Preliminarys:

B_+, B_- : subgroups of invertible upper & lower triangle matrices.

$\Delta_{i_1 \cdots i_k}(V)$: Plücker coordinates.

Fl_n / $Fl(k_1, k_2, \cdots, k_p; n)$.

$\mathring{R}_u^w = \mathring{X}_u \cap \mathring{X}^w$ open Richardson variety.

$R_u^w = X_u \cap X^w$ (closed) ...

$\Pi_u^w = \pi(R_u^w)$ $\pi: Fl_n \longrightarrow Fl(k_1, k_2, \cdots, k_p)$ projected.

$BS^\circ(i_1, \cdots, i_a) \subseteq Fl_n^{a+1}$ consisting $(F^0, F^1, \cdots, F^a)$

$F^0 = eB_+$ $F^{j-1} \xrightarrow{s_{i_j}} F^j \Rightarrow (F_i^{j-1} = F_i^j$ for $i \neq i_j$ & $F_{i_j}^{j-1} \neq F_{i_j}^j)$.

(open) Bott Samelson variety.

$F^0 = eB_+$ $1 \leq j \leq a$ either $F^{j-1} \xrightarrow{s_{i_j}} F^j$ or $F^{j-1} = F^j$.

(closed)

Deodhar decomposition,

$\mathring{R}_u^w$ is irreducible and its Deodhar pieces disjoint union, $\exists$ exactly one Zariski dense <u>Deodhar piece $m\downarrow = 0$</u>

Definitions:

Total positivity / totally nonnegative

Objects	condition.
Real matrix.	$\forall I, J$ $\Delta_{I,J}$ positive
$\Rightarrow$ Cauchy-Binet	$\Delta_{IK}(AB) = \sum_{I,J} \Delta_{IJ}(A)\Delta_{JK}(B)$
$\Rightarrow$ semigroup.	
Subspace.	$\forall$ Plücker coordinates positive (up to a globle sign flip)
Grassmannian. $G(k,n)_{\geq 0}$.	set of totally nonnegative subspaces of dim-k.

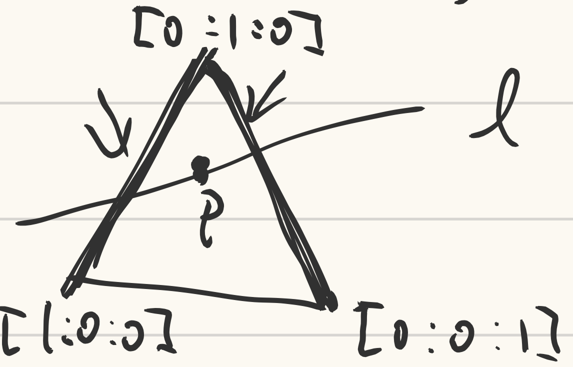
Flag $Fl_n^{\geq 0}$ $\quad F_1 \subset F_2 \subset \cdots \subset F_{n-1}$ F_k positive $\forall k$,

Example: $Fl_3(\mathbb{R})$: element: pair of a point & line

point $p = (\Delta_1 : \Delta_2 : \Delta_3)$ & line $\ell = \{(x_1 : x_2 : x_3) \mid \Delta_{23}x_1 - \Delta_{13}x_2 + \Delta_{12}x_3 = 0\}$

condition:

$\Delta_1, \Delta_2, \Delta_3 \geq 0$. $\Delta_{12}, \Delta_{23}, \Delta_{13} \geq 0$.



ℓ crosses the boundry of Δ on the left & right

Lemma 5.2: V t.n. subspace. $g \in GL_n$ t.n. matrix

$\Rightarrow gV$ t.n.

$F_\bullet$ t.n. flag $g \in GL_n$ t.n. matrix.

$\Rightarrow gF_\bullet$ t.n.

pf: Cauchy-Binet identity / g apply to every F_k.

Every t.n. subspace is included in a t.n flag.

