Plücker Algebra.

- Grassmannian & Plücker relations (straightening relations)
- Flag varieties
- Schubert / Richardson varieties standard monomials.

$Gr(k,n) = \{ k\text{-dim subspaces of } \mathbb{C}^n \}$

$$\underline{\underline{GL_k}} \backslash k\underbrace{\begin{bmatrix} & M & \end{bmatrix}}_{n} \hookrightarrow \mathbb{P}^{\binom{n}{k}-1} \quad \text{labelled by } I \in \binom{[n]}{k}.$$

$P_I = \det(M_I)$
$\hookrightarrow$ cols in I.

Defining equation of $Gr(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$

$$P_{145} \cdot P_{236} : \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & 6 \\ \hline \end{array} = - \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 5 & 3 \\ \hline 4 & 6 \\ \hline \end{array}$$

Plücker relation
$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 & 3 \\ \hline 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 4 & 1 \\ \hline 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 6 & 2 \\ \hline 4 & 3 \\ \hline 5 & 1 \\ \hline \end{array} = 0$$

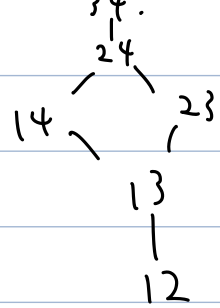
$\mathbb{C}[Gr(k,n)] = \mathbb{C}[P_I] / \langle \text{Plücker Relations} \rangle.$

$\swarrow$

Plücker poset

$$P_I \leq P_J \text{ if } \begin{array}{c} \{i_1 < i_2 < \dots < i_k\} \\ \wedge \quad \wedge \qquad \wedge \\ \{j_1 < j_2 < \dots < j_k\} \end{array} \Longleftrightarrow \text{SSYT}.$$

$|I| = |J|$ Young's lattice.



Take any linear extension of poset.
take degree revlex order on $\mathbb{C}[P_I]$.

Why this term order?

$$\text{init}_c \langle PR \rangle = \langle P_I P_J : I, J \text{ not comparable} \rangle.$$

$$\text{std mon} = P_{I^{(1)}} P_{I^{(2)}} \cdots P_{I^{(d)}} \iff k\boxed{}^{d} \text{ SSYT.}$$
$$I^{(1)} \le I^{(2)} \le \cdots \le I^{(d)}$$

Plücker Relations do not form a Gröbner basis.
but Straightening relations do!

$$\langle PR \rangle = \left\langle \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array} \right\rangle$$

2345 $\quad \hookrightarrow$ Gröbner basis.

complete

Flag varieties

$$Fl_n = \{ 0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{C}^n \}$$
$$\dim(V_i) = i$$

take V_k. $\downarrow$

$Gr(k,n)$

$$Fl_n \hookrightarrow Gr(1,n) \times Gr(2,n) \times \cdots \times Gr(n-1,n)$$
$$\mathbb{P}^{\binom{n}{1}-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$$

$$\mathbb{C}[P_I : I \subsetneq [n]] / I(Fl_n).$$

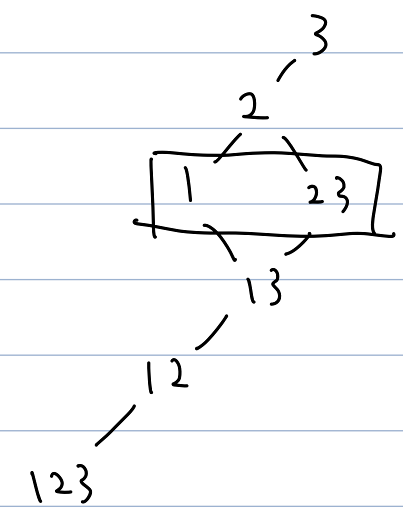
↳ gen by deg 2 straightening relations.

$P_{\dot I} P_{\dot J}$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline\end{array}$$

one of straightening relations.

$n=3$ $\quad 1,2,3,12,13,23,123$



$\begin{matrix} 2 & 2 \\ 3 & \end{matrix} \qquad \begin{matrix} 2 & 1 \\ 3 & \end{matrix}$ not SSYT.

total order if $|I|<|J|$ then $P_I > P_J$.

take deg revlex order

$\mathrm{init}(I(Fl_n)) =$ mon that's not SSYT

e.g. $P_{23} \cdot P_1$

std mon $=$ SSYT.

Fix w $\quad X^w$ $\quad \dim = \ell(w)$ $\qquad H^0(X^w_v, \mathcal{L}_\lambda)$

T-fixed pts $u \le w$ in Bruhat order.

Character : key polynomials $K_{w\lambda}$ ↙ Hilbert series degree

$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & \\ \hline 4 & & \\ \hline\end{array}$ is std mon of $X_w \iff \exists\ id \le u^{(1)} \le u^{(2)} \le u^{(3)} \le w$

s.t $u^{(1)}[3] = \{1,2,4\}$

$u^{(2)}[2] = \{2,3\}$

$u^{(3)}[1] = \{4\}$.

$\Updownarrow$

$w \ge 4231$.

Prop : $\forall u \in S_n$, $\forall k \in [n]$, $\forall I \subseteq \binom{[n]}{k}$ s.t $I \ge u[k]$, there is an unique w s.t $u \leq w$, $w[k] = 1$.

$id \leq 1243 \leq 2341 \leq 4231$

$= \sum$ std mon. whose unique smallest perm $= w'$.

$$\text{Key poly} = \sum_{\substack{\text{right key} \\ \leq \text{key tableaux}}} \text{Key atoms}$$