j, of $\dim\big(\mathrm{Span}(e_1, e_2, \ldots, e_{i-1}, e_{i'+1}, e_{i'+1}, \ldots, e_n) + F_j\big)$. If we let $F = gB_+$ then, equivalently, the knowledge of which Deodhar piece F is in is equivalent to the knowledge, for all $i \leq i'$ and all j, of the rank of the submatrix of g in rows $\{i, i+1, \ldots, i'\}$ and columns $\{1, 2, \ldots, j\}$.*