Prop 5.3: V t.n. subspace with dim k. $\exists$ t.n flag with $F_k = V$.

pf: $F_1 \subset F_2 \subset \cdots \subset F_{k-1} \subset F_k = V \subset F_{k+1} \subset \cdots \subset F_{n-1}$

construct F_{k-1} from V.

write $x_1, x_2, \cdots, x_n$ for the coordinates on $\mathbb{R}^n$

$\exists j$ the index s.t. $x_{j+1} = x_{j+2} = \cdots = x_n = 0$ & x_j is not identically 0 on V. $V' = V \cap \{x_j = 0\}$

V basis $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$

V' basis $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k-1}\}$. v_k form: $e_j + \sum_{i<j} c_i e_i$

Computing Plücker coordinates using the matrix whose clomns are $\vec{v}_i$.

$$\Delta_J(V') = \begin{cases} \Delta_{J\cup\{j\}}(V) & J \subseteq [j-1] \\ 0 & J \not\subseteq [j-1] \end{cases} \Rightarrow V' \text{ t.n. just take } F_{k-1} = V'$$

Another definition by Lusztig:

$B_-^{\geq 0}$ semigroup of t.n. matrice in B_-.

$Fl_n^{\geq 0}$ the closure of set of flags gB_+ for $g \in B_-^{\geq 0}$

$\Rightarrow$ All Equivalent (Lusztig).

t.n. part of flag variety is well behaved.

interacts Richardson variety & Deodhar decompositions

Let $\mathring{R}_u^{w,\geq 0} := \mathring{R}_u^w(\mathbb{R}) \cap Fl_n^{\geq 0}$.

Thm s.t. $u \leq w$ Then $\mathring{R}_u^{w,\geq 0} \cong \underline{\mathbb{R}_{>0}^{\ell(w)-\ell(u)}}$

More precisely w reduced word $(i_1, i_2, \cdots, i_a)$.

$\mathcal{D}$ the corresponding Deodhar torus in $\mathring{R}_u^w$.

$\mathcal{D}(\mathbb{R}) \cong (\mathbb{R}_{>0} \sqcup \mathbb{R}_{<0})^{\ell(w)-\ell(u)}$. $\mathring{R}_u^{w,\geq 0}$ is one of connected component $\mathcal{D}(\mathbb{R})$.

Remark: $\mathbb{R}$ and $\mathbb{R}_{>0}$ isomorphic as smooth manifolds.

we write $\mathbb{R}_{>0}^{\ell(w)-\ell(u)}$. $\mathring{R}_u^w$ natural coordinates are multipliative coordinates on a Deodhar torus.

Example: Consider the largest Richardson in Fl_3 $\mathring{R}_{123}^{321}$.

the t.n. points $\underbrace{(\Delta_1 : \Delta_2 : \Delta_3) \times (\Delta_{12} : \Delta_{13} : \Delta_{23})}$ >0 positive

321 has 2 reduced word: $s_1 s_2 s_1$ & $s_2 s_1 s_2$.

corresponding 2 parametrizations.

$$(t_1,t_2,t_3)\longmapsto\begin{bmatrix}1&&\\t_1&1&\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&t_2&1\end{bmatrix}\begin{bmatrix}1&&\\t_3&1&\\&&1\end{bmatrix}=\begin{bmatrix}1&0&0\\t_1+t_3&1&0\\t_2t_3&t_2&1\end{bmatrix}$$

$$(u_1,u_2,u_3)\longmapsto\begin{bmatrix}1&&\\&1&\\&u_1&1\end{bmatrix}\begin{bmatrix}1&&\\u_2&1&\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&u_3&1\end{bmatrix}=\begin{bmatrix}1&0&0\\u_2&1&0\\u_1u_2&u_1+u_3&1\end{bmatrix}$$

(t_1,t_2,t_3) (u_1,u_2,u_3) range over $\mathbb{R}^3_{>0}$. both maps will have the same image. $\mathring{R}_u^{w,\geq 0}$ is independent of the choice of reduced word for w. even though Deodhar pieces depend.

Remark: $u=e$. $\mathring{X}_e$ is the largest Schubert cell $\cong \mathbb{A}^{\binom{n}{2}}$ can be identified with U_- (lower unipotent group).

$u\longmapsto uB_+$ $\quad U_-\to\mathring{X}_e$.

The t.n flags in $\mathring{X}_e^{\geq 0}$ thus decompose into a union of the Richardson pieces. $\mathring{X}_e^{\geq 0}=\bigsqcup_{w\in S_n}\mathring{R}_e^{w,\geq 0}$.

w reduced $(s_{i_1},\cdots,s_{i_l})$. $u=e$. $J_+=\emptyset$.

so there only exists $y_i(t)$'s.

$(t_1,t_2,\cdots,t_l)\mapsto \underbrace{y_{i_1}(t_1)\,y_{i_2}(t_2)\cdots y_{i_l}(t_l)}B_+$

isomorphism $\mathbb{R}^l_{>0}\to\mathring{R}_e^{w,\geq 0}$ $\quad U_-^{\geq 0}$ semigroup.

the image of this map a unipotent cell.

independent of the choice of reduced word $\bigcup_{w\in S_n}U_-^{w,\geq 0}$

$u\neq e$ the image depends on the choice of reduced word. $\qquad\cong\mathbb{R}^{l(w)}_{>0}$.

example: $Fl_3^{\geq 0}$. $u=213$ $\quad w=321=(s_1s_2s_1)=(s_2s_1s_2)$.

parametrization matrix.

$$\begin{bmatrix}1&&\\t_1&1&\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&t_2&1\end{bmatrix}\begin{bmatrix}&-1&\\1&&\\&&1\end{bmatrix}=\begin{bmatrix}0&-1&0\\1&-t_1&0\\t_2&0&1\end{bmatrix}.$$

$$\begin{bmatrix}1&&\\&1&\\&u_1&1\end{bmatrix}\begin{bmatrix}&-1&\\1&&\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&u_3&1\end{bmatrix}=\begin{bmatrix}0&-1&0\\1&0&0\\u_1&u_3&1\end{bmatrix}.$$

Thm 5.9 $F_\bullet$ t.n. flag in $\mathring{R}_u^{w,\geq 0}$, w reduced word $(s_{i_1},\cdots,s_{i_a})$. $(F^0,F^1,\cdots,F^a)$ unique sequence of flags in $BS^\circ(i_1,i_2,\cdots,i_a)$ $F^a=F$. Then all the flags F^j are t.n.

pf: [MarshRietsch04] [Rietsch99].

$\mathring{R}_u^{w,\geq 0}$ Plücker coordinates. $J\in\binom{[n]}{k}$.

if $J=u[k]$ or $w[k]$ Δ_J nonzero on $\mathring{R}_u^w$.

if there is no v. $u\preceq v\preceq w$ and $v[k]=J$.

then Δ_J is identically 0 on $\mathring{R}_u^w$.

if there is v. $u\leq v\preceq w$ and $v[k]=J$ but $J\neq u[k].w[k]$ Δ_J is not identically 0 but maybe 0.

Thm 5.10. Let $u\preceq w$ $J\in\binom{[n]}{k}$.

If there exist v. with $u\preceq v\preceq w$ and $v[k]=J$.

then $\Delta_J>0$ everywhere on $\mathring{R}_u^{w,\geq 0}$

If there no such v. then $\Delta_J=0$ everywhere on $\mathring{R}_u^{w,\geq 0}$

pf: Fix w reduced word $(s_{i_1},s_{i_2},\cdots,s_{i_a})$ $u\preceq w$.

Put $w^j=s_{i_1}s_{i_2}\cdots s_{i_j}$ v^j the positive subsequence

$v^a=u$ $\quad v^{j-1}=\begin{cases}v^j s_{i_j} & v^j s_{i_j}<v_j\\ v^j & v^j s_{i_j}>v_j\end{cases}$

$\Rightarrow v^J = r_1 r_2 \cdots r_j \quad r_j = \begin{cases} e & j \in J_= \\ s_{i_j} & j \in J_{+} \end{cases}$

$\forall\, j \in J_=$ take a varible t_j. $h_j = \begin{cases} y_{i_j}(t_j) & j \in J_= \\ \dot{s}_{i_j} & j \in J_{+} \end{cases}$

so Thm 5.4 $\mathring{R}_u^{w,>0}$ parametrized

$(t_j)_{j\in J} \longrightarrow h_1 h_2 \cdots h_a B_+$.

Lemma 5.11 : $\forall\, 1 \le j \le a+1$ $\forall\, J \in \binom{[n]}{k}$

the minor $\Delta_J (h_j h_{j+1} \cdots h_a)$ is a polynomial in the t-varibles with nonnegative coefficents

Δ_J is identically 0 or it's positive.

2. $Fl_n^{\geq 0}$ into the pieces $\mathring{R}_u^{w,>0} \cong \mathbb{R}_{>0}^{\ell(w)-\ell(u)}$

$\Rightarrow Fl_n^{\geq 0}$ is a CW complex. (regular).

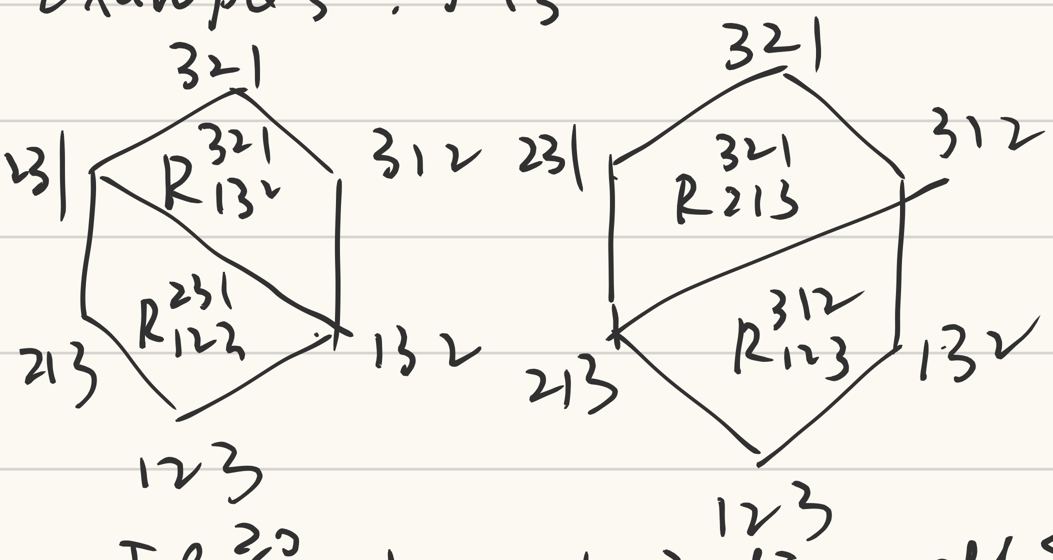
Rietsch & williams (Galashin, Karp, Lam).

Thm 5.13 $Fl_n^{\geq 0}$ is a regular CW complex, where the cells are the $\mathring{R}_u^{w,>0}$. $\overline{\mathring{R}_u^{w,>0}}$ is union of varion $\mathring{R}_{u'}^{w',>0}$

$(\overline{\mathring{R}_u^{w,>0}}, \mathring{R}_u^{w,>0})$ is homeomorphic (closed ball,

More precisely $\overline{\mathring{R}_u^{w,>0}} = \bigsqcup_{u \le u' \le w' \le w} \mathring{R}_{u'}^{w',>0}$ interior of ball).

Examples. $Fl_3^{\geq 0}$.



$Fl_3^{\geq 0}$ has 1 3-dim cell 4 2-dim cells

8 1-dim cells and 6 0-dim cells

regular CW complex:

$\Phi : D^n \to X$. Φ distrinct $\mathrm{Int}(D^n)$ is homeomorphism.

continuous &

Positivity in partial flag manifolds.

G general reductive group & parabolic subgroup $P \geq B_+$.

this applies to partial flag manifolds.

Let $F_{k_1} \subset F_{k_2} \subset \cdots \subset F_{k_r}$.

Then (Lusztig). $F_{k_1} \subset \cdots \subset F_{k_r}$ is t.n. if and only if the partial flag can be embedded in t.n. complete flag.

If is Grassmannian, flags is a single subspace

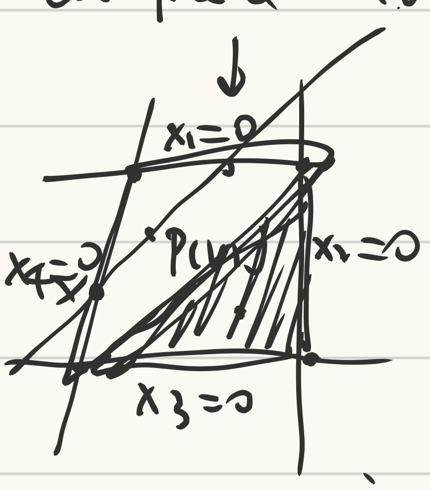
Prop 5.3. V is t.n. part of $G(k,n) \Leftrightarrow$ Plücker coordinates are nonnegative

Example. $V_1 \subset V_3$ a partial flag in $\mathbb{R}^4$.

V_1, V_3 $[\Delta_1 : \Delta_2 : \Delta_3 : \Delta_4]$ $[\Delta_{123} : \Delta_{124} : \Delta_{134} : \Delta_{234}]$.

with $\Delta_1\Delta_{234} - \Delta_2\Delta_{134} + \Delta_3\Delta_{124} - \Delta_4\Delta_{123} = 0$.

complete to $V_1 \subset V_2 \subset V_3$ t.n. flag $\Rightarrow \Delta_1\Delta_{234} - \Delta_2\Delta_{134} \geq 0$



(e_1, e_2, e_3, e_4) standard basis in $\mathbb{R}^4$.

x_1, x_2, x_3, x_4 the dual coordinates.

tetrahedron of $x_i \geq 0$ T.

V_1 t.n. $\Rightarrow \mathbb{P}(V_1)$ a point T.

V_3 t.n. $\Rightarrow \mathbb{P}(V_3)$ crosses T on the edge.

$\overline{e_1e_2}$ $\overline{e_2e_3}$ $\overline{e_3e_4}$ $\overline{e_4e_1}$

$\Delta_1\Delta_{234} - \Delta_2\Delta_{134} \geq 0$

$F_{k_1} \subset F_{k_2} \subset \cdots \subset F_{k_r}$ subspace t.n. $(k_1, k_2, \cdots, k_r)$.

[Bloch / Karp]. (Problem).

projected Richardson variety. $\pi : \mathring{R}^{w}_{u} \to \mathring{\Pi}^{w}_{u}$

define $\mathring{\Pi}^{w,\geq 0}_{u} := \pi(\mathring{R}^{w,\geq 0}_{u})$. $\mathring{\Pi}^{w,\geq 0}_{u}$ isomorphic $\mathbb{R}^{\ell(w)-\ell(u)}_{>0}$.

Thm. 5.18 $Fl(k_1, k_2, \cdots, k_r)^{\geq 0}$ a regular CW complex.

cells $\mathring{\Pi}^{w,\geq 0}_{u}$